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A DEFORMATION PROBLEM FOR GALOIS REPRESENTATIONS OVER IMAGINARY QUADRATIC FIELDS

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Abstract We prove the modularity of minimally ramified ordinary residually reducible p-adic Galois representations of an imaginary quadratic field F under certain assumptions. We first exhibit conditions under which the residual representation is unique up to isomorphism. Then we prove the existence of deformations arising from cuspforms on $\mathrm{GL}_2(\boldsymbol{A}_F)$ via the Galois representations constructed by Taylor *et al.* We establish a sufficient condition (in terms of the non-existence of certain field extensions which in many cases can be reduced to a condition on an L-value) for the universal deformation ring to be a discrete valuation ring and in that case we prove an $R = T$ theorem. We also study reducible deformations and show that no minimal characteristic 0 reducible deformation exists.

Keywords: Galois deformations; automorphic forms; imaginary quadratic fields; modularity

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1. Introduction

Starting with the work of Wiles [**35**,**44**] there has been a lot of progress in recent years on modularity results for two-dimensional p-adic Galois representations of totally real fields (see, for example, [**6**,**14**,**19**,**29**–**31**,**34**]). The goal of this paper is to prove such a result for imaginary quadratic fields, a case that requires new techniques since the associated symmetric space has no complex structure.

Let $F \neq \boldsymbol{Q}(\sqrt{-1}), \boldsymbol{Q}(\sqrt{-3})$ be an imaginary quadratic field of discriminant d_F. Under certain assumptions we prove an '$R = T$' theorem for residually reducible two-dimensional representations of the absolute Galois group of F. We pin down conditions (similar to [**29**], where an analogous problem is treated for representations of $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$) that determine our residual representation up to isomorphism and then study its minimal ordinary deformations. Modular deformations are constructed using the congruences involving Eisenstein cohomology classes of [**3**] and the result of Taylor on associating Galois representations to certain cuspidal automorphic representations over imaginary quadratic fields (using the improvements of [**5**]). The approach of [**29**] to prove the isomorphism between universal deformation ring and Hecke algebra fails in our case because of the non-existence of an ordinary reducible characteristic 0 deformation. This failure, however, allows under an additional assumption to show (using the method of [**1**]) that the Eisenstein deformation ring is a discrete valuation ring. As in [**7**] it is then easy to deduce an '$R = T$' theorem.

To give a more precise account, let c be the non-trivial automorphism of F, and let $p > 3$ be a prime split in the extension $F/\boldsymbol{Q}$. Fix embeddings $F \hookrightarrow \bar{\boldsymbol{Q}} \hookrightarrow \bar{\boldsymbol{Q}}_p \hookrightarrow \boldsymbol{C}$. Let F_Σ be the maximal extension of F unramified outside a finite set of places Σ and put $G_\Sigma = \mathrm{Gal}(F_\Sigma/F)$. Suppose $\boldsymbol{F}$ is a finite field of characteristic p and that $\chi_0 : G_\Sigma \to \boldsymbol{F}^\times$ is an anticyclotomic character ramified at the places dividing p. Suppose also that $\rho_0 : G_\Sigma \to \mathrm{GL}_2(\boldsymbol{F})$ is a continuous representation of the form

$$\rho_0 = \begin{pmatrix} 1 & * \\ 0 & \chi_0 \end{pmatrix}$$

and having scalar centralizer. Under certain conditions on χ_0 and Σ we show that ρ_0 is unique up to isomorphism (see § 3) and we fix a particular choice. This setup is similar to that of [**29**]. Note that, as explained in Remark 4.6, under our conditions ρ_0 does not arise as twist by a character of the restriction of a representation of $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$.

Following Mazur [**22**] we study ordinary deformations of ρ_0. Let $\mathcal{O}$ be a local complete Noetherian ring with residue field $\boldsymbol{F}$. An $\mathcal{O}$-deformation of ρ_0 is a local complete Noetherian $\mathcal{O}$-algebra A with residue field $\boldsymbol{F}$ and maximal ideal $\mathfrak{m}_A$ together with a strict equivalence class of continuous representations $\rho : G_\Sigma \to \mathrm{GL}_2(A)$ satisfying $\rho_0 = \rho \mod \mathfrak{m}_A$. An *ordinary* deformation is a deformation that satisfies

$$\rho|_{D_v} \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

for $v \mid p$, where $\chi_i|_{I_v} = \epsilon^{k_i}$ with integers $k_1 \geqslant k_2$ depending on v and ϵ is the p-adic cyclotomic character. Here D_v and I_v denote the decomposition group and the inertia group of $v \mid p$, corresponding to $F \hookrightarrow \bar{\boldsymbol{Q}}_p$ or the conjugate embedding, respectively.

To exhibit modular deformations we apply the cohomological congruences of [**3**] and the Galois representations constructed by Taylor *et al.* using a strengthening of Taylor's result in [**5**]. We also make use of a result of Urban [**40**] who proves that $\rho_\pi|_{D_v}$ is ordinary at $v \mid p$ if π is ordinary at v. We show that these results imply that there is an $\mathcal{O}$-algebra surjection

$$R \twoheadrightarrow T, \tag{1.1}$$

where R is the universal Σ-minimal deformation ring (cf. Definition 5.1) and T is a Hecke algebra acting on cuspidal automorphic forms of $\mathrm{GL}_2(\boldsymbol{A}_F)$ of weight 2 and fixed level.

As in [**7**] we can deduce that the surjection (1.1) is, in fact, an isomorphism if R is a discrete valuation ring (see Theorem 5.7). Using the method of [**1**] we prove in Proposition 5.8 that the latter reduces to the non-existence of reducible Σ-minimal deformations to $\mathrm{GL}_2(\mathcal{O}/\varpi^2\mathcal{O})$ (where ϖ denotes a uniformizer of $\mathcal{O}$). We then show (Theorem 5.12) that this last property can often be deduced from a condition on the L-value at 1 of a Hecke character of infinity type $z/\bar{z}$ which is related to χ_0. Finally, we combine these results in Theorem 5.16 to prove the modularity of certain residually reducible Σ-minimal G_Σ-representations. For an explicit numerical example where we can verify the conditions of Theorem 5.16 see Example 5.19.

To demonstrate our modularity result we give here the following special case.

Theorem 1.1. *Assume $\#\,\mathrm{Cl}_F = 1$, that p does not divide the class number of the ray class field of F of conductor p, and that any prime $q \mid d_F$ satisfies $q \not\equiv \pm 1 \pmod{p}$. Let $\mathfrak{p}$ be the prime of F over (p) corresponding to the embedding $F \hookrightarrow \bar{\boldsymbol{Q}} \hookrightarrow \bar{\boldsymbol{Q}}_p$ that we have fixed. Let τ be the unramified Hecke character of infinity type $\tau_\infty(z) = z/\bar{z}$ and let $\tau_\mathfrak{p} : G_\Sigma \to \boldsymbol{Z}_p^\times$ be the associated p-adic Galois character. Assume that $\mathrm{val}_p(L^{\mathrm{int}}(1,\tau)) = 1$. (For definitions see § 2.)*

Let $\rho : G_\Sigma \to \mathrm{GL}_2(\bar{\boldsymbol{Q}}_p)$ be a continuous irreducible representation that is ordinary at all places $v \mid p$. Suppose $\bar{\rho}^{\mathrm{ss}} \cong 1 \oplus \bar{\tau}_\mathfrak{p}$. If the following conditions are satisfied:

(1) $\Sigma \supset \{v \mid pd_F\}$,

(2) *if $v \in \Sigma$, $v \nmid p$, then $\bar{\tau}_\mathfrak{p}(\mathrm{Frob}_v) \neq (\#k_v)^{\pm 1}$ as elements of $\boldsymbol{F}_p$,*

(3) $\det(\rho) = \tau_\mathfrak{p}$,

(4) *ρ is Σ-minimal,*

then ρ is isomorphic to the Galois representation associated to a cuspform of $\mathrm{GL}_2(\boldsymbol{A}_F)$ of weight 2, twisted by the p-adic Galois character associated to a Hecke character of infinity type z.

We also study the existence of reducible deformations (see § 5.5). In contrast to the situation in [**29**] there exists no reducible Σ-minimal $\mathcal{O}$-deformation in our case, only a nearly ordinary (in the sense of Tilouine [**37**]) reducible deformation which is, however, not de Rham at one of the places above p. This means that the method of [**29**] to prove $R = T$ via the numerical criterion of Wiles and Lenstra [**20**, **44**] cannot be implemented despite having all the ingredients on the Hecke side (i.e. a lower bound on the congruence module measuring congruences between cuspforms and Eisenstein series).

