Homework 1

MATH 301

Due Wednesday, September 13, 2023

Instructions. *Read the Homework Guide to make sure you understand how to successfully complete the assignment. All claims must be sufficiently justified.*

***Exercise 1.** Prove that the composition of two surjective functions is surjective.

Exercise 2. Let $a, b \in \mathbb{Z}$. Prove that if $a \mid b$ and $b \mid a$, then either $a = b$ or $a = -b$.

Exercise 3. Let $a, b, c, m, n \in \mathbb{Z}$. Prove that if $a \mid b$ and $a \mid c$, then $a \mid (mb + nc)$.

Exercise 4. Complete the following exercises from Section 2.4 in the course textbook:

14, 15, 16, 18, *20, 27

Definition. Given two nonzero integers a and b, an integer c is a *common multiple* of a and b if $a \mid c$ and $b \mid c$. The *least common multiple* of a and b, denoted $\operatorname{lcm}(a, b)$, is the smallest positive common multiple of a and b.

***Exercise 5.** Let a and b be nonzero integers.

(1) Prove that the least common multiple of a and b exists.

(2) Prove that if $k \in \mathbb{Z}$ is a common multiple of a and b, then $\operatorname{lcm}(a, b)$ divides k. (Hint: divide k by $\operatorname{lcm}(a, b)$ using the division algorithm.)

****Exercise 6.** Let a and b be nonzero integers.

(1) Prove that the product of $\operatorname{lcm}(a, b)$ and $\gcd(a, b)$ is equal to $|ab|$. (Hint: the product ab is divisible by $d = \gcd(\mathrm{a}, \mathrm{b})$. Let $m = |ab|/d$. Now, let k be a common multiple of a and b. Write d as a linear combination in a and b, and use this to express the fraction k/m as an integer.)

(2) Prove that $\operatorname{lcm}(a, b) = |ab|$ if and only if $\gcd(a, b) = 1$.