Homework 8

MATH 301/601

Due Wednesday, April 3, 2024

Instructions. *Read the appropriate homework guide (Homework Guide for 301 or Homework Guide for 601) to make sure you understand how to successfully complete the assignment. All claims must be sufficiently justified.*

Exercise 1. Complete the following exercises from Section 6.5 in the course textbook:

1, 3, 4, 5, 6, 8, 11, 12, 17, ***18**,

Exercise 2. Let H be a subgroup of a group G. Fix $g \in G$, and define $\varphi_g\colon H \to gH$ by $\varphi_g(h) = gh$. Prove that φ_g is a bijection.

***Exercise 3.** Let $p \in \mathbb{N}$ be prime. How many subgroups does $\mathbb{Z}_{2p}$ have? Prove it.

Exercise 4. Let G be a group. Define the relation $\sim$ on G as follows: $a \sim b$ if and only if b is a conjugate of a (that is, there exists $g \in G$ such that $b = gag^{-1}$). Prove that $\sim$ is an equivalence relation.

***Exercise 5.** Prove that the 3-cycles $(1\ 2\ 3)$ and $(1\ 3\ 2)$ are not conjugate in A_4.

Double-star problem set up[1]

Definition 1. Let G be a group, and let X be a set. A *group action* of G on X is a function $\phi\colon G \times X \to X$ satisfying:

(i) $\phi(e, x) = x$ for all $x \in X$, and

(ii) $\phi(gh, x) = \phi(g, \phi(h, x))$ for all $g, h \in G$ and for all $x \in X$.

Usually the group action is clear from context and we simply write gx or $g \cdot x$ instead of $\phi(g, x)$. In this notation, (i) says $e \cdot x = x$ for all $x \in X$, and (ii) says $(gh) \cdot x = g \cdot (h \cdot x)$. Again suppressing the function ϕ, we generally write $G \curvearrowright X$ to denote the fact that the group G is acting on the set X.

In the "real world", we generally think about a group by the way it acts on some set. For example, we think about the dihedral groups via their action on regular polygons, and we think of matrix groups via their action on vector spaces.

Definition 2. Let $G \curvearrowright X$. Given $x \in X$, the *orbit* of x, denoted $\mathcal{O}_x$, is the subset of X given by

$$\mathcal{O}_x = \{g \cdot x : g \in G\}$$

and the *stabilizer* of x, denoted $\mathrm{Stab}_G(x)$ is the subgroup[2] of G given by

$$\mathrm{Stab}_G(x) = \{g \in G : g \cdot x = x\}.$$

[1] See Section 14.1

[2] You should convince yourself that this is indeed a subgroup.

The goal of the next exercise is to prove the following:

Theorem 3 (Orbit–Stabilizer Theorem)**.** *Let G be a group acting on a set X. If $x \in X$, then $|G| = |\mathcal{O}_x| \cdot |\mathrm{Stab}_G(x)|$.*

The orbit–stabilizer theorem should be viewed as a generalization of Lagrange's theorem (which we will use to prove the orbit–stabilizer theorem). Indeed, let H be a subgroup of G, and let $\mathcal{L}_H$ be the left cosets of H. Then G acts on $\mathcal{L}_H$ by $g \cdot (aH) = (ga)H$, with $\mathrm{Stab}_G(H) = H$ and $\mathcal{O}_H = \mathcal{L}_H$.

****Exercise 6.** Let G be a group acting on a set X. Let $x \in X$.

(a) Let $g, h \in G$. Prove that $gx = hx$ if and only if $h^{-1}g \in \mathrm{Stab}_G(x)$.

(b) Let $\mathcal{L}$ be the set of left cosets of $\mathrm{Stab}_G(x)$ in G. Let $\psi\colon \mathcal{L} \to \mathcal{O}_x$ be given by $\psi(g\mathrm{Stab}_G(x)) = gx$.

  (i) Prove that ψ is a well-defined.

  (ii) Prove that ψ is bijective.

(c) The previous part implies that $|\mathcal{O}_x| = [G : \mathrm{Stab}_G(x)]$. Apply Lagrange's theorem to obtain $|G| = |\mathcal{O}_x| \cdot |\mathrm{Stab}_G(x)|$.

(d) Now suppose G is a finite group, and let G act on itself by conjugation, that is, the action is given by $g \cdot a = gag^{-1}$. Apply the orbit–stabilizer theorem to show that, for $a \in G$, the cardinality of the set $\{gag^{-1} : g \in G\}$ (that is, the conjugacy class of a) divides $|G|$.