The assumption on χ_0 being anticyclotomic is not necessary for the methods of this paper or the congruence result we use (see [**3**, Theorem 13]) but is necessary for constructing the modular deformations, and is related to a condition on the central character in Taylor's result on associating Galois representation to cuspforms. The restrictions in Definition 3.2 on the places contained in Σ and on the class group of the splitting field of χ_0 are similar to those of [**29**] and are essential for the uniqueness of ρ_0. Our methods do not allow to go beyond the Σ-minimal case (to achieve that in the $\boldsymbol{Q}$-case, [**29**] uses Proposition 1 of [**35**], but its analogue fails for imaginary quadratic fields) or treat residually irreducible Galois representations. To complement our study of the *absolute* deformation problem of a residually reducible Galois representation the reader is referred to the analysis of the nearly ordinary *relative* deformation problem in [**9**].

2. Notation and terminology

2.1. Galois groups

Let F be an imaginary quadratic extension of $\boldsymbol{Q}$ of discriminant $d_F \neq 3, 4$ and $p > 3$ a rational prime which splits in F. Fix a prime $\mathfrak{p}$ of F lying over (p) and denote the other prime of F over (p) by $\bar{\mathfrak{p}}$. Let Cl_F denote the class group of F. We assume that $p \nmid \# \mathrm{Cl}_F$ and that any prime $q \mid d_F$ satisfies $q \not\equiv \pm 1 \pmod{p}$.

For a field K write G_K for the Galois group $\mathrm{Gal}(\bar{K}/K)$. If K is a finite extension of $\boldsymbol{Q}_\ell$ for some rational prime ℓ, we write $\mathcal{O}_K$ (respectively ϖ_K, and $\boldsymbol{F}_K$) for the ring of integers of K (respectively for a uniformizer of K, and $\mathcal{O}_K/\varpi_K\mathcal{O}_K$).

If $K \supset F$ is a number field, $\mathcal{O}_K$ will denote its ring of integers. If v is a place of K, we write K_v for the completion of K with respect to the absolute value $|\cdot|_v$ determined by v and set $\mathcal{O}_{K,v} = \mathcal{O}_{K_v}$ (if v is archimedean, we set $\mathcal{O}_{K,v} = K_v$). We also write ϖ_v for a uniformizer of K_v, $\mathfrak{P}_v$ for the maximal ideal of $\mathcal{O}_{K,v}$, and k_v for its residue field.

Fix once and for all compatible embeddings $i_v : \bar{F} \hookrightarrow \bar{F}_v$ and $\bar{F}_v \hookrightarrow \boldsymbol{C}$, for every prime v of F, so we will often regard elements of $\bar{F}_v$ as complex numbers without explicitly mentioning it. If w is a place of $K \subset \bar{F}$, it determines a place v of F, and we always regard K_w as a subfield of $\bar{F}_v$ as determined by the embedding i_v. This also allows us to identify G_{K_w} with the decomposition group $D_{\bar{v}} \subset G_K$ of a place $\bar{v}$ of $\bar{F}$. We will denote that decomposition group by D_v. Abusing notation somewhat we will denote the image of D_v in any quotient of G_K also by D_v. We write $I_v \subset D_v$ for the inertia group.

Let Σ be a finite set of places of K. Then K_Σ will denote the maximal Galois extension of K unramified outside the primes in Σ. We also write G_Σ for G_{F_Σ}. Moreover, for a positive integer n, denote by μ_n the group of nth roots of unity.

2.2. Hecke characters

For a number field K, denote by $\boldsymbol{A}_K$ the ring of adeles of K and set $\boldsymbol{A} = \boldsymbol{A}_{\boldsymbol{Q}}$. By a *Hecke character* of K we mean a continuous homomorphism

$$\lambda : K^\times \setminus \boldsymbol{A}_K^\times \to \boldsymbol{C}^\times.$$

For a place v of K write $\lambda^{(v)}$ for the restriction of λ to K_v and $\lambda^{(\infty)}$ for the restriction of λ to $\prod_{v|\infty} K_v$. The latter will be called the *infinity type* of λ. We also usually write $\lambda(\varpi_v)$ to mean $\lambda^{(v)}(\varpi_v)$. Given λ there exists a unique ideal $\mathfrak{f}_\lambda$ of K largest with respect to the following property: $\lambda^{(v)}(x) = 1$ for every finite place v of K and $x \in \mathcal{O}_{K,v}^\times$ such that $x - 1 \in \mathfrak{f}_\lambda \mathcal{O}_{K,v}$. The ideal $\mathfrak{f}_\lambda$ is called the *conductor* of λ. If $K = F$, there is only one archimedean place, which we will simply denote by ∞. For a Hecke character λ of F, one has $\lambda^{(\infty)}(z) = z^m \bar{z}^n$ with $m, n \in \boldsymbol{R}$. If $m, n \in \boldsymbol{Z}$, we say that λ is of type (A_0). We always assume that our Hecke characters are of type (A_0). Write $L(s, \lambda)$ for the Hecke L-function of λ.

Let λ be a Hecke character of infinity type $z^a(z/\bar{z})^b$ with conductor prime to p. Assume $a, b \in \boldsymbol{Z}$ and $a > 0$ and $b \geqslant 0$. Put

$$L^{\mathrm{alg}}(0, \lambda) := \Omega^{-a-2b} \left(\frac{2\pi}{\sqrt{d_F}} \right)^b \Gamma(a+b) L(0, \lambda),$$

where Ω is a complex period. In most cases, this normalization is integral, i.e. lies in the integer ring of a finite extension of $F_{\mathfrak{p}}$. See [**4**, Theorem 3] for the exact statement. Put

$$L^{\mathrm{int}}(0,\lambda) = \begin{cases} L^{\mathrm{alg}}(0,\lambda) & \text{if } \mathrm{val}_p(L^{\mathrm{alg}}(0,\lambda)) \geqslant 0, \\ 1 & \text{otherwise.} \end{cases}$$

For $z \in \boldsymbol{C}$ we write $\bar{z}$ for the complex conjugate of z. The action of complex conjugation extends to an automorphism of $\boldsymbol{A}_F^\times$ and we will write $\bar{x}$ for the image of $x \in \boldsymbol{A}_F^\times$ under that automorphism.

For a Hecke character λ of F, we denote by λ^{c} the conjugate Hecke character of F defined by $\lambda^{\mathrm{c}}(x) = \lambda(\bar{x})$.

2.3. Galois representations

For a field K and a topological ring R, by a *Galois representation* we mean a continuous homomorphism $\rho : G_K \to \mathrm{GL}_n(R)$. If $n = 1$ we usually refer to ρ as a *Galois character*. We write $K(\rho)$ for the fixed field of $\ker\rho$ and call it the *splitting field of* ρ. If ρ is a Galois character and M is an R-module, we denote by $M(\rho)$ the $R[G_K]$-module M with the G_K-action given by ρ. If K is a number field and v is a finite prime of K with inertia group I_v we say that ρ is unramified at v if $\rho|_{I_v} = 1$. If Σ is a finite set of places of K such that ρ is unramified at all $v \notin \Sigma$, then ρ can be regarded as a representation of G_{K_Σ}.

Let E be a finite extension of $\boldsymbol{Q}_p$. Every Galois representation $\rho : G_K \to \mathrm{GL}_n(E)$ can be conjugated (by an element $M \in \mathrm{GL}_n(E)$) to a representation $\rho_M : G_K \to \mathrm{GL}_n(\mathcal{O}_E)$. We denote by $\bar{\rho}_M : G_K \to \mathrm{GL}_n(\boldsymbol{F}_E)$ its reduction modulo $\varpi_E\mathcal{O}_E$. It is sometimes called a *residual representation* of ρ. The isomorphism class of its semisimplification $\bar{\rho}_M^{\mathrm{ss}}$ is independent of the choice of M and we simply write $\bar{\rho}^{\mathrm{ss}}$.

Let $\epsilon : G_F \to \boldsymbol{Z}_p^\times$ denote the p-adic cyclotomic character. For any subgroup $G \subset G_F$ we will also write ϵ for $\epsilon|_G$. Our convention is that the Hodge–Tate weight of ϵ at $\mathfrak{p}$ is 1.

