Homework 9

MATH 301

Due Wednesday, April 26, 2023.

Instructions. *Read the Homework Guide to make sure you understand how to successfully complete the assignment. All claims must be sufficiently justified.*

Exercise 1. Let $h\colon G \to G'$ be a surjective homomorphism. Prove that h is an isomorphism if and only if the kernel of h is trivial.

Definition 1. For $m, n \in \mathbb{N}$, we define addition on the cartesian product $\mathbb{Z}_m \times \mathbb{Z}_n = \{(a, b) : a \in \mathbb{Z}_m, b \in \mathbb{Z}_n\}$ by $(a, b) + (c, d) = (a + c, b + d)$, where the addition is modulo m in the first coordinate and modulo n in the second. The cartesian product $\mathbb{Z}_m \times \mathbb{Z}_n$ together with this coordinate-wise addition is a group (it is an example of a direct product, which we will discuss next week). Note that $\mathbb{Z}_m \times \mathbb{Z}_n$ has order $m \cdot n$.

Exercise 2. Prove that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_6$.

Exercise 3. (a) Prove that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4$.

(b) Let G be a group of order 4. Prove that G is isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$.

Exercise 4. Let G be a group, and given $g \in G$, define $\lambda_g\colon G \to G$ by $\lambda_g(h) = gh$ for $h \in G$. Prove that:

(a) λ_g is a bijection for every $g \in G$, and

(b) $\psi\colon G \to \mathrm{Sym}(G)$ given by $\psi(g) = \lambda_g$ is an injective homomorphism.

Definition 2. An *automorphism* of a group G is an isomorphism $G \to G$.

Exercise 5. Let G be a group. Prove that the set of automorphisms of G, denoted $\mathrm{Aut}(G)$, is a group with respect to function composition (this group is called *the automorphism group of* G).

Definition 3. See Example 3.15 (in Section 3.2) for the definition of the *quaternion group* Q_8. (You will need this group in Exercise 43 below.)

Exercise 6. Complete the following exercises from Section 9.4:

8, 38, 39, 41, 43