Worksheet/Notes for Wednesday, April 17 MATH 301/601

Instructions. Please carefully read these notes by **Wednesday, May 1** and make a serious attempt to understand the content. The exercises will be part of Homework 12 (due May 8), but to truly understand what you are reading, you will have to at least think about them before then. I recommend meeting with your peers during our cancelled class on Wednesday, April 17 to think through some portion of the exercises.

The goal of these notes is to introduce you to two new definitions: normal subgroups and homomorphisms. In the textbook, normal subgroups are introduced in Section 10.1 and homomorphisms are introduced in Section 11.1. We will not cover the entirety of those sections here, so when we meet again we will discuss those sections in more depth. Here, we will explore the definitions, basic examples, and basic properties. When we return after the break, we see how these two notions are deeply connected.

Normal subgroups

Given a subgroup H of a group G, recall that the left coset of H with representative g is the set $gH = \{gh : h \in H\}$, and the right coset of H with representative g is the set $Hg = \{hg : h \in H\}$. We have seen that it is not always the case that gH and Hg are equal, so when this happens we give the subgroup a special designation.

Definition 1 (Normal subgroup)**.** A subgroup N of a group G is *normal*, written $N \triangleleft G$, if $gN = Ng$ for all $g \in G$.

After digesting the definition, it should be clear that every subgroup in an abelian group is normal, so the definition is most interesting in the setting of non-abelian groups.

Example 1. Let N be an index two subgroup of a group G. It was a homework problem to show that, under this hypothesis, $gN = Ng$ for all $g \in G$; in other words, you proved that every index two subgroup is normal. For instance, A_n is normal in S_n, as $[S_n : A_n] = 2$. Similarly, the subgroup of rotations in D_n has index two, and hence it is normal.

Example 2. The special linear group $\mathrm{SL}(\mathrm{n}, \mathbb{R})$ (i.e., the group of determinant one invertible $n \times n$ matrices) is a normal subgroup of the general linear group $\mathrm{GL}(\mathrm{n}, \mathbb{R})$ (i.e., the group of $n \times n$ invertible matrices). Can you prove it? (Below we give a simpler way of checking whether a subgroup is normal.)

Normally (no pun intended), I think about normal subgroups in terms of conjugation. Recall that to conjugate a group element a by another group element g means to perform the operation gag^{-1}. We can also conjugate an entire subgroup: if H is a subgroup of a group G and $g \in G$, then H conjugated by g is the subgroup $gHg^{-1} = \{ghg^{-1} : h \in H\}$ (technically, you should check that gHg^{-1} is in fact a subgroup). The following proposition tells us that a subgroup is normal if and only if it is invariant under conjugation.

Proposition 2. *Let N be a subgroup of a group G. Then the following are equivalent:*

(i) N is normal.

(ii) $gNg^{-1} \subset N$ *for all* $g \in G$.

(iii) $gNg^{-1} = N$ *for all* $g \in G$.

Proof. First, assume that N is normal. Let $n \in N$, and let $g \in G$. Then, as $gN = Ng$, there exists $n' \in N$ such that $gn = n'g$, and hence $gng^{-1} = n'$. This implies that $gng^{-1} \in N$, and as both n and g were arbitrary, we have that $gNg^{-1} \subset N$, establishing that (i) implies (ii).

Now, let us assume that $gNg^{-1} \subset N$ for all $g \in G$. We will establish the equality of these two sets for each $g \in G$. Fix $g \in G$. Let $n \in N$. By assumption, $g^{-1}Ng \subset N$, so $g^{-1}ng \in N$, implying that there exists $n' \in N$ such that $g^{-1}ng = n'$. It follows that $n = gn'g^{-1} \in gNg^{-1}$. As both n and g were arbitrary, we have established that $N \subset gNg^{-1}$ for all $g \in G$. Therefore, $N = gNg^{-1}$ for all $g \in G$, establishing (ii) implies (iii).

To finish, assume $gNg^{-1} = N$ for all $g \in G$. Fix $g \in G$. Let $n \in N$. Then $gng^{-1} \in gNg^{-1} = N$, so there exists $n' \in N$ such that $gng^{-1} = n'$, implying $gn = n'g$. In particular, $gn \in Ng$, and hence, $gN \subset Ng$. A similar argument (replacing g with g^{-1} and vice versa) shows that $Ng \subset gN$, and hence $gN = Ng$ for all $g \in G$. Therefore, N is normal in G, establishing (iii) implies (i). □

Morally, the proposition says that the normal subgroups are those that have a "coordinate-free" definition. This might make sense if we consider an example of a subgroup that is not normal.

Exercise 1. Let H be the subgroup of S_n defined by $H = \{\sigma \in S_n : \sigma(1) = 1\}$. Now, suppose $\mu \in S_n$ such that $\mu(1) = 2$. Prove that $\mu H \mu^{-1} = \{\sigma \in S_n : \sigma(2) = 2\}$.

The above exercise tells us that H is not normal, as it fails condition (iii) of the proposition: relabeling the elements of $\{1, 2, \dots n\}$ did not preserve the subgroup H.