Let λ be a Hecke character of F of type (A_0) and $\Sigma_\lambda = \{v \mid p\mathfrak{f}_\lambda\}$. We define (following Weil) a $\mathfrak{p}$-adic Galois character

$$\lambda_{\mathfrak{p}} : G_{\Sigma_\lambda} \to \bar{F}_{\mathfrak{p}}^\times$$

associated to λ by the following rule. For a finite place $v \nmid p\mathfrak{f}_\lambda$ of F, put $\lambda_{\mathfrak{p}}(\mathrm{Frob}_v) = i_{\mathfrak{p}}(i_\infty^{-1}(\lambda(\varpi_v)))$ where Frob_v denotes the *arithmetic* Frobenius at v. It takes values in the integer ring of a finite extension of $F_{\mathfrak{p}}$.

Definition 2.1. For a topological ring R we call a Galois representation $\rho : G_\Sigma \to \mathrm{GL}_2(R)$ *ordinary* if

$$\rho|_{D_v} \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

for $v \mid p$, where $\chi_i|_{I_v} = \epsilon^{k_i}$ with integers $k_1 \geqslant k_2$ depending on v.

2.4. Automorphic representations of $\boldsymbol{A}_F$ and their Galois representations

Set $G = \mathrm{Res}_{F/\boldsymbol{Q}}\,\mathrm{GL}_2$. For K_f an open compact subgroup of $G(\boldsymbol{A}_f)$, denote by $S_2(K_f)$ the space of cuspidal automorphic forms of $G(\boldsymbol{A})$ of weight 2, right-invariant under K_f

(for more details see § 3.1 of [**38**]). For ψ a finite order Hecke character write $S_2(K_f,\psi)$ for the forms with central character ψ. This is isomorphic as a $G(\boldsymbol{A}_f)$-module to $\bigoplus \pi_f^{K_f}$ for automorphic representations π of a certain infinity type (see Theorem 2.2 below) with central character ψ. Here π_f denotes the restriction of π to $\mathrm{GL}_2(\boldsymbol{A}_f)$ and $\pi_f^{K_f}$ stands for the K_f-invariants.

For $g \in G(\boldsymbol{A}_f)$ we have the usual Hecke action of $[K_f g K_f]$ on $S_2(K_f)$ and $S_2(K_f,\psi)$. For primes v such that the vth component of K_f is $\mathrm{GL}_2(\mathcal{O}_{F,v})$ we define

$$T_v = \left[K_f \begin{bmatrix} \varpi_v & \\ & 1 \end{bmatrix} K_f\right].$$

Combining the work of Taylor, Harris and Soudry with results of Friedberg–Hoffstein and Laumon or Weissauer, one can show the following (see [**5**] for general case of cuspforms of weight k).

Theorem 2.2 (Berger and Harcos [5, Theorem 1.1]). *Given a cuspidal automorphic representation π of $\mathrm{GL}_2(\boldsymbol{A}_F)$ with π_∞ isomorphic to the principal series representation corresponding to*

$$\begin{bmatrix} t_1 & * \\ & t_2 \end{bmatrix} \mapsto \left(\frac{t_1}{|t_1|}\right)\left(\frac{|t_2|}{t_2}\right)$$

and cyclotomic central character ψ (i.e. $\psi^{\mathrm{c}} = \psi$), let Σ_π denote the set consisting of the places of F lying above p, the primes where π or π^{c} is ramified, and the primes ramified in $F/\boldsymbol{Q}$.

Then there exists a finite extension E of $F_{\mathfrak{p}}$ and a Galois representation

$$\rho_\pi : G_{\Sigma_\pi} \to \mathrm{GL}_2(E)$$

such that if $v \notin \Sigma_\pi$, then ρ_π is unramified at v and the characteristic polynomial of $\rho_\pi(\mathrm{Frob}_v)$ is $x^2 - a_v(\pi)x + \psi(\varpi_v)(\# k_v)$, where $a_v(\pi)$ is the Hecke eigenvalue corresponding to T_v. Moreover, ρ_π is absolutely irreducible.

Remark 2.3. Taylor has some additional technical assumption in [**33**] and only showed the equality of the Hecke and Frobenius polynomial outside a set of places of zero density. Conjecture 3.2 in [**8**] describes a conjectural extension of Taylor's theorem.

Urban studied in [**39**] the case of ordinary automorphic representations π, and together with results in [**40**] on the Galois representations attached to ordinary Siegel modular forms proved the following theorem.

Theorem 2.4 (Urban [40, Corollary 2]). *Let v be a prime of F lying over p. If π is unramified at v and ordinary at v, i.e. $|a_v(\pi)|_v = 1$, then the Galois representation ρ_π is ordinary at v. Moreover,*

$$\rho_\pi|_{D_v} \cong \begin{bmatrix} \Psi_1 & * \\ & \Psi_2 \end{bmatrix},$$

where $\Psi_2|_{I_v} = 1$ and $\Psi_1|_{I_v} = \det \rho_\pi|_{I_v} = \epsilon$.

Definition 2.5. Let E be a finite extension of $F_{\mathfrak{p}}$ and $\rho : G_\Sigma \to \mathrm{GL}_2(E)$ a Galois representation for a finite set of places Σ. We say that ρ is *modular* if there exists a cuspidal automorphic representation π as in Theorem 2.2, such that $\rho \cong \rho_\pi$ (possibly after enlarging E).

From now on we fix a finite extension E of $F_{\mathfrak{p}}$ which we assume to be sufficiently large (see § 4.2 and Remark 4.5, where this condition is made more precise). To simplify notation we put $\mathcal{O} := \mathcal{O}_E$, $\boldsymbol{F} = \boldsymbol{F}_E$ and $\varpi = \varpi_E$.

3. Uniqueness of a certain residual Galois representation

Let Σ be a finite set of finite primes of F containing the primes lying over p. In this section we study residual Galois representations $\rho_0 : G_\Sigma \to \mathrm{GL}_2(\boldsymbol{F})$ of the form

$$\rho_0 = \begin{bmatrix} 1 & * \\ & \chi_0 \end{bmatrix}$$

having scalar centralizer for a certain class of characters χ_0 (cf. Definition 3.2). We show that for a fixed χ_0 there exists at most one such representation up to isomorphism (Corollary 3.7). The existence of such a representation follows from the global Euler characteristic formula [**24**, Theorem 5.1], which implies

$$\dim_{\boldsymbol{F}}(H^1(G_\Sigma, \boldsymbol{F}(\chi_0^{-1}))) \geqslant 1.$$

In § 4 we show that there exists an ordinary one provided that $\mathrm{val}_p(L^{\mathrm{int}}(0,\phi)) > 0$ for a certain Hecke character ϕ of F such that the reduction of $\phi_{\mathfrak{p}}\epsilon$ is χ_0. Alternatively, one could invoke the generalizations of Kummer's criterion to imaginary quadratic fields (see, for example, [**11**, **17**, **21**, **45**]).

Definition 3.1. Let R be a commutative ring, $J \subset R$ an ideal, M a free R-module and N a submodule of M. We will say that N is *saturated with respect to J* if

$$N = \{m \in M \mid mJ \subset N\}.$$

Let $\chi_0 : G_\Sigma \to \boldsymbol{F}^\times$ be a Galois character. Let S_p be the set of primes of $F(\chi_0)$ lying over p. Write M_{χ_0} for $\prod_{v \in S_p}(1 + \mathfrak{P}_v)$ and T_{χ_0} for its torsion submodule. The quotient M_{χ_0}/T_{χ_0} is a free $\boldsymbol{Z}_p$-module of finite rank. Let $\bar{\mathcal{E}}_{\chi_0}$ be the closure in M_{χ_0}/T_{χ_0} of the image of $\mathcal{E}_{\chi_0}$, the group of units of the ring of integers of $F(\chi_0)$ which are congruent to 1 modulo every prime in S_p.

We will now restrict ourselves to studying a certain class of characters χ_0, which we will call *Σ-admissible*. The definition of Σ-admissibility (Definition 3.2) pins down conditions under which we are able to prove that the space $\mathrm{Ext}^1_{G_\Sigma}(\mathbf{1}, \chi_0)$, where $\mathbf{1}$ denotes the trivial character, is one dimensional, i.e. that ρ_0 is unique up to isomorphism (Corollary 3.7).

Definition 3.2. We say that χ_0 is *Σ-admissible* if all of the following conditions are satisfied.

(1) χ_0 is ramified at $\mathfrak{p}$.

(2) If $v \in \Sigma$, then either χ_0 is ramified at v or $\chi_0(\mathrm{Frob}_v) \neq (\# k_v)^{\pm 1}$ (as elements of $\boldsymbol{F}$).

