IHARA'S LEMMA FOR IMAGINARY QUADRATIC FIELDS

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ABSTRACT. An analogue over imaginary quadratic fields of a result in algebraic number theory known as Ihara's lemma is established. More precisely, we show that for a prime ideal $\mathfrak{p}$ of the ring of integers of an imaginary quadratic field F, the kernel of the sum of the two standard $\mathfrak{p}$-degeneracy maps between the cuspidal sheaf cohomology $H^1_!(X_0, \tilde{M}_0)^2 \to H^1_!(X_1, \tilde{M}_1)$ is Eisenstein. Here X_0 and X_1 are analogues over F of the modular curves $X_0(N)$ and $X_0(Np)$, respectively. To prove our theorem we use the method of modular symbols and the congruence subgroup property for the group SL_2 which is due to Serre.

1. INTRODUCTION

Ihara's lemma in the version stated in [6] asserts that the kernel of the map $\alpha : J_0(N)^2 \to J_0(Np)$ is Eisenstein if $(N,p) = 1$. Here $J_0(N')$ denotes the Jacobian of the compactified modular curve $\Gamma_0(N')\backslash\mathbf{H}$, and α is the sum of the two standard p-degeneracy maps from $J_0(N)$ to $J_0(Np)$. The original proof of the result is due to Ihara [4] and uses algebraic geometry. In [6] Ribet gave a different proof without appealing to algebro-geometric methods. The result was later improved upon by Khare [5] to dispose of the condition that N be coprime to p. Khare also gives a rearranged proof in the case when $(N,p) = 1$ using the method of modular symbols (cf. [5], Remark 4). We will use his approach to generalize the result to imaginary quadratic fields, where algebro-geometric techniques are not available.

Let F denote an imaginary quadratic extension of $\mathbf{Q}$ and $\mathcal{O}_F$ its ring of integers. The reason why over F the algebro-geometric machinery is not available is the fact that the symmetric space on which automorphic forms are defined is the hyperbolic-3-space, the product of $\mathbf{C}$ and $\mathbf{R}_+$, and the analogues X_n of the modular curves are not algebraic varieties (cf. section 2). However, [5] uses only group cohomology and his method may be adapted to the situation over an imaginary quadratic field. In this setting the Jacobians are replaced with certain sheaf cohomology groups $H^1_!(X_n, \tilde{M}_n)$ and for a prime $\mathfrak{p} \subset \mathcal{O}_F$ we have analogues of the two standard $\mathfrak{p}$-degeneracy maps whose sum $H^1_!(X_0, \tilde{M}_0)^2 \to H^1_!(X_1, \tilde{M}_1)$ we will call α. (For precise definitions see section 2.) The main result of this note (Theorem 3.1) then asserts that the kernel of α is Eisenstein (for definition of "Eisenstein" see section 3).

Originally Ihara's lemma had been used by Ribet [6] to prove the existence of congruences between modular forms of level N and those of level Np. His result, valid for forms of weight 2, was later generalized to arbitrary weight by Diamond [2], who used the language of cohomology like we chose to. A crucial ingredient in Diamond's proof is the self-duality of $H^1(\Gamma_0(N), M)$. Over imaginary quadratic fields, as over $\mathbf{Q}$, there is a connection between the space of automorphic

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forms and the cohomology groups $H^1_!(X_n, \tilde{M}_n)$ called the Eichler-Shimura-Harder isomorphism (cf. [10]). However, there seems to be no obvious way to adapt the approach of Ribet and Diamond to our situation as $H^1_!(X_n, \tilde{M}_n)$ is not self-dual.

Ihara's lemma was also used in the proof of modularity of Galois representations attached to elliptic curves over $\mathbf{Q}$ ([11], [1]). Thanks to the work of Taylor [9] one can attach Galois representations to a certain class of automorphic forms on $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_{2/F})$. One could hope that Ihara's lemma in our formulation could be useful in proving the converse to Taylor's theorem, i.e., that ordinary Galois representations of $\mathrm{Gal}(\overline{F}/F)$ (satisfying appropriate conditions) arise from automorphic forms, but at this moment this is a mere speculation as too many other important ingredients of a potential proof seem to be missing.

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2. Preliminaries

Let F be an imaginary quadratic extension of $\mathbf{Q}$ and denote by $\mathcal{O}_F$ its ring of integers. Let $\mathfrak{N}$ be an ideal of $\mathcal{O}_F$ such that the $\mathbf{Z}$-ideal $\mathfrak{N} \cap \mathbf{Z}$ has a generator greater than 3. Let $\mathfrak{p}$ be a prime ideal such that $\mathfrak{p} \nmid \mathfrak{N}$. Denote by Cl_F the class group of F and choose representatives of distinct ideal classes to be prime ideals $\mathfrak{p}_i$, $i = 1, \dots, \#\,\mathrm{Cl}_F$, relatively prime to both $\mathfrak{N}$ and $\mathfrak{p}$. Let π, (resp. π_i) be a uniformizer of the completion $F_\mathfrak{p}$ (resp. $F_{\mathfrak{p}_i}$) of F at the prime $\mathfrak{p}$ (resp. $\mathfrak{p}_i$), and put $\tilde{\pi}$ (resp. $\tilde{\pi}_i$) to be the idele $(\dots, 1, \pi, 1, \dots) \in \mathbf{A}_F^\times$ (resp. $(\dots, 1, \pi_i, 1, \dots) \in \mathbf{A}_F^\times$), where π (resp. π_i) occurs at the $\mathfrak{p}$-th place (resp. $\mathfrak{p}_i$-th place). We also put $\mathcal{O}_{(\mathfrak{p})} := \bigcup_{j=0}^\infty \mathfrak{p}^{-j}\mathcal{O}_F$.

