Homework 10

MATH 301

Due Wednesday, December 6, 2023

Instructions. *Read the Homework Guide to make sure you understand how to successfully complete the assignment. All claims must be sufficiently justified.*

Exercise 1. Complete the following exercises from Section 11.4:

9, 10, 11, 12, 13, ***16**, 18

***Exercise 2.** Let G and H be finite groups whose orders are relatively prime. Prove that every homomorphism $G \to H$ is trivial.

For Exercises 3 and 4, you will want to use Exercise 11 in §11.4.

Exercise 3. Find all homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}_{12}$. Justify that you have them all.

***Exercise 4.** Find all homomorphisms from $\mathbb{Z}_{24}$ to $\mathbb{Z}_{18}$. Justify that you have them all.

Exercise 5. Up to isomorphism, determine the groups H for which there exists a surjective homomorphism from D_4 onto H.

Exercise 6. A 2×2 rotation matrix is a *rotation matrix* of the form

$$R_\theta = \begin{bmatrix} \cos(2\pi\theta) & -\sin(2\pi\theta) \\ \sin(2\pi\theta) & \cos(2\pi\theta) \end{bmatrix}$$

where $\theta \in \mathbb{R}$. The two-dimensional special orthogonal group $SO(2)$ is the set of 2×2 rotation matrices equipped with matrix multiplication.

(a) Prove that $f\colon \mathbb{R} \to SO(2)$ given by $f(\theta) = R_\theta$ is a group homomorphism.

(b) Find the kernel of f.

(c) Prove that $SO(2)$ is isomorphic to $\mathbb{R}/\mathbb{Z}$.

****Exercise 7.** If H is a subgroup of G such that the product of two left cosets of H in G is again a left coset of H in G, prove that H is normal in G.

****Exercise 8.** Suppose that N and M are two normal subgroups of G with $N \cap M = \{e\}$. Show that $nm = mn$ for every $n \in N$ and $m \in M$

****Exercise 9.** The subgroup of a group G generated by the set $U = \{xyx^{-1}y^{-1} : x, y \in G\}$ is called the *commutator subgroup of* G and is denoted G'.

(a) Prove that G' is normal in G.

(b) Prove that G/G' is abelian.

(c) Prove that if N is a normal subgroup of G and G/N is abelian, then $G' \subset N$.