(3) χ_0 is anticyclotomic, i.e. $\chi_0(c\sigma c) = \chi_0(\sigma)^{-1}$ for every $\sigma \in G_\Sigma$ and c the generator of $\mathrm{Gal}(F/\boldsymbol{Q})$.

(4) The $\boldsymbol{Z}_p$-submodule $\bar{\mathcal{E}}_{\chi_0} \subset M_{\chi_0}/T_{\chi_0}$ is saturated with respect to the ideal $p\boldsymbol{Z}_p$.

(5) The χ_0^{-1}-eigenspace of the p-part of $\mathrm{Cl}_{F(\chi_0)}$ is trivial.

Remark 3.3. Note that Conditions (1) and (3) of Definition 3.2 imply that χ_0 is also ramified at $\bar{\mathfrak{p}}$. Moreover, observe that χ_0 is Σ-admissible if and only if χ_0^{-1} is. Indeed, Conditions (1)–(4) in Definition 3.2 are invariant under taking the inverse. Moreover, since χ_0 is anticyclotomic, the extension $F(\chi_0)/\boldsymbol{Q}$ is Galois, hence the χ_0^{-1}-eigenspace of the p-part of $\mathrm{Cl}_{F(\chi_0)}$ vanishes if and only if the χ_0-eigenspace does.

Remark 3.4. Condition (4) in Definition 3.2 implies that every $\epsilon \in \bar{\mathcal{E}}_{\chi_0}$ which is a pth power of an element of M_{χ_0}/T_{χ_0} must be a pth power of an element of $\bar{\mathcal{E}}_{\chi_0}$. In particular the map

$$\bar{\mathcal{E}}_{\chi_0} \otimes_{\boldsymbol{Z}_p} \bar{\boldsymbol{F}}_p \to (M_{\chi_0}/T_{\chi_0}) \otimes_{\boldsymbol{Z}_p} \bar{\boldsymbol{F}}_p$$

is injective. One way to check Condition (4) in practice is to compute the p-part C of the class group of $F(\chi_0)(\mu_p)$ as a $\mathrm{Gal}(F(\chi_0)(\mu_p)/F(\chi_0))$-module. In particular, if ω denotes the character of $\mathrm{Gal}(F(\chi_0)(\mu_p)/F(\chi_0))$ giving the action on μ_p, then Kummer theory implies that Condition (4) is satisfied if the ω-part of C is trivial.

Let

$$\rho_0 : G_\Sigma \to \mathrm{GL}_2(\boldsymbol{F})$$

be a Galois representation satisfying both of the following two conditions

(Red) $\rho_0 = \begin{bmatrix} 1 & * \\ & \chi_0 \end{bmatrix}$ for a Σ-admissible character χ_0;

(Sc) ρ_0 has scalar centralizer.

We have the following tower of fields: $F \subset F(\chi_0) \subset F(\rho_0)$. Note that p does not divide $[F(\chi_0) : F]$. Moreover, $F(\rho_0)/F(\chi_0)$ is an abelian extension of exponent p, hence $\mathrm{Gal}(F(\rho_0)/F(\chi_0))$ can be regarded as an $\boldsymbol{F}_p$-vector space V_0 on which the group $G := \mathrm{Gal}(F(\chi_0)/F)$ operates $\boldsymbol{F}_p$-linearly by conjugation and thus defines a representation

$$r_0 : G \to \mathrm{GL}_{\boldsymbol{F}_p}(V_0),$$

which is isomorphic to the irreducible $\boldsymbol{F}_p$-representation associated with χ_0^{-1}.

Let L denote the maximal abelian extension of $F(\chi_0)$ unramified outside the set Σ and such that p annihilates $\mathrm{Gal}(L/F(\chi_0))$. Then, as before, $V := \mathrm{Gal}(L/F(\chi_0))$ is an $\boldsymbol{F}_p$-vector space endowed with an $\boldsymbol{F}_p$-linear action of G, and one has

$$V \otimes_{\boldsymbol{F}_p} \bar{\boldsymbol{F}}_p \cong \bigoplus_{\varphi \in \mathrm{Hom}(G, \bar{\boldsymbol{F}}_p^\times)} V^\varphi,$$

where for a $\boldsymbol{Z}_p[G]$-module N and an $\bar{\boldsymbol{F}}_p$-valued character φ of G, we write

$$N^\varphi = \{n \in N \otimes_{\boldsymbol{Z}_p} \bar{\boldsymbol{F}}_p \mid \sigma n = \varphi(\sigma) n \text{ for every } \sigma \in G\}. \tag{3.1}$$

Note that $V_0 \otimes_{\boldsymbol{F}_p} \bar{\boldsymbol{F}}_p$ is a direct summand of $V^{\chi_0^{-1}}$.

Theorem 3.5. *If χ_0 is Σ-admissible, then* $\dim_{\bar{\boldsymbol{F}}_p} V^{\chi_0^{-1}} = 1$.

Proof. Let L_0 be the maximal abelian extension of $F(\chi_0)$ of exponent p unramified outside the set Σ and such that G acts on $\mathrm{Gal}(L_0/F(\chi_0))$ via the irreducible $\boldsymbol{F}_p$-representation associated with χ_0^{-1}. It is enough to show that

$$\dim_{\bar{\boldsymbol{F}}_p}(\mathrm{Gal}(L_0/F(\chi_0)) \otimes \bar{\boldsymbol{F}}_p) \leqslant 1.$$

Condition (2) of Definition 3.2 ensures that $L_0/F(\chi_0)$ is unramified outside the set $\{\mathfrak{p}, \bar{\mathfrak{p}}\}$. Hence it is enough to study the extensions $L/F(\chi_0)$ and $L_0/F(\chi_0)$ with $\Sigma = \{\mathfrak{p}, \bar{\mathfrak{p}}\}$. For $\mathfrak{p}_0 \in \{\mathfrak{p}, \bar{\mathfrak{p}}\}$ let $S_{\mathfrak{p}_0}$ be the set of primes of $F(\chi_0)$ lying over $\mathfrak{p}_0$. Then $S_p := S_{\mathfrak{p}} \cup S_{\bar{\mathfrak{p}}}$. Write M (respectively T, $\bar{\mathcal{E}}$) for M_{χ_0} (respectively T_{χ_0}, $\bar{\mathcal{E}}_{\chi_0}$). By Condition (5) of Definition 3.2 and class field theory (see, for example, Corollary 13.6 in [**42**]) one has $\mathrm{Gal}(L/F(\chi_0)) \cong (M/\bar{\mathcal{E}}) \otimes \boldsymbol{F}_p$. Hence $\mathrm{Gal}(L_0/F(\chi_0))$ is a quotient of $(M/\bar{\mathcal{E}}) \otimes \boldsymbol{F}_p$. On the other hand, using Condition (3) of Definition 3.2 one can show that $\mathrm{Gal}(L_0/F(\chi_0))$ is a quotient of $(M/T) \otimes \boldsymbol{F}_p$. This follows from the fact that T is a product of the groups μ_p; thus χ_0 being anticyclotomic by Condition (3) of Definition 3.2 cannot occur in T. We will now study both $(M/T) \otimes \bar{\boldsymbol{F}}_p$ and $(M/\bar{\mathcal{E}}) \otimes \bar{\boldsymbol{F}}_p$, beginning with the former one.

Let $G^\vee := \mathrm{Hom}(G, \bar{\boldsymbol{F}}_p^\times)$. Since G is abelian, $(M/T) \otimes \bar{\boldsymbol{F}}_p$ decomposes into a direct sum of $\bar{\boldsymbol{F}}_p[G]$-modules

$$(M/T) \otimes \bar{\boldsymbol{F}}_p = \bigoplus_{\psi \in G^\vee} (M/T)^\psi,$$

with $(M/T)^\psi$ defined as in (3.1). Note that we can refine this by writing

$$M/T = \prod_{\mathfrak{p}_0 \in \{\mathfrak{p}, \bar{\mathfrak{p}}\}} M_{\mathfrak{p}_0}/T_{\mathfrak{p}_0},$$

where $M_{\mathfrak{p}_0} = \prod_{v \in S_{\mathfrak{p}_0}} (1 + \mathfrak{P}_v)$ and $T_{\mathfrak{p}_0}$ is the torsion subgroup of $M_{\mathfrak{p}_0}$. Each $M_{\mathfrak{p}_0}/T_{\mathfrak{p}_0}$ is G-stable.

