Homework 1

MATH 301/601

Due Wednesday, February 7, 2024

Instructions. *Read the appropriate homework guide (Homework Guide for 301 or Homework Guide for 601) to make sure you understand how to successfully complete the assignment. All claims must be sufficiently justified.*

Exercise 1. Complete the following exercises from Section 1.4 in the course textbook:

22, 24

***Exercise 2.** (1) Give an example of a function $\mathbb{N} \to \mathbb{N}$ that is injective but not surjective. (2) Give an example of a function $\mathbb{N} \to \mathbb{N}$ that is surjective but not injective. (3) Give an example of a bijection from $\mathbb{N} \to \mathbb{Z}$.

Exercise 3. Let $a, b, c, m, n \in \mathbb{Z}$. Prove that if $a \mid b$ and $a \mid c$, then $a \mid (mb + nc)$.

Exercise 4. Let $a, b \in \mathbb{Z}$. Prove that if $a \mid b$ and $b \mid a$, then either $a = b$ or $a = -b$.

***Exercise 5.** Let $S \subset \mathbb{N}$ such that $1 \in S$ and $n + 1 \in S$ whenever $n \in S$. Prove that $S = \mathbb{N}$. (Hint: Use the well-ordering principle.)

***Exercise 6.** Let $n \in \mathbb{N}$. Prove that the remainder obtained from dividing n^2 by 4 is either 0 or 1.

****Exercise 7.** Define the ordering $<$ on $\mathbb{N} \times \mathbb{N}$ by $(a, b) < (c, d)$ if $a < c$ or $a = c$ and $b < d$ (this is called the *lexicographical ordering*). Prove that $(\mathbb{N} \times \mathbb{N}, <)$ is well ordered, that is, show that given a nonempty subset S of $\mathbb{N} \times \mathbb{N}$ there exists $s \in S$ such that $s < s'$ for all $s' \in S \setminus \{s\}$.