For each $n \in \mathbf{Z}_{\geq 0}$, we define compact open subgroups of $\mathrm{GL}_2(\mathbf{A}_{F,f})$

$$K_n := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \prod_{\mathfrak{q} \nmid \infty} \mathrm{GL}_2(\mathcal{O}_{F,\mathfrak{q}}) \mid c \in \mathfrak{N}\mathfrak{p}^n \right\}.$$

Here $\mathbf{A}_{F,f}$ denotes the finite adeles of F and $\mathcal{O}_{F,\mathfrak{q}}$ the ring of integers of $F_\mathfrak{q}$. For $n \geq 0$ we also set $K_n^\mathfrak{p} = \begin{bmatrix} \tilde{\pi} & \\ & 1 \end{bmatrix} K_n \begin{bmatrix} \tilde{\pi} & \\ & 1 \end{bmatrix}^{-1}$.

For any compact open subgroup K of $\mathrm{GL}_2(\mathbf{A}_{F,f})$ we put

$$X_K = \mathrm{GL}_2(F) \setminus \mathrm{GL}_2(\mathbf{A}_F)/K \cdot U_2(\mathbf{C}) \cdot Z_\infty,$$

where $Z_\infty = \mathbf{C}^\times$ is the center of $\mathrm{GL}_2(\mathbf{C})$ and $U(2) := \{M \in \mathrm{GL}_2(\mathbf{C}) \mid M\overline{M}^t = I_2\}$ (here 'bar' denotes complex conjugation and I_2 stands for the 2×2-identity matrix). If K is sufficiently large (which will be the case for all compact open subgroups we will consider) this space is a disjoint union of $\#\,\mathrm{Cl}_F$ connected components $X_K = \coprod_{i=1}^{\#\,\mathrm{Cl}_F} (\Gamma_K)_i \setminus \mathcal{Z}$, where $\mathcal{Z} = \mathrm{GL}_2(\mathbf{C})/U_2(\mathbf{C})\mathbf{C}^\times$ and $(\Gamma_K)_i = \mathrm{GL}_2(F) \cap \begin{bmatrix} \tilde{\pi}_i & \\ & 1 \end{bmatrix} K \begin{bmatrix} \tilde{\pi}_i & \\ & 1 \end{bmatrix}^{-1}$. To ease notation we put $X_n := X_{K_n}$, $X_n^\mathfrak{p} := X_{K_n^\mathfrak{p}}$, $\Gamma_{n,i} := (\Gamma_{K_n})_i$ and $\Gamma_{n,i}^\mathfrak{p} := (\Gamma_{K_n^\mathfrak{p}})_i$.

We have the following diagram:

$$
\begin{tikzcd}
\cdots \arrow[r, "\subset"] & K_{n+1} \arrow[r, "\subset"] \arrow[d, "\wr"] & K_n \arrow[r, "\subset"] \arrow[d, "\wr"] & K_{n-1} \arrow[r, "\subset"] \arrow[d, "\wr"] & \cdots \\
\cdots \arrow[r, "\subset"] & K^{\mathfrak{p}}_{n+1} \arrow[r, "\subset"] \arrow[ur, "\subset"] & K^{\mathfrak{p}}_n \arrow[r, "\subset"] \arrow[ur, "\subset"] & K^{\mathfrak{p}}_{n-1} \arrow[r, "\subset"] & \cdots
\end{tikzcd}
\tag{2.1}
$$

where the horizontal and diagonal arrows are inclusions and the vertical arrows are conjugation by the maps $\begin{bmatrix} \tilde{\pi} & \\ & 1 \end{bmatrix}$. Diagram (2.1) is not commutative, but it is "vertically commutative", by which we mean that given two objects in the diagram, two directed paths between those two objects define the same map if and only if the two paths contain the same number of vertical arrows.

Diagram (2.1) induces the following vertically commutative diagram of the corresponding symmetric spaces:

$$
\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r] \arrow[d, "\wr"] & X_n \arrow[r] \arrow[d, "\wr"] & X_{n-1} \arrow[r] \arrow[d, "\wr"] & \cdots \\
\cdots \arrow[r] & X^{\mathfrak{p}}_{n+1} \arrow[r] \arrow[ur] & X^{\mathfrak{p}}_n \arrow[r] \arrow[ur] & X^{\mathfrak{p}}_{n-1} \arrow[r] & \cdots
\end{tikzcd}
\tag{2.2}
$$

The horizontal and diagonal arrows in diagram (2.2) are the natural projections and the vertical arrows are maps given by $(g_\infty, g_f) \mapsto \left(g_\infty, g_f \left[\begin{smallmatrix} \tilde{\pi} & \\ & 1 \end{smallmatrix}\right]^{-1}\right)$.

Let M be a torsion abelian group of exponent relatively prime to $\#\mathcal{O}_F^\times$ endowed with a $\mathrm{GL}_2(F)$-action. Denote by $\tilde{M}_K$ the sheaf of continuous sections of the topological covering $\mathrm{GL}_2(F) \setminus (\mathrm{GL}_2(\mathbf{A}_F)/K \cdot U_2(\mathbf{C}) \cdot Z_\infty) \times M \to X_K$, where $\mathrm{GL}_2(F)$ acts diagonally on $(\mathrm{GL}_2(\mathbf{A}_F)/KU_2(\mathbf{C})Z_\infty) \times M$. Here M is equipped with the discrete topology. Since we will only be concerned with the case when M is a trivial $\mathrm{GL}_2(F)$-module, we assume it from now on. This means that $\tilde{M}_K$ is a constant sheaf. As above, we put $\tilde{M}_n := \tilde{M}_{K_n}$ and $\tilde{M}^{\mathfrak{p}}_n := \tilde{M}_{K^{\mathfrak{p}}_n}$.