Going back to my comment before the proposition, in my own work, if I want to check if a subgroup N of a group G is normal, I will check that $gng^{-1} \in N$ for all $n \in N$ and all $g \in G$, which is just checking condition (ii) in the proposition. Why don't you practice:

Exercise 2. Use condition (ii) in Proposition 2 to prove the following statements:

(a) A_n is a normal subgroup of S_n.

(b) $\mathrm{SL}(\mathrm{n}, \mathbb{R})$ is a normal subgroup of $\mathrm{GL}(\mathrm{n}, \mathbb{R})$ (you have to remember the basic properties of the determinant).

(c) The intersection of two normal subgroups is normal.

Homomorphisms

Whenever you are studying some class of mathematical objects, there is some associated natural class of functions between such objects. In the setting of group theory, these are the homomorphisms.

Definition 3 (Homomorphism)**.** Let G_1 and G_2 be groups. A function $\varphi\colon G_1 \to G_2$ is a *homomorphism* if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G$.

This definition should feel familiar as an isomorphism is a bijective homomorphism, and so we have already been working with a special class of homomorphisms. Let us look at some other examples.

Example 3.

- Let A be an $m \times n$ matrix. Then $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ given by $T_A(v) = Av$ is a homomorphism, when we consider $\mathbb{R}^n$ and $\mathbb{R}^m$ as groups with respect to coordinate-wise addition. It is a homomorphism because

$$T_A(u+v) = A(u+v) = Au + Av = T_A(u) + T_A(v)$$

  (of course slightly more is true as T_A is linear).

- If $\varphi_n\colon \mathbb{Z} \to \mathbb{Z}_n$ is the function defined by $\varphi(k) = \bar{k}$, then φ is a homomorphism, since

$$\varphi(k+\ell) = \overline{k+\ell} = \bar{k} + \bar{\ell} = \varphi(k) + \varphi(\ell)$$

- The determinant function $\det\colon \mathrm{GL(n,\mathbb{R})} \to \mathbb{R}^\times$ is a homomorphism as $\det(AB) = \det(A)\det(B)$.

Exercise 3. Recall that given $z \in \mathbb{C}$, there exist real numbers x and y such that $z = x+iy$. The *complex conjugate* of z, denoted $\bar{z}$, is the complex number $\bar{z} = x - iy$. The *magnitude* of a complex number z, denoted $|z|$, is the real number $\sqrt{z\bar{z}} = \sqrt{x^2+y^2}$.

(i) Prove that complex conjugation is a homomorphism $\mathbb{C}^\times \to \mathbb{C}^\times$, i.e., prove that $\overline{zw} = \bar{z}\bar{w}$ for any $z, w \in \mathbb{C}$ (in fact, it is an isomorphism).

(ii) Prove that $|\cdot|\colon \mathbb{C}^\times \to \mathbb{R}^\times$ is a homomorphism.

Next, we record the fact that homomorphisms have some the same basic properties of isomorphisms:

Proposition 4. *Let $\varphi\colon G_1 \to G_2$ be a homomorphism.*

(i) $\varphi(e_{G_1}) = e_{G_2}$.

(ii) $\varphi(a^{-1}) = \varphi(a)^{-1}$ *for all* $a \in G$.

(iii) $\varphi(a^n) = \varphi(a)^n$ *for all* $a \in G$ *and for all* $n \in \mathbb{Z}$. □

If you remember from linear algebra, the null space of a matrix plays an important role. A homomorphism has an equivalent notion of null space, called the kernel.

Definition 5 (Kernel)**.** Let $\varphi\colon G_1 \to G_2$ be a homomorphism. The *kernel* of φ, denoted $\ker\varphi$, is defined to be the set $\ker\varphi = \{a \in G_1 : \varphi(a) = e_{G_2}\}$, or equivalently, $\ker\varphi = \varphi^{-1}(e_{G_2})$.

The next two exercises establishes a critical link between homomorphisms and normal subgroups.

Exercise 4. Prove that the kernel of a homomorphism is a normal subgroup.

By definition, $\mathrm{SL}(\mathrm{n}, \mathbb{R})$ is the kernel of the determinant, and hence by the exercise, it is a normal subgroup. We will see when we next meet that every normal subgroup can be realized as the kernel of some homomorphism.

Exercise 5. Let $\varphi\colon S_n \to \mathbb{Z}_2$ be given by

$$\varphi(\sigma) = \begin{cases} \bar{0} & \text{if } \sigma \text{ is even} \\ \bar{1} & \text{otherwise} \end{cases}$$

(i) Prove that φ is a homomorphism.

(ii) Prove that $\ker \varphi = A_n$.

We finish by seeing that the kernel can detect whether a homomorphism is an isomorphism.

Exercise 6. Let $\varphi\colon G_1 \to G_2$ be a homomorphism. Prove that φ is an isomorphism if and only if $\ker \varphi$ is trivial (that is, $\ker \varphi = \{e_{G_1}\}$).