Homework 2

MATH 301

Due Wednesday, September 20, 2023

Instructions. *Read the Homework Guide to make sure you understand how to successfully complete the assignment. All claims must be sufficiently justified.*

Exercise 1. Use induction to prove that

$$\frac{x^n - 1}{x - 1} = \sum_{i=0}^{n-1} x^i$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{Z}$. (Note: you are inducting on n, not x.)

Exercise 2. Complete the following exercises from Section 2.4 in the course textbook:

1, 5, 9, *27, 28 (you will need to use Exercise 1), 31

Exercise 3. Complete exercise #1 from Section 3.5 in the course textbook.

Definition 1. An *equivalence relation* on a set S is a binary relation $\sim$ that is:

(i) *reflexive*, that is, $a \sim a$ for all $a \in S$;

(ii) *symmetric*, that is, $a \sim b$ implies $b \sim a$ for all $a, b \in S$; and

(iii) *transitive*, that is, $a \sim b$ and $b \sim c$ implies $a \sim c$ for all $a, b, c \in S$.

Exercise 4. Let $n \in \mathbb{N}$. Prove that equivalence modulo n is an equivalence relation on $\mathbb{Z}$.

***Exercise 5.** Let $n \in \mathbb{N}$. Prove that given any $m \in \mathbb{Z}$ there exists a unique element $a \in \{0, 1, 2, \ldots, n-1\}$ such that $m \equiv a \pmod{n}$. (Hint: Use the division algorithm.)

Exercise 6. Let $n \in \mathbb{N}$, and let $a, b \in \mathbb{Z}$. Prove that if $a \equiv b \pmod{n}$, then

$$\gcd(a, n) = \gcd(b, n).$$