Given a surjective map $\phi : X_K \to X_{K'}$, we get an isomorphism of sheaves $\phi^{-1}\tilde{M}_{K'} \xrightarrow{\sim} \tilde{M}_K$, which yields a map on cohomology

$$H^q(X_{K'}, \tilde{M}_{K'}) \to H^q(X_K, \phi^{-1}\tilde{M}_{K'}) \cong H^q(X_K, \tilde{M}_K).$$

Hence diagram (2.2) gives rise to a vertically commutative diagram of cohomology groups:

$$
\begin{tikzcd}
\cdots & H^q(X_{n+1}, \tilde{M}_{n+1}) \arrow[l] & H^q(X_n, \tilde{M}_n) \arrow[l, "\alpha_1^{n,n+1}"'] \arrow[dl, "\alpha_1^{n,(n+1)\mathfrak{p}}"'] & H^q(X_{n-1}, \tilde{M}_{n-1}) \arrow[l, "\alpha_1^{n-1,n}"'] \arrow[dl, "\alpha_1^{n-1,n\mathfrak{p}}"'] & \cdots \arrow[l] \\
\cdots & H^q(X^{\mathfrak{p}}_{n+1}, \tilde{M}^{\mathfrak{p}}_{n+1}) \arrow[l] \arrow[u, "\alpha_{\mathfrak{p}}^{n+1}", "\wr"'] & H^q(X^{\mathfrak{p}}_n, \tilde{M}^{\mathfrak{p}}_n) \arrow[l, "\alpha_1^{n\mathfrak{p},(n+1)\mathfrak{p}}"'] \arrow[u, "\alpha_{\mathfrak{p}}^{n}", "\wr"'] & H^q(X^{\mathfrak{p}}_{n-1}, \tilde{M}^{\mathfrak{p}}_{n-1}) \arrow[l, "\alpha_1^{(n-1)\mathfrak{p},n\mathfrak{p}}"'] \arrow[u, "\alpha_{\mathfrak{p}}^{n-1}", "\wr"'] & \cdots \arrow[l]
\end{tikzcd}
\tag{2.3}
$$

These sheaf cohomology groups can be related to the group cohomology of $\Gamma_{n,i}$ and $\Gamma^{\mathfrak{p}}_{n,i}$ with coefficients in M. In fact, for each compact open subgroup K with corresponding decomposition $X_K = \coprod_{i=1}^{\# \mathrm{Cl}_F} (\Gamma_K)_i \setminus \mathcal{Z}$, we have the following commutative diagram in which the horizontal maps are inclusions:

$$
\begin{tikzcd}
H^q_!(X_K, \tilde{M}_K) \arrow[r] & H^q(X_K, \tilde{M}_K) \\
\bigoplus_{i=1}^{\# \mathrm{Cl}_F} H^q_P((\Gamma_K)_i, M) \arrow[r] \arrow[u] & \bigoplus_{i=1}^{\# \mathrm{Cl}_F} H^q((\Gamma_K)_i, M) \arrow[u]
\end{tikzcd}
\tag{2.4}
$$

Here $H^q_!(X_K, \tilde{M}_K)$ denotes the image of the cohomology with compact support $H^q_c(X_K, \tilde{M}_K)$ inside $H^q(X_K, \tilde{M}_K)$ and H^q_P denotes the parabolic cohomology, i.e., $H^q_P((\Gamma_K)_i, M) := \ker(H^q((\Gamma_K)_i, M) \to \bigoplus_{B \in \mathcal{B}} H^q((\Gamma_K)_{i,B}, M))$, where $\mathcal{B}$ is the set of Borel subgroups of $GL_2(F)$ and $(\Gamma_K)_{i,B} := (\Gamma_K)_i \cap B$. The vertical arrows in diagram (2.4) are isomorphisms provided that there exists a torsion-free normal subgroup of $(\Gamma_K)_i$ of finite index relatively prime to the exponent of M. If $K = K_n$ or $K = K_n^{\mathfrak{p}}$, $n \geq 0$, this condition is satisfied because of our assumption that $\mathfrak{N} \cap \mathbf{Z}$ has a generator greater than 3 and the exponent of M is relatively prime to $\# \mathcal{O}_F^\times$ (cf. [10], section 2.3). In what follows we may therefore identify the sheaf cohomology with the group cohomology. Note that all maps in diagram (2.3) preserve parabolic cohomology. The maps $\alpha_1^{*,*}$ are the natural restriction maps on group cohomology, so in particular they preserve the decomposition $\bigoplus_{i=1}^{\# \mathrm{Cl}_F} H^q((\Gamma_K)_i, M)$.

Using the identifications of diagram (2.4) we can prove the following result which will be useful later:

Lemma 2.1. *The map* $\alpha_1^{0,1} : H^1_!(X_0, \tilde{M}_0) \to H^1_!(X_1, \tilde{M}_1)$ *is injective.*

Proof. Using the isomorphism between group and sheaf cohomology all we need to prove is that the restriction maps $\mathrm{res}_i : H^1(\Gamma_{0,i}, M) \to H^1(\Gamma_{1,i}, M)$ are injective. Since M is a trivial $\Gamma_{0,i}$-module, the cohomology groups are just Homs, so it is enough to show the following statement: if G denotes the smallest normal subgroup of $\Gamma_{0,i}$ containing $\Gamma_{1,i}$, then $G = \Gamma_{0,i}$. For this we use the decomposition

$$\Gamma_{0,i} = \coprod_{\substack{k \in R(\mathcal{O}_F/\mathfrak{p}) \\ k \in \mathfrak{N}}} \Gamma_{1,i} \begin{bmatrix} 1 & \\ k & 1 \end{bmatrix} \sqcup \Gamma_{1,i} \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where the matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is chosen so that C and D are relatively prime elements of $\mathcal{O}_F$ with $C \in \mathfrak{N}$, $D \in \mathfrak{p}$, and $A \in \mathcal{O}_F$, $B \in \mathfrak{p}_i$ satisfy $AD - BC = 1$. Here $R(\mathcal{O}_F/\mathfrak{p})$ denotes a set of representatives in $\mathcal{O}_F$ of the distinct residue classes of $\mathcal{O}_F/\mathfrak{p}$. Let $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_{1,i}$ with $d \in \mathfrak{p}$. Then for any $k \in \mathfrak{N}\mathfrak{p}_i^{-1}\mathcal{O}_F$ we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} 1 + bdk & -b^2 k \\ d^2 k & 1 - bdk \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \\ k & 1 \end{bmatrix}$$