Lemma 3.6. *Let $\mathfrak{p}_0 \in \{\mathfrak{p}, \bar{\mathfrak{p}}\}$. For every $\psi \in G^\vee$, we have*

$$\dim_{\bar{\boldsymbol{F}}_p} (M_{\mathfrak{p}_0}/T_{\mathfrak{p}_0})^\psi = 1.$$

Proof of Lemma 3.6. Note that to decompose $(M_{\mathfrak{p}_0}/T_{\mathfrak{p}_0})\otimes\bar{\boldsymbol{F}}_p$ it is enough to decompose $\prod_{v\in S_{\mathfrak{p}_0}}\mathfrak{P}_v\otimes\bar{\boldsymbol{F}}_p$, since $(1+\mathfrak{P}_v)/(\text{torsion})\cong\mathfrak{P}_v$ as $\boldsymbol{Z}_p[D_v]$-modules. It is not difficult to see that

$$\prod_{v\in S_{\mathfrak{p}_0}}\mathfrak{P}_v\otimes\bar{\boldsymbol{F}}_p\cong\bigoplus_{\phi\in G^\vee}\bar{\boldsymbol{F}}_p(\phi),$$

where $\bar{\boldsymbol{F}}_p(\phi)$ denotes the one-dimensional $\bar{\boldsymbol{F}}_p$-vector space on which G acts via ϕ. The lemma follows easily. □

We are now ready to complete the proof of Theorem 3.5. Recall that the tensor product $\operatorname{Gal}(L_0/F(\chi_0))\otimes\bar{\boldsymbol{F}}_p$ is both a quotient of $(M/T)\otimes\bar{\boldsymbol{F}}_p$ and of $(M/\bar{\mathcal{E}})\otimes\bar{\boldsymbol{F}}_p$. Since $(M/T)\otimes\bar{\boldsymbol{F}}_p=(M_{\mathfrak{p}}/T_{\mathfrak{p}})\otimes\bar{\boldsymbol{F}}_p\times(M_{\bar{\mathfrak{p}}}/T_{\bar{\mathfrak{p}}})\otimes\bar{\boldsymbol{F}}_p$, Lemma 3.6 implies that

$$(M/T)\otimes\bar{\boldsymbol{F}}_p=\bigoplus_{\psi\in G^\vee}(\bar{\boldsymbol{F}}_p(\psi)\oplus\bar{\boldsymbol{F}}_p(\psi)).$$

On the other hand, one has

$$\bar{\mathcal{E}}\otimes\bar{\boldsymbol{F}}_p=\bigoplus_{\psi\in G^\vee\setminus\{\mathbf{1}\}}\bar{\boldsymbol{F}}_p(\psi),$$

where $\mathbf{1}$ denotes the trivial character. Now, Condition (4) of Definition 3.2 ensures that the map

$$\bar{\mathcal{E}}\otimes\bar{\boldsymbol{F}}_p\to(M/T)\otimes\bar{\boldsymbol{F}}_p$$

is an injection (see Remark 3.4). So, $\operatorname{Gal}(L_0/F(\chi_0))\otimes\bar{\boldsymbol{F}}_p$ is a quotient of

$$((M/T)\otimes\bar{\boldsymbol{F}}_p)/(\bar{\mathcal{E}}\otimes\bar{\boldsymbol{F}}_p)\cong\bar{\boldsymbol{F}}_p(\mathbf{1})\oplus\bar{\boldsymbol{F}}_p(\mathbf{1})\oplus\bigoplus_{\psi\in G^\vee\setminus\{\mathbf{1}\}}\bar{\boldsymbol{F}}_p(\psi).$$

Since $\chi_0\neq\mathbf{1}$, we have $\dim_{\bar{\boldsymbol{F}}_p}(\operatorname{Gal}(L_0/F(\chi_0))\otimes\bar{\boldsymbol{F}}_p)\leqslant 1$, which is what we wanted to show. □

Corollary 3.7. *Suppose $\rho':G_\Sigma\to\mathrm{GL}_2(\boldsymbol{F})$ is a Galois representation satisfying conditions* (Red) *and* (Sc). *Then $\rho'\cong\rho_0$.*

Proof. As ρ_0 and ρ' correspond to elements in $\operatorname{Ext}_{\bar{\boldsymbol{F}}_p[G_\Sigma]}(\mathbf{1},\chi_0)\cong H^1(G_\Sigma,\bar{\boldsymbol{F}}_p(\chi_0^{-1}))$, it is enough to show that $H^1(G_\Sigma,\bar{\boldsymbol{F}}_p(\chi_0^{-1}))$ is one dimensional as an $\bar{\boldsymbol{F}}_p$-vector space. Using the inflation-restriction sequence we see that $H^1(G_\Sigma,\bar{\boldsymbol{F}}_p(\chi_0^{-1}))$ is isomorphic to

$$\operatorname{Hom}_G(\ker\chi_0,\bar{\boldsymbol{F}}_p(\chi_0^{-1}))\cong\operatorname{Hom}_{\bar{\boldsymbol{F}}_p[G]}(V^{\chi_0^{-1}},\bar{\boldsymbol{F}}_p(\chi_0^{-1})).$$

Since $V^{\chi_0^{-1}}\cong\bar{\boldsymbol{F}}_p(\chi_0^{-1})$ by Theorem 3.5, the corollary follows. □

4. Modular forms and Galois representations

In this section we exhibit irreducible ordinary Galois representations that are residually reducible and arise from weight 2 cuspforms.

4.1. Eisenstein congruences

Let ϕ_1, ϕ_2 be two Hecke characters with infinity types $\phi_1^{(\infty)}(z) = z$ and $\phi_2^{(\infty)}(z) = z^{-1}$. Put $\gamma = \phi_1\phi_2$.

Denote by $\mathfrak{S}$ the finite set of places where both ϕ_i are ramified, but $\phi = \phi_1/\phi_2$ is unramified. Write $\mathfrak{M}_i$ for the conductor of ϕ_i. For an ideal $\mathfrak{N}$ in $\mathcal{O}_F$ and a finite place v of F put $\mathfrak{N}_v = \mathfrak{N}\mathcal{O}_{F,v}$. We define

$$K^1(\mathfrak{N}_v) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F,v}) \;\middle|\; a-1,\ c \equiv 0 \mod \mathfrak{N}_v \right\},$$

and

$$U^1(\mathfrak{N}_v) = \{k \in \mathrm{GL}_2(\mathcal{O}_{F,v}) \mid \det(k) \equiv 1 \mod \mathfrak{N}_v\}.$$

Now put

$$K_f := \prod_{v\in\mathfrak{S}} U^1(\mathfrak{M}_{1,v}) \prod_{v\notin\mathfrak{S}} K^1((\mathfrak{M}_1\mathfrak{M}_2)_v) \subset G(\boldsymbol{A}_f). \tag{4.1}$$

From now on, let Σ be a finite set of places of F containing

$$S_\phi := \{v \mid \mathfrak{M}_1\mathfrak{M}_1^c\mathfrak{M}_2\mathfrak{M}_2^c\} \cup \{v \mid pd_F\}.$$

We denote by $\boldsymbol{T}(\Sigma)$ the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(S_2(K_f,\gamma))$ generated by the Hecke operators T_v for all places $v \notin \Sigma$. Following [**32**, p. 107] we define idempotents $e_{\mathfrak{p}}$ and $e_{\bar{\mathfrak{p}}}$, commuting with each other and with $\boldsymbol{T}(\Sigma)$ acting on $S_2(K_f,\gamma)$. They are characterized by the property that any element $h \in X := e_{\mathfrak{p}}e_{\bar{\mathfrak{p}}}S_2(K_f,\gamma)$ which is an eigenvector for $T_{\mathfrak{p}}$ and $T_{\bar{\mathfrak{p}}}$ satisfies $|a_{\mathfrak{p}}(h)|_p = |a_{\bar{\mathfrak{p}}}(h)|_p = 1$, where $a_{\mathfrak{p}}(h)$ (respectively $a_{\bar{\mathfrak{p}}}(h)$) is the $T_{\mathfrak{p}}$-eigenvalue (respectively $T_{\bar{\mathfrak{p}}}$-eigenvalue) corresponding to h. Let $\boldsymbol{T}^{\mathrm{ord}}(\Sigma)$ denote the quotient algebra of $\boldsymbol{T}(\Sigma)$ obtained by restricting the Hecke operators to X.

