Homework 5

MATH 301

Due Wednesday, October 18, 2023

Instructions. *Read the Homework Guide to make sure you understand how to successfully complete the assignment. All claims must be sufficiently justified.*

Exercise 1. Complete the following exercises from Section 4.5 in the course textbook:

#1(a,b,c,d), 2(a,e,f), 3(b,c,e), 4(a,b,c), 9, 11, 27, 30, 31, 39

Exercise 2. (a) Compute the center of $\mathrm{GL}(2,\mathbb{R})$. (Hint: use the following test matrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.)

(b) Compute the center of $\mathrm{SL}(2,\mathbb{R})$.

***Exercise 3.** Let G be a group. Let $a \in G \setminus \{e\}$ such that the order of a, denoted $|a|$, is m (recall, this means the cyclic subgroup $\langle a \rangle$ is order m). The goal of this exercise is to prove that $|a| = \min\{n \in \mathbb{N} : a^n = e\}$.

(a) Show that the set $S = \{n \in \mathbb{N} : a^n = e\}$ is not empty. (Hint: argue that there exists $j, k \in \{1, 2, \ldots, m+1\}$ such that $a^j = a^k$.) Then, by the well-ordering principle, S has a least element, call it ℓ.

(b) Show that $a^k \neq a^j$ if $0 \leq j < k < \ell$.

(c) Given $k \in \mathbb{Z}$, use the division algorithm to show that $a^k \in \{e, a, a^2, \ldots, a^{\ell-1}\}$.

(d) Conclude that $\langle a \rangle = \{e, a, a^2, \ldots, a^{\ell-1}\}$, and hence $|a| = \ell$.

Exercise 4. Let $a, b \in \mathbb{Z}$.

(a) Let $\langle a, b \rangle = \{as + bt : s, t \in \mathbb{Z}\}$. Prove that $\langle a, b \rangle$ is a subgroup of $\mathbb{Z}$.

(b) Show that if H is a subgroup such that $a, b \in H$, then $\langle a, b \rangle < H$ (this says that $\langle a, b \rangle$ is the subgroup generated by a and b).

(c) Find $n \in \mathbb{N} \cup \{0\}$ such that $\langle a, b \rangle = n\mathbb{Z}$. Justify your answer. (It might help to try some concrete values for a and b if you are not sure what n should be.)

***Exercise 5.** Let G be a finite group. Show that there exists an integer N such that $g^N = e$ for every $g \in G$.

***Exercise 6.** Suppose G is a nontrivial group in which the only two subgroups of G are itself and the trivial subgroup.

(a) Prove that G is cyclic.

(b) Using part (a), prove that G is a finite group of prime order.