and the matrix $\left[\begin{smallmatrix} 1+bdk & -b^2 k \\ d^2 k & 1-bdk \end{smallmatrix}\right] \in \Gamma_{i,1}$, hence G contains $\coprod_{\substack{k \in R(\mathcal{O}_F/\mathfrak{p}) \\ k \in \mathfrak{N}}} \Gamma_{1,i} \left[\begin{smallmatrix} 1 & \\ k & 1 \end{smallmatrix}\right]$, and thus $G = \Gamma_{i,0}$. $\square$

We can augment diagram (2.1) on the right by introducing one more group:

$$K_{-1} := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(F_{\mathfrak{p}}) \times \prod_{\mathfrak{q} \nmid \mathfrak{p}\infty} \mathrm{GL}_2(\mathcal{O}_{F,\mathfrak{q}}) \mid c \in \mathfrak{N},\, ad - bc \in \prod_{\mathfrak{q} \nmid \infty} \mathcal{O}_{F,\mathfrak{q}}^\times \right\}.$$

The group K_{-1} is not compact, but we can still define

$$\Gamma_{-1,i} := \mathrm{GL}_2(F) \cap \begin{bmatrix} \tilde{\pi}_i & \\ & 1 \end{bmatrix} K_{-1} \begin{bmatrix} \tilde{\pi}_i & \\ & 1 \end{bmatrix}^{-1}$$

for $i = 1, \dots \# \mathrm{Cl}_F$. After identifying the sheaf cohomology groups $H^1_!(X_0, \tilde{M}_0)$ and $H^1_!(X_0^{\mathfrak{p}}, \tilde{M}_0^{\mathfrak{p}})$ with the groups $\bigoplus_i H^1_P(\Gamma_{0,i}, M)$ and $\bigoplus_i H^1_P(\Gamma^{\mathfrak{p}}_{0,i}, M)$, respectively, using diagram (2.4), we can augment diagram (2.3) on the right in the following way
(2.5)

$$\begin{tikzcd}
\cdots & H^q_!(X_1, \tilde{M}_1) \arrow[l] & H^q_!(X_0, \tilde{M}_0) \arrow[l, "\alpha_1^{0,1}"'] \arrow[dl, "\alpha_1^{0,1\mathfrak{p}}"'] & H^q_!(X_{-1}, \tilde{M}_{-1}) \arrow[l, "\alpha_1^{-1,0}"'] \arrow[dl, "\alpha_1^{-1,0\mathfrak{p}}"'] \\
\cdots & H^q_!(X_1^{\mathfrak{p}}, \tilde{M}_1^{\mathfrak{p}}) \arrow[l] \arrow[u, "\alpha_{\mathfrak{p}}^1", "\wr"'] & H^q_!(X_0^{\mathfrak{p}}, \tilde{M}_0^{\mathfrak{p}}) \arrow[l, "\alpha_1^{0\mathfrak{p},1\mathfrak{p}}"'] \arrow[u, "\alpha_{\mathfrak{p}}^0", "\wr"'] &
\end{tikzcd}$$

Here we put $H^q_!(X_{-1}, \tilde{M}_{-1}) := \bigoplus_i H^1_P(\Gamma_{-1,i}, M)$ and the maps $\alpha_1^{-1,0}$ and $\alpha_1^{-1,0\mathfrak{p}}$ are direct sums of the restriction maps.

The sheaf and group cohomologies are in a natural way modules over the corresponding Hecke algebras. (For the definition of the Hecke action on cohomology, see [10] or [3]). Here we will only consider the subalgebra $\mathbf{T}$ of the full Hecke algebra which is generated over $\mathbf{Z}$ by the double cosets $T_{\mathfrak{p}'} := K \begin{bmatrix} \pi' & \\ & 1 \end{bmatrix} K$ and $T_{\mathfrak{p}',\mathfrak{p}'} := K \begin{bmatrix} \pi' & \\ & \pi' \end{bmatrix} K$ for π' a uniformizer of $F_{\mathfrak{p}'}$ with $\mathfrak{p}'$ running over prime ideals of $\mathcal{O}_F$ such that $\mathfrak{p}' \nmid \mathfrak{N}\mathfrak{p}$. The algebra $\mathbf{T}$ acts on all the cohomology groups in diagram (2.5). Moreover if $\mathfrak{p}'$ is principal, the induced action of $T_{\mathfrak{p}'}$ and $T_{\mathfrak{p}',\mathfrak{p}'}$ on the group cohomology respects the decomposition $\bigoplus_i H^q_*((\Gamma_K)_i, M)$, where $* = \emptyset$ or P.

3. Main result

We will say that a maximal ideal $\mathfrak{n}$ of the Hecke algebra $\mathbf{T}$ is *Eisenstein* if $T_{\mathfrak{l}} \equiv N\mathfrak{l} + 1 \pmod{\mathfrak{n}}$ for all ideals $\mathfrak{l}$ of $\mathcal{O}_F$ which are trivial as elements of the ray class group of conductor $\mathfrak{n}$. Such ideals $\mathfrak{l}$ are principal and have a generator l with $l \equiv 1 \pmod{\mathfrak{n}}$. Here $N\mathfrak{l}$ denotes the ideal norm.

From now on we fix a non-Eisenstein maximal ideal $\mathfrak{m}$ of the Hecke algebra $\mathbf{T}$. Our main result is the following theorem.

