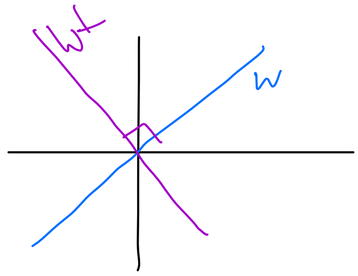
Recap · $v, w \in \mathbb{R}^n$ are orthogonal if $v \cdot w = 0$.

· Given $W \subset \mathbb{R}^n$ subspace, its orthogonal complement of W

$$W^\perp = \{ v \in \mathbb{R}^n \mid v \cdot w = 0 \ \forall w \in W \}$$



Recall: Given a matrix A, the row space of A is the subspace $\text{row}(A) = \text{col}(A^T)$ = "subspace spanned by the rows of A"

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & -1 & 5 \end{bmatrix}$$

$$\text{row}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} \right\}$$

Thm 6.3 $\text{row}(A)^\perp = \text{null}(A)$ and $\text{col}(A)^\perp = \text{null}(A^T)$.

Proof NTS: $\text{null}(A) \subset \text{row}(A)^\perp$ and $\text{row}(A)^\perp \subset \text{null}(A)$

Take $v \in \text{null}(A)$, and we'll show that $v \in \text{row}(A)^\perp$.

$\Rightarrow Av = 0$. If $r_1, r_2, \ldots, r_m$ are the rows of A, then

$$Av = \begin{bmatrix} r_1 \cdot v \\ r_2 \cdot v \\ \vdots \\ r_m \cdot v \end{bmatrix} \Rightarrow r_1 \cdot v = r_2 \cdot v = \cdots = r_m \cdot v = 0$$

Now, if $w \in row(A)$, then $w = c_1 r_1 + c_2 r_2 + \cdots + c_m r_m$

$\Rightarrow \quad v \cdot w = c_1 \underbrace{v \cdot r_1}_{0} + c_2 \underbrace{v \cdot r_2}_{0} + \cdots + c_m \underbrace{v \cdot r_m}_{0} = 0$

$\Rightarrow \quad v \in row(A)^{\perp}$

$\Rightarrow \quad null(A) \subset row(A)^{\perp}$

Conversely, suppose $w \in row(A)^{\perp}$, then we need to show that $w \in null(A)$.

$$Aw = \begin{bmatrix} r_1 \cdot w \\ r_2 \cdot w \\ \vdots \\ r_m \cdot w \end{bmatrix} = 0 \Rightarrow w \in null(A)$$

$\Rightarrow \quad row(A)^{\perp} \subset null(A) \Rightarrow row(A)^{\perp} = null(A)$.

To finish, we want $col(A)^{\perp} = null(A^T)$.

$null(A^T) = row(A^T)^{\perp} = col(A)^{\perp}$. □

Def A set of vectors is <u>orthogonal</u> if each pair of distinct vectors is orthogonal.

Ex. $u_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$, $u_3 = \begin{bmatrix} -2 \\ 10 \\ 6 \end{bmatrix}$

$$u_1 \cdot u_2 = 3 \cdot 1 + 0 \cdot 2 + 1 \cdot (-3) = 0$$

$$u_1 \cdot u_3 = 3(-2) + 0 \cdot 10 + 1 \cdot 6 = 0$$

$$u_2 \cdot u_3 = 1 \cdot (-2) + 2(10) + (-3) \cdot 6 = 0$$

$\Rightarrow \{u_1, u_2, u_3\}$ is orthogonal.

Recall: $\{u_1, u_2, \dots, u_p\}$ is <u>linearly</u> <u>independent</u> if

$$0 = c_1 u_1 + c_2 u_2 + \dots + c_p u_p$$

has only the trivial solution.

Thm 6.4 Let $S = \{u_1, u_2, \dots, u_p\} \subset \mathbb{R}^n$ be an orthogonal set of non-zero vectors.

Then, S is linearly independent, and hence S is a basis for span S

Proof Suppose

$$0 = c_1 u_1 + c_2 u_2 + \cdots + c_p u_p.$$

Take the dot product of each side w/ u_1.

$$\Rightarrow \quad 0 = c_1 u_1 \cdot u_1 + c_2 \underbrace{u_2 \cdot u_1}_{} + \cdots + c_p \underbrace{u_p \cdot u_1}_{0}$$

$$\Rightarrow \quad 0 = c_1 \|u_1\|^2$$

Since $u_1 \neq 0$, we know that $\|u_1\|^2 \neq 0$.

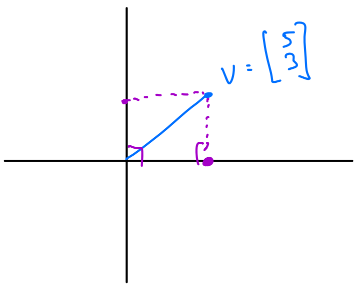
$\Rightarrow$ $c_1 = 0$.

Similarly, $c_2 = c_3 = \cdots = c_p = 0$.

$\Rightarrow$ S is linearly independent. $\square$

Def An orthogonal basis of a subspace $W \subset \mathbb{R}^n$ is a basis for W that is also an orthogonal set.

Ex The standard basis $e_1, \dots, e_n$ for $\mathbb{R}^n$ is an orthogonal basis.



$v = 5e_1 + 3e_2$ Observation: $5 = v \cdot e_1$

$3 = v \cdot e_2$

$v = (v \cdot e_1)e_1 + (v \cdot e_2)e_2$

Thm 6.5 Let $\{u_1, u_2, \dots, u_p\}$ be an orthogonal basis for subspace $W \subset \mathbb{R}^n$. For each $y \in W$,

$$y = c_1 u_1 + c_2 u_2 + \dots + c_p u_p$$

with $c_j = \dfrac{y \cdot u_j}{\|u_j\|^2}$ $\forall\, j \in \{1, \dots, p\}$.

Proof

$$y \cdot u_1 = c_1 u_1 \cdot u_1 + c_2 \underbrace{u_2 \cdot u_1}_{0} + \dots + c_p \underbrace{u_p \cdot u_1}_{0}$$

$$= c_1 \|u_1\|^2$$

$$\Rightarrow c_1 = \frac{y \cdot u_1}{\|u_1\|^2}.$$

Similarly for $c_2, \dots, c_p$. ∎

Ex. $u_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$, $u_3 = \begin{bmatrix} -2 \\ 10 \\ 6 \end{bmatrix}$

$\Rightarrow S = \{u_1, u_2, u_3\}$ is an orthogonal basis for $\mathbb{R}^3$

Q: Find the S-coordinates for $v = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$.

A: $\|u_1\|^2 = u_1 \cdot u_1 = 3^2 + 0^2 + 1^2 = 10$

$\|u_2\|^2 = 14$

$\|u_3\|^2 = 140$

$$v = \frac{v \cdot u_1}{\|u_1\|^2} u_1 + \frac{v \cdot u_2}{\|u_2\|^2} \cdot u_2 + \frac{v \cdot u_3}{\|u_3\|^2} u_3$$ by Thm 6.5

$$= \frac{-1}{10} u_1 + \frac{3}{14} u_2 + \frac{1}{2} u_3$$