Let $J(\Sigma) \subset \boldsymbol{T}(\Sigma)$ be the ideal generated by

$$\{T_v - \phi_1(\varpi_v)\#k_v - \phi_2(\varpi_v) \mid v \notin \Sigma\}.$$

Definition 4.1. Denote by $\mathfrak{m}(\Sigma)$ the maximal ideal of $\boldsymbol{T}^{\mathrm{ord}}(\Sigma)$ containing the image of $J(\Sigma)$. We set $\boldsymbol{T}_\Sigma := \boldsymbol{T}^{\mathrm{ord}}(\Sigma)_{\mathfrak{m}(\Sigma)}$. Moreover, set $J_\Sigma := J(\Sigma)\boldsymbol{T}_\Sigma$. We refer to J_Σ as the *Eisenstein ideal of* $\boldsymbol{T}_\Sigma$.

Theorem 4.2 (Berger [2, Theorem 6.3], [3, Theorem 14]). *Let ϕ be an unramified Hecke character of infinity type $\phi^{(\infty)}(z) = z^2$. There exist Hecke characters ϕ_1, ϕ_2 with $\phi_1/\phi_2 = \phi$ such that their conductors are divisible only by ramified primes or inert primes not congruent to $\pm 1 \mod p$ and such that*

$$\#(\boldsymbol{T}_\Sigma/J_\Sigma) \geqslant \#(\mathcal{O}/(L^{\mathrm{int}}(0,\phi))).$$

Proof. Theorem 14 in [**3**] states this inequality for the Hecke algebra $\boldsymbol{T}(\Sigma)$. However, the Eisenstein cohomology class used in the proof of Theorem 14 in [**3**] is ordinary because by [**4**, Lemma 9] its $T_{\mathfrak{p}}$-eigenvalue (respectively $T_{\bar{\mathfrak{p}}}$-eigenvalue) is the p-adic unit $p\phi_1(\mathfrak{p}) + \phi_2(\mathfrak{p})$ (respectively $p\phi_1(\bar{\mathfrak{p}}) + \phi_2(\bar{\mathfrak{p}})$). Therefore, one can prove the statement for the ordinary cuspidal Hecke algebra. ∎

Remark 4.3. If ϕ is unramified then $\overline{\phi_{\mathfrak{p}}\epsilon}$ is anticyclotomic (see [**3**, Lemma 1]). The condition on the conductor of the auxiliary character ϕ_1 together with our assumption on the discriminant of F therefore ensure that for $\chi_0 = \overline{\phi_{\mathfrak{p}}\epsilon}$, Condition (2) of Σ-admissibility is automatically satisfied for all primes $v \in S_\phi$.

The assumption on the ramification of ϕ can be relaxed. For example, Proposition 16 and Theorem 28 of [**4**] and Proposition 9 and Lemma 11 of [**3**] imply the following theorem.

Theorem 4.4. *Let ϕ_1, ϕ_2 be as at the start of this section. Assume both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are coprime to (p) and divisible only by primes split in $F/\boldsymbol{Q}$ and that $p \nmid \#(\mathcal{O}_F/\mathfrak{M}_1\mathfrak{M}_2)^\times$. Suppose $(\phi_1/\phi_2)^{\mathrm{c}} = \overline{\phi_1/\phi_2}$. If the torsion part of $H^2_c(S_{K_f}, \boldsymbol{Z}_p)$ is trivial, where*

$$S_{K_f} = G(\boldsymbol{Q})\backslash G(\boldsymbol{A})/K_f U(2)\boldsymbol{C}^\times$$

then

$$\#(\boldsymbol{T}_\Sigma/J_\Sigma) \geqslant \#(\mathcal{O}/(L^{\mathrm{int}}(0, \phi_1/\phi_2))).$$

Remark 4.5. In fact, by replacing $\boldsymbol{Z}_p$ by the appropriate coefficient system, the result is true for characters ϕ_1, ϕ_2 of infinity type $z\bar{z}^{-m}$ and z^{-m-1}, respectively, for $m \geqslant 0$. For Theorems 4.2 and 4.4, the field E needs to contain the values of the finite parts of ϕ_1 and ϕ_2 as well as $L^{\mathrm{int}}(0, \phi_1/\phi_2)$.

We will from now on assume that we are either in the situation of Theorem 4.2 or Theorem 4.4 and fix the characters ϕ_1, ϕ_2 and $\phi = \phi_1/\phi_2$, with corresponding conditions on the set Σ and definitions of K_f, $\boldsymbol{T}_\Sigma$ and J_Σ. We also assume from now on that $\mathrm{val}_p(L^{\mathrm{int}}(0, \phi)) > 0$. Put $\chi_0 = \overline{\phi_{\mathfrak{p}}\epsilon}$ and assume that χ_0 is Σ-admissible. If we are in the situation of Theorem 4.4, then suppose also that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are not divisible by any primes v such that $\#k_v \equiv 1 \mod p$. (This last assumption is only used in the proof of Theorem 5.2.)

4.2. Residually reducible Galois representations

Write

$$S_2(K_f, \gamma)_{\mathfrak{m}(\Sigma)} = \bigoplus_{\pi \in \Pi_\Sigma} \pi_f^{K_f}$$

for a finite set Π_Σ of ordinary cuspidal automorphic representations with central character γ, such that $\pi_f^{K_f} \neq 0$. The set Π_Σ is non-empty by Theorem 4.2 under our assumption that $\mathrm{val}_p(L^{\mathrm{int}}(0, \phi)) > 0$.

Let $\pi \in \Pi_\Sigma$. Let $\rho_\pi : G_\Sigma \to \mathrm{GL}_2(E)$ be the Galois representation attached to π by Theorem 2.2. (This is another point where we assume that E is large enough.) The condition on the central character in Theorem 2.2 can be satisfied (after possibly twisting with a finite character) under our assumptions on ϕ (see [**3**, Lemma 8]). The representation ρ_π is unramified at all $v \notin S_\phi$, and satisfies

$$\mathrm{tr}\, \rho_\pi(\mathrm{Frob}_v) = a_v(\pi)$$

and

$$\det \rho_\pi(\mathrm{Frob}_v) = \gamma(\varpi_v)\# k_v.$$

By definition, $\boldsymbol{T}_\Sigma$ injects into $\prod_{\pi\in\Pi_\Sigma} \mathrm{End}_{\mathcal{O}}(\pi^{K_f})$. Since T_v acts on π by multiplication by $a_v(\pi) \in \mathcal{O}$ the Hecke algebra $\boldsymbol{T}_\Sigma$ embeds, in fact, into $B = \prod_{\pi\in\Pi_\Sigma} \mathcal{O}$.

Observe that $(\operatorname{tr}\rho_\pi(\sigma))_{\pi\in\Pi_\Sigma} \in \boldsymbol{T}_\Sigma \subset B$ for all $\sigma \in G_\Sigma$. This follows from the Chebotarev density theorem and the continuity of ρ_π (note that $\boldsymbol{T}_\Sigma$ is a finite $\mathcal{O}$-algebra).

Fix $\pi \in \Pi_\Sigma$ for the rest of this subsection. Define $\rho'_\pi := \rho_\pi \otimes \phi_{2,\mathfrak{p}}^{-1}$. Then ρ'_π satisfies

$$\operatorname{tr}\rho'_\pi(\mathrm{Frob}\, v) \equiv 1 + (\phi_{\mathfrak{p}}\epsilon)(\mathrm{Frob}\, v) \pmod{\varpi} \quad \text{for } v \notin S_\phi,$$

and

$$\det \rho'_\pi = \gamma\phi_{2,\mathfrak{p}}^{-2}\epsilon = \phi_{\mathfrak{p}}\epsilon.$$

By choosing a suitable lattice Λ one can ensure that ρ'_π has image inside $\mathrm{GL}_2(\mathcal{O})$. The Chebotarev density theorem and the Brauer–Nesbitt theorem imply that

$$(\bar{\rho}'_\pi)^{\mathrm{ss}} \cong 1 \oplus \bar{\phi}_{\mathfrak{p}}\bar{\epsilon}.$$

By Theorem 2.2 ρ'_π is irreducible, so a standard argument (see, for example, Proposition 2.1 in [**27**]) shows the lattice Λ may be chosen in such a way that $\bar{\rho}'_\pi$ is not semisimple and

$$\bar{\rho}'_\pi = \begin{bmatrix} 1 & * \\ & \bar{\phi}_{\mathfrak{p}}\bar{\epsilon} \end{bmatrix}. \tag{4.2}$$