Theorem 3.1. *Consider the map $H^1_!(X_0, \tilde{M}_0)^2 \xrightarrow{\alpha} H^1_!(X_1, \tilde{M}_1)$ defined as $\alpha : (f,g) \mapsto \alpha_1^{0,1} f + \alpha^1_{\mathfrak{p}} \alpha_1^{0,1\mathfrak{p}} g$. The localization $\alpha_{\mathfrak{m}}$ of α is injective.*

We prove Theorem 3.1 in two steps. Define a map $\beta : H^1_!(X_{-1}, \tilde{M}_{-1}) \to H^1_!(X_0, \tilde{M}_0)^2$ by $g' \mapsto (-\alpha^0_{\mathfrak{p}} \alpha_1^{-1,0\mathfrak{p}} g', \alpha_1^{-1,0} g')$ and note that $\alpha\beta = 0$ by the vertical commutativity of diagram (2.5), i.e., $\ker \alpha \supset \beta\big(H^1_!(X_{-1}, \tilde{M}_{-1})\big)$. We first prove

Proposition 3.2. $\ker\alpha = \beta\big(H^1_!(X_{-1}, \tilde{M}_{-1})\big)$.

Then we show

Proposition 3.3. $H^1_!(X_{-1}, \tilde{M}_{-1})_{\mathfrak{m}} = 0$.

Propositions 3.2 and 3.3 imply Theorem 3.1.

The idea of the proof is due to Khare [5] and uses modular symbols, which we now define. Let D denote the free abelian group on the set $\mathcal{B}$ of all Borel subgroups of $\mathrm{GL}_2(F)$. The action of $\mathrm{GL}_2(F)$ on $\mathcal{B}$ by conjugation gives rise to a $\mathbf{Z}$-linear action of $\mathrm{GL}_2(F)$ on D. We sometimes identify $\mathcal{B}$ with $\mathbf{P}^1(F) = \big\{\frac{a}{c} \mid a \in \mathcal{O}_F, c \in \mathcal{O}_F \setminus \{0\}\big\} \cup \{\infty\}$, on which $\mathrm{GL}_2(F)$ acts by the linear fractional transformations. Let $D_0 := \{\sum n_i B_i \in D \mid n_i \in \mathbf{Z}, B_i \in \mathcal{B}, \sum n_i = 0\}$ be the subset of elements of degree zero. If $X_K = \coprod_i \Gamma_i \setminus \mathcal{Z}$, then for each Γ_i the exact sequence

$$0 \to \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}, M) \to \mathrm{Hom}_{\mathbf{Z}}(D, M) \to \mathrm{Hom}_{\mathbf{Z}}(D_0, M) \to 0$$

gives rise to an exact sequence

$$\begin{aligned}(3.1)\quad 0 \to M \to \mathrm{Hom}_{\mathbf{Z}[\Gamma_i]}(D, M) \to \mathrm{Hom}_{\mathbf{Z}[\Gamma_i]}(D_0, M) \to \\ \to H^1(\Gamma_i, M) \to H^1(\Gamma_i, \mathrm{Hom}_{\mathbf{Z}}(D, M)).\end{aligned}$$

The group $\mathrm{Hom}_{\mathbf{Z}[\Gamma_i]}(D_0, M)$ is called the group of *modular symbols.*

Lemma 3.4. *Let Γ be a group acting on the set $\mathcal{B}$ of Borel subgroups of* $\mathrm{GL}_2(F)$ *and let C denote a set representatives for the Γ-orbits of $\mathcal{B}$. Then for any trivial Γ-module W,*

$$H^1(\Gamma, \mathrm{Hom}_{\mathbf{Z}}(D, W)) = \bigoplus_{c\in C} H^1(\Gamma_c, W),$$

where Γ_c is the stabilizer of c in Γ.

Proof. The Γ-module structure on $\mathrm{Hom}_{\mathbf{Z}}(D, W)$ is defined via $\phi^\gamma(x) = \phi(\gamma^{-1}x)$ and on $\bigoplus_{c\in C} \mathrm{Ind}^{\Gamma}_{\Gamma_c} W$ via

$$(f_{c_1}, \dots, f_{c_n})^\gamma(\gamma_{c_1}, \dots, \gamma_{c_n}) = (f_{c_1}, \dots, f_{c_n})(\gamma_{c_1}\gamma, \dots, \gamma_{c_n}\gamma).$$

Note that we have a Γ-module isomorphism $\Phi : \mathrm{Hom}_{\mathbf{Z}}(D, W) \xrightarrow{\sim} \bigoplus_{c\in C} \mathrm{Ind}^{\Gamma}_{\Gamma_c} W$, given by $\Phi(\phi)_c(\gamma) = \phi(\gamma^{-1}c)$. Thus

$$H^1(\Gamma, \mathrm{Hom}_{\mathbf{Z}}(D, W)) \simeq H^1\big(\Gamma, \bigoplus_{c\in C} \mathrm{Ind}^{\Gamma}_{\Gamma_c} W\big) \simeq \bigoplus_{c\in C} H^1(\Gamma, \mathrm{Ind}^{\Gamma}_{\Gamma_c} W)$$

since the action of Γ stabilizes $\mathrm{Ind}^{\Gamma}_{\Gamma_c} W$ for every $c \in C$. The last group is in turn isomorphic to $\bigoplus_{c\in C} H^1(\Gamma_c, W)$ by Shapiro's Lemma. $\square$

By taking the direct sum of the exact sequences (3.1) and using Lemma 3.4, we obtain the exact sequence

$$\begin{aligned}(3.2)\quad 0 \to \bigoplus_i M \to \bigoplus_i \mathrm{Hom}_{\mathbf{Z}[\Gamma_i]}(D, M) \to \\ \to \bigoplus_i \mathrm{Hom}_{\mathbf{Z}[\Gamma_i]}(D_0, M) \to \bigoplus_i H^1_P(\Gamma_i, M) \to 0,\end{aligned}$$

where the last group is isomorphic to $H^1_!(X_K, \tilde{M}_K)$.

