Homework 10

MATH 301/601

Due Wednesday, May 1, 2024

Instructions. *Read the appropriate homework guide (Homework Guide for 301 or Homework Guide for 601) to make sure you understand how to successfully complete the assignment. All claims must be sufficiently justified.*

Exercise 1. Complete the following exercises from Section 9.4 in the course textbook:

17, 19 (just the first part, and see Example 9.25), ***22**†, 24, 25, 48

†For #22, do not use the classification of finite abelian groups.

***Exercise 2.** Let G be a group, and let H and K be subgroups of G. Prove that if G is the internal direct product of H and K, then G is isomorphic to the external direct product $H \times K$. (Hint: show that the map $\varphi\colon H \times K \to G$ given by $\varphi((h,k)) = hk$ is an isomorphism.)

***Exercise 3.** The goal of this exercise is to prove that every group of order four is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let $G = \{e, a_1, a_2, a_3\}$ be a group with four elements, where e is the identity. We already know that every cyclic group of order four is isomorphic to $\mathbb{Z}_4$, so for the following parts assume that G is an order four group that is not cyclic.

(a) Prove that $g^2 = e$ for all $g \in G$.

(b) Prove that $a_i a_j = a_k$ whenever i, j, and k are all distinct.

(c) Let $G' = \{e', a_1', a_2', a_3'\}$ be another non-cyclic order four group. Define the function $\varphi\colon G \to G'$ by $\varphi(e) = e'$, $\varphi(a_1) = a_1'$, $\varphi(a_2) = a_2'$, and $\varphi(a_3) = a_3'$. Prove that φ is an isomorphism. (In particular, setting $G' = \mathbb{Z}_2 \times \mathbb{Z}_2$, we see that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.)

Exercise 4. Read the notes for the missed class on April 17. The notes can be found at `http://qc.edu/~nvlamis/301S24/Notes_Week12.pdf`.

****Exercise 5.** Let N be a group, and let H be a subgroup of Aut(N), the automorphism group of N. The *(external) semidirect product* of N and H is the group $N \rtimes H$ whose underlying set is $N \times H$ and whose group operation is defined by $(a, \varphi)(b, \psi) = (a\varphi(b), \varphi \circ \psi)$.

(a) Prove that $N \rtimes H$ is a group (yes, I said it was a group in the definition, but that needs a proof).

(b) Let G be a group, and let N and H be subgroups of G such that

  (i) $N \cap H = \{e\}$,

  (ii) $G = NH = \{nh : n \in N, h \in H\}$, and

  (iii) $hnh^{-1} \in N$ for all $n \in N$ and all $h \in H$.

(Note that condition (ii) and (iii) together imply that N is a normal subgroup of G, see the Week 12 notes.) Condition (iii) says that each element of H induces an automorphism of N via conjugation, that is, for $h \in H$ we can define $\varphi_h \in \mathrm{Aut(N)}$ by $\varphi_h(n) = hnh^{-1}$ for all $n \in N$. Identifying h with φ_h, we can view H as a subgroup of Aut(N).

Under these hypotheses and under this identification of H with a subgropup of Aut(N), prove that $G \cong N \rtimes H$. Here, we say G is the *(internal) semidirect product* of N and H.

(Remark: the map $h \mapsto \varphi_h$ may fail to be an isomorphism, so that H may not actually be a subgroup of Aut(N). However, this map is a homomorphism (see Week 12 notes): there is no need to worry about this inaccuracy. The definition of semidirect product given above is actually stricter than the actual definition, which is actually in terms of homomorphisms.)