Hence $\bar{\rho}'_\pi$ satisfies conditions (Red) and (Sc) of § 3. By Theorem 2.4, ρ'_π is ordinary which combined with (4.2) implies that

$$\bar{\rho}'_\pi|_{D_{\bar{\mathfrak{p}}}} \cong \begin{bmatrix} 1 & \\ & (\bar{\phi}_{\mathfrak{p}}\bar{\epsilon})|_{D_{\bar{\mathfrak{p}}}} \end{bmatrix}. \tag{4.3}$$

We put

$$\rho_0 := \bar{\rho}'_\pi. \tag{4.4}$$

Remark 4.6. It follows from the proof of Theorem 3.5 that since ρ_0 splits when restricted to $D_{\bar{\mathfrak{p}}}$ (see (4.3)), the extension $F(\rho_0)/F(\chi_0)$ is totally ramified at $\mathfrak{p}$ and split at $\bar{\mathfrak{p}}$, so, in particular, there exists $\tau \in I_{\mathfrak{p}}$ such that

$$\rho_0(\tau) = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}.$$

This implies that no twist of ρ_0 by a character is invariant under $c \in \mathrm{Gal}(F/\boldsymbol{Q})$ and so no character twists of ρ_0 and the deformations of ρ_0 considered in the following sections arise from base change.

Furthermore, the ordinary modular deformations of ρ_0 in § 5.2 cannot be induced from a character of a quadratic extension of F because such representations split when restricted to the decomposition groups D_v for $v \mid p$. This follows from Urban's result (Theorem 2.3) and the restriction of these characteristic 0 representations being semisimple on an open subgroup of each of the decomposition groups.

5. Deformations of ρ_0

Let Σ, ϕ, χ_0 and ρ_0 be as in § 4. Recall that we have assumed that χ_0 is Σ-admissible and have shown in § 4.2 that ρ_0 satisfies conditions (Red) and (Sc) of § 3. Hence by Corollary 3.7, ρ_0 is unique up to isomorphism. By Remark 4.6 the extension $F(\rho_0)/F(\chi_0)$ is ramified at $\mathfrak{p}$ but splits at $\bar{\mathfrak{p}}$. In this section we study deformations of ρ_0.

5.1. Definitions

Denote the category of local complete Noetherian $\mathcal{O}$-algebras with residue field $\boldsymbol{F}$ by $\mathrm{LCN}(E)$. An $\mathcal{O}$-deformation of ρ_0 is a pair consisting of $A \in \mathrm{LCN}(E)$ and a strict equivalence class of continuous representations $\rho : G_\Sigma \to \mathrm{GL}_2(A)$ such that $\rho_0 = \rho \pmod{\mathfrak{m}_A}$, where $\mathfrak{m}_A$ is the maximal ideal of A. As is customary we will denote a deformation by a single member of its strict equivalence class. Note that the Hodge–Tate weights of $\phi_{\mathfrak{p}}\epsilon$ are -1 at $\mathfrak{p}$ and $+1$ at $\bar{\mathfrak{p}}$.

Following [**29**] we make the following definition.

Definition 5.1. We say that an $\mathcal{O}$-deformation $\rho : G_\Sigma \to \mathrm{GL}_2(A)$ of ρ_0 is *Σ-minimal* if ρ is ordinary,

$$\det \rho = \phi_{\mathfrak{p}}\epsilon,$$

and for all primes $v \in \Sigma$ such that $\# k_v \equiv 1 \pmod{p}$ one has

$$\rho|_{I_v} \cong \begin{bmatrix} 1 & \\ & \phi_{\mathfrak{p}}|_{I_v} \end{bmatrix}.$$

Note that by our assumption on the conductor of ϕ, we in fact have $\phi_{\mathfrak{p}}|_{I_v} = 1$ for v as above. Also the ordinarity condition means in this case that

$$\rho|_{I_{\mathfrak{p}}} \cong \begin{bmatrix} 1 & * \\ & \epsilon^{-1} \end{bmatrix} \quad \text{and} \quad \rho|_{I_{\bar{\mathfrak{p}}}} \cong \begin{bmatrix} \epsilon & * \\ & 1 \end{bmatrix}.$$

Since ρ_0 has a scalar centralizer and Σ-minimality is a deformation condition in the sense of [**22**], there exists a universal deformation ring which we will denote by $R_{\Sigma,\mathcal{O}} \in \mathrm{LCN}(E)$, and a universal Σ-minimal $\mathcal{O}$-deformation $\rho_{\Sigma,\mathcal{O}} : G_\Sigma \to \mathrm{GL}_2(R_{\Sigma,\mathcal{O}})$ such that for every $A \in \mathrm{LCN}(E)$ there is a one-to-one correspondence between the set of $\mathcal{O}$-algebra maps $R_{\Sigma,\mathcal{O}} \to A$ (inducing identity on $\boldsymbol{F}$) and the set of Σ-minimal deformations $\rho : G_\Sigma \to \mathrm{GL}_2(A)$ of ρ_0.

5.2. Irreducible modular deformations of ρ_0

The arguments from § 4.2 together with the uniqueness of ρ_0 (Corollary 3.7) can now be reinterpreted as the following theorem.

Theorem 5.2. *For any $\pi \in \Pi_\Sigma$ there is an $\mathcal{O}$-algebra homomorphism $r_\pi : R_{\Sigma,\mathcal{O}} \twoheadrightarrow \mathcal{O}$ inducing ρ'_π.*

Proof. The only property left to be checked is Σ-minimality. This is clear since ρ_π is unramified away from S_ϕ, and no $v \in S_\phi$ satisfies $\#k_v \equiv 1 \pmod p$ by construction (if we are in the case of Theorem 4.2) or assumption (in the case of Theorem 4.4). $\square$

Remark 5.3. The assumption on the conductors of ϕ_1, ϕ_2 made at the end of § 4.1 could be relaxed if local–global compatibility was known for the Galois representations constructed by Taylor. For a discussion of the Langlands conjecture for imaginary quadratic fields see [**8**, Conjecture 3.2].

Proposition 5.4. *There does not exist any non-trivial upper-triangular Σ-minimal deformation of ρ_0 to $\mathrm{GL}_2(\boldsymbol{F}[x]/x^2)$.*

Proof. Let $\rho : G_\Sigma \to \mathrm{GL}_2(\boldsymbol{F}[x]/x^2)$ be an upper-triangular Σ-minimal deformation. Then ρ has the form

$$\begin{bmatrix} 1 + x\alpha & * \\ & \chi_0 + x\beta \end{bmatrix}$$

for $\alpha : G_\Sigma \to \boldsymbol{F}^+$ a group homomorphism (here $\boldsymbol{F}^+$ denotes the additive group of $\boldsymbol{F}$) and $\beta : G_\Sigma \to \boldsymbol{F}$ a function.

By ordinarity of ρ we have $\det \rho = \chi_0$, which forces $\beta = -\alpha\chi_0$. Let v be a prime of F and consider the restriction of α to I_v. If $v \in \Sigma$, $v \nmid p$ and $\#k_v \not\equiv 1 \mod p$, one must have (by local class field theory) that $\alpha(I_v) = 0$. If $v \in \Sigma$ and $\#k_v \equiv 1 \bmod p$ (respectively $v = \mathfrak{p}$), then Σ-minimality (respectively ordinarity at $\mathfrak{p}$) implies that $\alpha(I_v) = 0$. Thus α can only be ramified at $\bar{\mathfrak{p}}$. However, since ρ is ordinary at $\bar{\mathfrak{p}}$, $\rho|_{I_{\bar{\mathfrak{p}}}}$ can be conjugated to a representation of the form

$$\begin{bmatrix} 1 & \\ * & \chi_0 \end{bmatrix}.$$

This, together with the fact that χ_0 is ramified at $\bar{\mathfrak{p}}$ (see the remark after Definition 3.2) easily implies that α must be unramified at $\bar{\mathfrak{p}}$. Since $p \nmid \#\,\mathrm{Cl}_F$, we must have $\alpha = 0$. Hence ρ is of the form

$$\begin{bmatrix} 1 & * \\ & \chi_0 \end{bmatrix}$$

and for $G' = \ker(\chi_0) \subset G_\Sigma$ we have

$$\rho|_{G'} = \begin{bmatrix} 1 & b_0 + xb_1 \\ & 1 \end{bmatrix}$$

for $b_0, b_1 : G' \to \boldsymbol{F}^+$ group homomorphisms. Note that $F(\rho)/F(\chi_0)$ is thus an abelian extension unramified outside Σ which is annihilated by p. Moreover, $\mathrm{Gal}(F(\chi_0)/F)$ acts on $\mathrm{Gal}(F(\rho)/F(\chi_0))$ via χ_0^{-1}. Hence $\mathrm{Gal}(F(\rho)/F(\chi_0)) \otimes_{\boldsymbol{F}_p} \bar{\boldsymbol{F}}_p$ is a quotient of $V^{\chi_0^{-1}}$ with $V^{\chi_0^{-1}}$ as in § 3. By an argument analogous to that in the proof of Corollary 3.7, it follows from Theorem 3.5 that ρ must be the trivial deformation. $\square$