Remark 3.5. The space of modular symbols $\bigoplus_i \mathrm{Hom}_{\mathbf{Z}[\Gamma_i]}(D_0, M)$ is also a Hecke module in a natural way. In fact it can be shown (at least if $\mathfrak{N}$ is square-free) that the localized map $\left(\bigoplus_i \mathrm{Hom}_{\mathbf{Z}[\Gamma_i]}(D, M)\right)_{\mathfrak{m}} \to \left(\bigoplus_i \mathrm{Hom}_{\mathbf{Z}[\Gamma_i]}(D_0, M)\right)_{\mathfrak{m}}$ is an isomorphism, but we will not need this fact.

4. Proof of Proposition 3.2

Suppose $(f, g) \in \ker \alpha$, i.e., $\alpha_1^{0,1} f = -\alpha_{\mathfrak{p}}^1 \alpha_1^{0,1\mathfrak{p}} g$. Let $h_1 = -g \in H^1(X_0, \tilde{M}_0)$ and $h_2 \in H^1(X_0^{\mathfrak{p}}, \tilde{M}_0^{\mathfrak{p}})$ be the pre-image of f under the isomorphism $\alpha_0^{\mathfrak{p}}$. Then $\alpha_1^{0,1} \alpha_{\mathfrak{p}}^0 h_2 = \alpha_{\mathfrak{p}}^1 \alpha_1^{0,1\mathfrak{p}} h_1$. By the vertical commutativity of diagram (2.3), we have $\alpha_1^{0,1} \alpha_{\mathfrak{p}}^0 = \alpha_{\mathfrak{p}}^1 \alpha_1^{0\mathfrak{p},1\mathfrak{p}}$, whence $\alpha_{\mathfrak{p}}^1 \alpha_1^{0\mathfrak{p},1\mathfrak{p}} h_2 = \alpha_{\mathfrak{p}}^1 \alpha_1^{0,1\mathfrak{p}} h_1$. Since $\alpha_{\mathfrak{p}}^1$ is an isomorphism, we get $\alpha_1^{0\mathfrak{p},1\mathfrak{p}} h_2 = \alpha_1^{0,1\mathfrak{p}} h_1 \in H^1(X_1^{\mathfrak{p}}, \tilde{M}_1^{\mathfrak{p}})$.

For $K \subset K'$ two compact open subgroups of $\mathrm{GL}_2(\mathbf{A}_{F,f})$ for which $X_K = \coprod_i \Gamma_i \setminus \mathcal{Z}$ and $X_{K'} = \coprod_i \Gamma_i' \setminus \mathcal{Z}$ with $\Gamma_i, \Gamma_i' \in \mathcal{G}_i$, $\Gamma_i \subset \Gamma_i'$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Z}[\Gamma_i]}(D_0, M) & \xrightarrow{\phi_{\Gamma_i}} & H_P^1(\Gamma_i, M) \\ \text{inclusion} \uparrow & & \uparrow \text{res} \\ \mathrm{Hom}_{\mathbf{Z}[\Gamma_i']}(D_0, M) & \xrightarrow{\phi_{\Gamma_i'}} & H_P^1(\Gamma_i', M) \end{array} \tag{4.1}$$

where the maps ϕ_{Γ_i} and $\phi_{\Gamma_i'}$ denote the appropriate connecting homomorphisms from exact sequence (3.2). So far we have shown that

$$-\alpha_1^{0,1\mathfrak{p}} g = \alpha_1^{0,1\mathfrak{p}} h_1 = \alpha_1^{0\mathfrak{p},1\mathfrak{p}} h_2. \tag{4.2}$$

We identify g with a tuple $(g_i)_i \in \bigoplus_i H_P^1(\Gamma_{0,i}, M)$ and define $(h_1)_i \in H_P^1(\Gamma_{0,i}, M)$ and $(h_2)_i \in H_P^1(\Gamma_{0,i}^{\mathfrak{p}}, M)$ similarly. Equality (4.2) translates to

$$-g_i|_{\Gamma_{1,i}^{\mathfrak{p}}} = (h_1)_i|_{\Gamma_{1,i}^{\mathfrak{p}}} = (h_2)_i|_{\Gamma_{1,i}^{\mathfrak{p}}} \tag{4.3}$$

Fix $g_{\mathrm{mod},i} \in \phi_{\Gamma_{1,i}^{\mathfrak{p}}}^{-1}(g)$ and regard it as an element of $\mathrm{Hom}_{\mathbf{Z}}(D_0, M)$ invariant under $\Gamma_{0,i}$. Using diagram (4.1) with $\Gamma_i = \Gamma_{1,i}^{\mathfrak{p}}$ and $\Gamma_i' = \Gamma_{0,i}^{\mathfrak{p}}$ and equality (4.3) we conclude that there exists $(h_2)_{\mathrm{mod},i} \in \phi_{\Gamma_{0,i}^{\mathfrak{p}}}^{-1}\left((h_2)_i\right)$ such that $(h_2)_{\mathrm{mod},i} = -g_{\mathrm{mod},i}$ regarded as elements of $\mathrm{Hom}_{\mathbf{Z}}(D_0, M)$. Hence $g_{\mathrm{mod},i}$ is invariant under both $\Gamma_{0,i}$ and $\Gamma_{0,i}^{\mathfrak{p}}$.

