Homework 7

MATH 301/601

Due Wednesday, March 27, 2024

Instructions. *Read the appropriate homework guide (Homework Guide for 301 or Homework Guide for 601) to make sure you understand how to successfully complete the assignment. All claims must be sufficiently justified.*

Exercise 1. Complete the following exercises from Section 5.4 in the course textbook:

7, 8, 9, 11, 14, 18, 23, ***25**, 37(a,b)

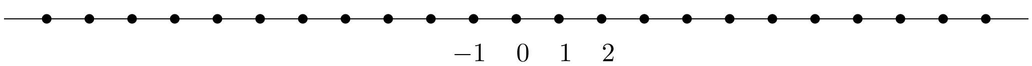
Exercise 2. Let $n \in \mathbb{N}$.

(a) Prove that A_n is a subgroup of S_n.

(b) Explain why the subset of odd permutations in S_n is not a subgroup of S_n.

***Exercise 3.** From a previous homework exercise, $|S_4| = 4! = 24$. Show that for any divisor d of 24 there exists a subgroup H such that $|H| = d$.

***Exercise 4.** Let $\Gamma = (V, E)$ be the graph with $V = \mathbb{Z}$ and $(m, n) \in \mathbb{Z}$ if and only if $|m - n| = 1$. So, Γ is just the number line (a portion of which is drawn here):



The *infinite dihedral group*, denoted D_∞, is the automorphism group of the graph Γ. Let $\tau, \rho \in D_\infty$ be given by $\tau(n) = n + 1$ and $\rho(n) = -n$ for $n \in \mathbb{Z}$.

(a) For $k \in \mathbb{Z}$, write down a formula for τ^k.

(b) Prove that if $f \in D_\infty$ such that $f(0) = 0$ and $f(1) = 1$, then f is the identity. (Hint: Let's first focus on the natural numbers. Use strong induction: Let $k \in \mathbb{N} \setminus \{1\}$. Suppose that $f(j) = j$ for all $0 \leq j < k$ and prove that $f(k) = k$. A similar argument works for the negative integers.)

(c) Prove that every element of D_∞ can be written as either τ^k or $\tau^k \rho$ for some $k \in \mathbb{Z}$. (Hint: let $f \in D_\infty$. Use a power of τ to get $f(0)$ back to 0, and then use ρ to get 1 back to itself if necessary.)

****Exercise 5.** Let $\sigma = (1\ 2\ 3\ 4\ 5) \in A_9$ and $\tau = (5\ 6\ 7\ 8\ 9) \in A_9$. Prove that the permutation $(1\ 2\ 3)$ can be written as a word in $\{\sigma, \tau\}$, i.e., there exists $r \in \mathbb{N}$ and $n_1, \ldots, n_r, m_1, \ldots m_r \in \mathbb{Z}$ such that $(1\ 2\ 3) = \sigma^{n_1}\tau^{m_1}\sigma^{n_2}\tau^{m_2} \cdots \sigma^{n_r}\tau^{m_r}$.

(Hint: It will help to visualize the problem. Look at the figure on the next page. Imagine σ as a counterclockwise rotation of the left circle and τ as a counterclockwise rotation of the right circle. Imagine playing a game where you are rotating the two circles to get the points in the desired position. Record the moves you take.)

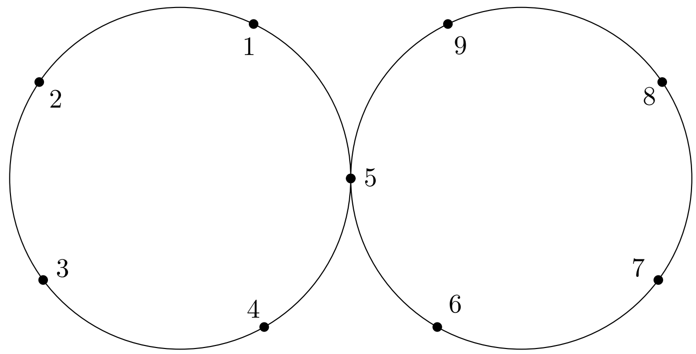


Figure 1: Visualizing Exercise 5

(From Exercise 5 you should feel confident in the fact that every 3-cycle can be expressed as a word in $\{\sigma, \tau\}$, and hence by §5.4 #25, every permutation in A_9 can be expressed as a word in σ and τ).