Lemma 4.1. *For $i = 1, \dots, \# \mathrm{Cl}_F$ the groups $\Gamma_{0,i}$ and $\Gamma_{0,i}^{\mathfrak{p}}$ generate $\Gamma_{-1,i}$.*

Proof. This is an immediate consequence of Theorem 3 on page 110 in [8]. □

Using Lemma 4.1 we conclude that $g_{\mathrm{mod},i} \in \mathrm{Hom}_{\mathbf{Z}[\Gamma_{-1,i}]}(D_0, M)$. Put $g_i' = \phi_{\Gamma_{-1,i}}(g_{\mathrm{mod},i})$. Again, by the commutativity of diagram (4.1) with $\Gamma_i' = \Gamma_{-1,i}$ and $\Gamma_i = \Gamma_{0,i}$ we have $g_i'|_{\Gamma_{0,i}} = g_i$. Hence $g' := (g_i')_i \in \bigoplus_i H_P^1(\Gamma_{-1,i}, M) \cong H_!^1(X_{-1}, \tilde{M}_{-1})$ satisfies $g = \alpha_1^{-1,0} g'$.

Thus $0 = \alpha_1^{0,1} f + \alpha_{\mathfrak{p}}^1 \alpha_1^{0,1\mathfrak{p}} g = \alpha_1^{0,1} f + \alpha_{\mathfrak{p}}^1 \alpha_1^{0,1\mathfrak{p}} \alpha_1^{-1,0} g'$. By the vertical commutativity of diagram (2.5) we have $\alpha_{\mathfrak{p}}^1 \alpha_1^{0,1\mathfrak{p}} \alpha_1^{-1,0} = \alpha_1^{0,1} \alpha_{\mathfrak{p}}^0 \alpha_1^{-1,0\mathfrak{p}}$, so $0 = \alpha_1^{0,1} f + \alpha_1^{0,1} \alpha_{\mathfrak{p}}^0 \alpha_1^{-1,0\mathfrak{p}} g'$. Since $\alpha_1^{0,1}$ is injective by Lemma 2.1, this implies that $f = -\alpha_{\mathfrak{p}}^0 \alpha_1^{-1,0\mathfrak{p}} g'$. Hence $(f, g) = (-\alpha_{\mathfrak{p}}^0 \alpha_1^{-1,0\mathfrak{p}} g', \alpha_1^{-1,0} g') \in \beta(H_!^1(X_{-1}, \tilde{M}_{-1}))$, completing the proof of Proposition 3.2.

5. Proof of Proposition 3.3

In this section we prove that for a principal ideal $\mathfrak{l} = (l) \subset \mathcal{O}_F$ such that $l \equiv 1 \pmod{\mathfrak{N}}$ we have $T_{\mathfrak{l}} f = (N\mathfrak{l} + 1)f$ on elements $f \in H^1_!(X_{-1}, \tilde{M}_{-1}) \cong \bigoplus_i H^1_P(\Gamma_{-1,i}, M)$. For such an ideal $\mathfrak{l}$, the operators $T_{\mathfrak{l}}$ preserve each direct summand $H^1_P(\Gamma_{-1,i}, M)$. The restriction of $T_{\mathfrak{l}}$ to $H^1_P(\Gamma_{-1,i}, M)$ is given by the usual action of the double coset $\Gamma_{-1,i} \left[\begin{smallmatrix} 1 & \\ & l \end{smallmatrix}\right] \Gamma_{-1,i}$ on group cohomology (see, e.g., [3]). For $k, l \in \mathcal{O}_F$ we put $\sigma_{k,l} := \left[\begin{smallmatrix} 1 & k \\ & l \end{smallmatrix}\right]$, $\sigma_l := \left[\begin{smallmatrix} l & \\ & 1 \end{smallmatrix}\right]$. To describe the action of $T_{\mathfrak{l}}$ explicitly we use the following lemma.

Lemma 5.1. *Let $\mathfrak{l} = (l)$ be a principal ideal of $\mathcal{O}_F$ and $n \geq -1$. Then*

$$\Gamma_{n,i} \begin{bmatrix} 1 & \\ & l \end{bmatrix} \Gamma_{n,i} = \coprod_{\substack{k \in R(\mathcal{O}_F/\mathfrak{l}) \\ k \in \mathfrak{p}_i}} \Gamma_{n,i}\sigma_{k,l} \sqcup \Gamma_{n,i}\sigma_l,$$

where $R(\mathcal{O}_F/\mathfrak{l})$ denotes a set of representatives of $\mathcal{O}_F/\mathfrak{l}$ in $\mathcal{O}_F$.

Proof. This is easy. $\square$

Lemma 5.2. *Let $n \geq 3$ be an odd integer. Every ideal class c of F contains infinitely many prime ideals $\mathfrak{q}$ such that $(N\mathfrak{q} - 1, n) = 1$.*

Proof. We assume $F \cap \mathbf{Q}(\zeta_n) = \mathbf{Q}$, the other case being easier. Let $G = \mathrm{Gal}(F/\mathbf{Q})$, $N = \mathrm{Gal}(\mathbf{Q}(\zeta_n)/\mathbf{Q})$ and $C = \mathrm{Gal}(H/F) \cong \mathrm{Cl}_F$, where H denotes the Hilbert class field of F. We have the following diagram of fields

(5.1)
$$\begin{tikzcd}
& & F_nH \arrow[dl, dash, "C"'] \arrow[dr, dash, "N"] & \\
& F_n = F\mathbf{Q}(\zeta_n) \arrow[dl, dash, "G"'] \arrow[dr, dash, "N"] & & H \arrow[dl, dash, "C"] \\
\mathbf{Q}(\zeta_n) \arrow[dr, dash, "N"'] & & F \arrow[dl, dash, "G"] & \\
& \mathbf{Q} & &
\end{tikzcd}$$

Choose $(\sigma, \tau) \in \mathrm{Gal}(F_nH/H) \times \mathrm{Gal}(F_nH/F_n) \cong N \times C$, such that

$$\sigma \in \mathrm{Gal}(F_nH/H) \cong (\mathbf{Z}/n\mathbf{Z})^\times$$

corresponds to an element $\tilde{\sigma} \in (\mathbf{Z}/n\mathbf{Z})^\times$ with $\tilde{\sigma} \not\equiv 1$ modulo any of the divisors of n, and $\tau \in \mathrm{Gal}(F_nH/F_n) \cong C \cong \mathrm{Cl}_F$ corresponds to the ideal class c. By the Chebotarev density theorem there exist infinitely many primes $\mathfrak{Q}$ of the ring of integers of F_nH such that $\mathrm{Frob}_{\mathfrak{Q}} = (\sigma, \tau)$. Then the infinite set of primes $\mathfrak{q}$ of $\mathcal{O}_F$ lying under such $\mathfrak{Q}$ satisfy the condition of the lemma, i.e., $\mathfrak{q} \in c$ and $(N\mathfrak{q} - 1, n) = 1$. $\square$

By Lemma 5.2 we may assume that the ideals $\mathfrak{p}_i$ were chosen so that $N\mathfrak{p}_i - 1$ is relatively prime to the exponent of M for all $i = 1, \dots, \#\,\mathrm{Cl}_F$.

Proof of Proposition 3.3. Let $f \in H^1_P(\Gamma_{-1,i}, M)$ and let $\mathfrak{l} = (l)$ be a principal ideal of $\mathcal{O}_{(\mathfrak{p})}$ with $l \equiv 1 \pmod{\mathfrak{N}}$. We will prove that $f_{\mathfrak{l}} := T_{\mathfrak{l}} f - (N\mathfrak{l} + 1) = 0$. By the definition of parabolic cohomology, we have $f_{\mathfrak{l}}(\Gamma_{-1,i} \cap B) = 0$ for all $B \in \mathcal{B}$.

Moreover, as the exponent of M is relatively prime to $\#\mathcal{O}_F^\times$, it is enough to prove that $f_\mathfrak{l}(\tilde{\Gamma}_i) = 0$, where $\tilde{\Gamma}_i := \Gamma_{-1,i} \cap \mathrm{SL}_2(F)$. Put

$$\mathrm{SL}_2(\mathcal{O}_{(\mathfrak{p})})_i := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(F) \mid a, d \in \mathcal{O}_{(\mathfrak{p})}, b \in \mathfrak{p}_i\mathcal{O}_{(\mathfrak{p})}, c \in \mathfrak{p}_i^{-1}\mathcal{O}_{(\mathfrak{p})} \right\}.$$

We first show that $f_\mathfrak{l} = 0$ on the i-th principal congruence subgroup

$$\Gamma_{\mathfrak{N},i} := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathcal{O}_{(\mathfrak{p})})_i \mid b, c \in \mathfrak{N}\mathcal{O}_{(\mathfrak{p})}, a \equiv d \equiv 1 \mod \mathfrak{N}\mathcal{O}_{(\mathfrak{p})} \right\}.$$

If $x \in \mathfrak{N}\mathfrak{p}_i\mathcal{O}_{(\mathfrak{p})}$, and $g \in \mathrm{SL}_2(\mathcal{O}_{(\mathfrak{p})})$, then $g \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g^{-1} \in \mathrm{SL}_2(\mathcal{O}_{(\mathfrak{p})}) \cap \Gamma_{-1,i}$ and $f_\mathfrak{l}\left(g \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g^{-1}\right) = 0$ by the definition of parabolic cohomology. So $f_\mathfrak{l} = 0$ on the smallest normal subgroup H of $\mathrm{SL}_2(\mathcal{O}_{(\mathfrak{p})})$ containing matrices of the form $\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}$ with $x \in \mathfrak{N}\mathfrak{p}_i\mathcal{O}_{(\mathfrak{p})}$. By a theorem of Serre [7],

$$H = \Gamma_{\mathfrak{N}\mathfrak{p}_i} := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathcal{O}_{(\mathfrak{p})}) \mid b, c \in \mathfrak{N}\mathfrak{p}_i\mathcal{O}_{(\mathfrak{p})}, a \equiv d \equiv 1 \mod \mathfrak{N}\mathfrak{p}_i\mathcal{O}_{(\mathfrak{p})} \right\}.$$

Thus $f_\mathfrak{l} = 0$ on $\Gamma_{\mathfrak{N}\mathfrak{p}_i}$. Put

$$\Gamma'_{\mathfrak{N}\mathfrak{p}_i} := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathcal{O}_{(\mathfrak{p})})_i \mid b, c \in \mathfrak{N}\mathfrak{p}_i\mathcal{O}_{(\mathfrak{p})} \right\}.$$

Since $\Gamma_{\mathfrak{N}\mathfrak{p}_i}$ is a normal subgroup of $\Gamma'_{\mathfrak{N}\mathfrak{p}_i}$ of index $N\mathfrak{p}_i - 1$ we have $f_\mathfrak{l}\left(\Gamma'_{\mathfrak{N}\mathfrak{p}_i}\right) = 0$ by Lemma 5.2 and our choice of $\mathfrak{p}_i$. On one hand $f_\mathfrak{l}$ is zero on the elements of the form $\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}$, $c \in \mathfrak{N}\mathfrak{p}_i^{-1}\mathcal{O}_{(\mathfrak{p})}$, (again by the definition of parabolic cohomology) and on the other hand elements of this form together with $\Gamma'_{\mathfrak{N}\mathfrak{p}_i}$ generate $\Gamma_{\mathfrak{N},i}$, so $f_\mathfrak{l}(\Gamma_{\mathfrak{N},i}) = 0$, as asserted.

Thus $f_\mathfrak{l}$ descends to the quotient $\tilde{\Gamma}_i/\Gamma_{\mathfrak{N},i}$. However, on this quotient all $\sigma_{k,l}$ and σ_l act as the identity, since $l \equiv 1 \pmod{\mathfrak{N}}$ and we can always choose $k \in \mathfrak{N}\mathfrak{p}_i$. Thus $f_\mathfrak{l} = 0$. $\square$

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