Proposition 5.5. *The universal deformation ring $R_{\Sigma,\mathcal{O}}$ is generated as an $\mathcal{O}$-algebra by traces.*

Proof. We follow the argument of [**7**, Lemma 4.2]. If suffices to show that any non-trivial deformation of ρ_0 to $\mathrm{GL}_2(\boldsymbol{F}[x]/x^2)$ is generated by traces. Let ρ be such a deformation. Observe that for $\sigma \in \mathrm{Gal}(\bar{\boldsymbol{Q}}/F(\chi_0))$ the element $\rho(\sigma)$ can be written as

$$\begin{pmatrix} 1+xa(\sigma) & b_0(\sigma)+xb_1(\sigma) \\ xc(\sigma) & 1+xd(\sigma) \end{pmatrix},$$

so $\det(\rho)(\sigma) - \mathrm{tr}(\rho)(\sigma) = -1 - xb_0(\sigma)c(\sigma)$. Since c is non-trivial by Proposition 5.4, the Chebotarev density theorem implies there exists a σ such that $xb_0(\sigma)c(\sigma) \neq 0$. Since $\det(\rho)(\sigma) = 1$, it follows that the traces of ρ generate $\boldsymbol{F}[x]/x^2$. □

Lemma 5.6. *The image of the map $R_{\Sigma,\mathcal{O}} \to \prod_{\pi\in\Pi_\Sigma} \mathcal{O}$ given by $x \mapsto (r_\pi(x))_\pi$ is $\boldsymbol{T}_\Sigma$.*

Proof. The $\mathcal{O}$-algebra $R_{\Sigma,\mathcal{O}}$ is generated by the set $\{\mathrm{tr}\,\rho_{\Sigma,\mathcal{O}}(\mathrm{Frob}_v) \mid v \notin \Sigma\}$. For $v \notin \Sigma$, we have

$$r_\pi(\mathrm{tr}\,\rho_{\Sigma,\mathcal{O}}(\mathrm{Frob}_v)) = \phi_{2,\mathfrak{p}}(\mathrm{Frob}_v)^{-1}a_v(\pi).$$

Hence the image of the map in the lemma is the closure of the $\mathcal{O}$-subalgebra of $\prod_{\pi\in\Pi_\Sigma}\mathcal{O}$ generated by the set $\{\phi_{2,\mathfrak{p}}(\mathrm{Frob}_v)^{-1}T_v \mid v \notin \Sigma\}$ which is the same as the closure of the $\mathcal{O}$-subalgebra of $\prod_{\pi\in\Pi_\Sigma}\mathcal{O}$ generated by the set $\{T_v \mid v \notin \Sigma\}$ which in turn is $\boldsymbol{T}_\Sigma$. □

By Lemma 5.6 we obtain a surjective $\mathcal{O}$-algebra homomorphism $r : R_{\Sigma,\mathcal{O}} \twoheadrightarrow \boldsymbol{T}_\Sigma$. As in [**7**] we can now deduce the following theorem.

Theorem 5.7. *If $R_{\Sigma,\mathcal{O}}$ is a discrete valuation ring and if*

$$\mathrm{val}_p(L^{\mathrm{int}}(0,\phi)) > 0,$$

then the map $r : R_{\Sigma,\mathcal{O}} \to \boldsymbol{T}_\Sigma$ defined above is an isomorphism.

Proof. This follows easily because the Hecke algebra is torsion free since by definition it acts faithfully on a vector space of characteristic 0. □

5.3. When is $R_{\Sigma,\mathcal{O}}$ a discrete valuation ring?

Set $\Psi := \phi_\mathfrak{p}\epsilon$ and write Ψ_2 for $\Psi \pmod{\varpi^2}$.

Proposition 5.8. *Assume that ρ_0 does not admit any Σ-minimal upper-triangular deformation to $\mathrm{GL}_2(\mathcal{O}/\varpi^2\mathcal{O})$. Then $R_{\Sigma,\mathcal{O}}$ is a discrete valuation ring.*

Remark 5.9. The condition on the non-existence of a Σ-minimal upper-triangular deformation of ρ_0 to $\mathrm{GL}_2(\mathcal{O}/\varpi^2\mathcal{O})$ follows from the following condition on the character ϕ (or, which is the same, on the splitting field $F(\Psi_2)$ of Ψ_2): there does not exist an abelian p-extension L of $F(\Psi_2)$, unramified outside $\mathfrak{p}$ such that $\mathrm{Gal}(L/F(\Psi_2))$ is isomorphic to a $\boldsymbol{Z}[\mathrm{Gal}(F(\Psi_2)/F)]$-submodule of $(\mathcal{O}/\varpi^2\mathcal{O})(\Psi_2^{-1})$ on which $\mathrm{Gal}(F(\Psi_2)/F)$ operates faithfully. Indeed, as in the proof of Proposition 5.4, the condition of Σ-minimality forces any such deformation to be of the form

$$\begin{bmatrix} 1 & * \\ 0 & \Psi_2 \end{bmatrix}$$

with $*$ corresponding to an extension of $F(\Psi_2)$ unramified away from $\mathfrak{p}$.

Proof of Proposition 5.8. We briefly recall some general facts about Eisenstein representations from § 3 of [**7**] and § 2 of [**1**]. Let $(A, \mathfrak{m}, k)$ be a local p-adically complete ring. Let G be a topological group and consider a continuous representation $\rho : G \to \mathrm{GL}_2(A)$ such that $\mathrm{tr}(\rho) \mod \mathfrak{m}$ is the sum of two distinct characters $\tau_i : G \to k^\times$, $i = 1, 2$.

Definition 5.10. The *ideal of reducibility* of A is the smallest ideal I of A such that $\mathrm{tr}(\rho) \bmod I$ is the sum of two characters.

Lemma 5.11 (Bellaïche and Chenevier [1, Corollaire 2], Calegari [7, Lemma 3.4]). *Suppose A is Noetherian, that the ideal of reducibility is maximal, and that*

$$\dim_k \mathrm{Ext}^1_{\mathrm{cts}, k[G]}(\tau_1, \tau_2) = \dim_k \mathrm{Ext}^1_{\mathrm{cts}, k[G]}(\tau_2, \tau_1) = 1.$$

If A admits a surjective map to a ring of characteristic 0, *then A is a discrete valuation ring.*

We apply this lemma for $G = G_\Sigma$, $A = R_{\Sigma,\mathcal{O}}$, $\tau_1 = 1$ and $\tau_2 = \chi_0$. Σ-admissibility of χ_0 (which implies Σ-admissibility of its inverse by Remark 3.3) guarantees that the dimension condition in Lemma 5.11 is satisfied. Moreover, since $R_{\Sigma,\mathcal{O}} \to \boldsymbol{T}_\Sigma$ is surjective and $\boldsymbol{T}_\Sigma$ is a ring of characteristic zero, we infer that $R_{\Sigma,\mathcal{O}}$ is a discrete valuation ring whenever the ideal of reducibility I of $R_{\Sigma,\mathcal{O}}$ is maximal. This is the case if and only if there does not exist a surjection $R_{\Sigma,\mathcal{O}}/I \twoheadrightarrow \boldsymbol{F}[x]/x^2$ or $R_{\Sigma,\mathcal{O}}/I \twoheadrightarrow \mathcal{O}/\varpi^2\mathcal{O}$, or, by the universality of $R_{\Sigma,\mathcal{O}}$ if ρ_0 does not admit any non-trivial Σ-minimal deformations of ρ_0 to $\mathrm{GL}_2(\boldsymbol{F}[x]/x^2)$ or $\mathrm{GL}_2(\mathcal{O}/\varpi^2\mathcal{O})$ that are upper-triangular. The latter cannot occur by assumption and the former by Proposition 5.4. $\square$

Note that $\mathrm{Gal}(F(\Psi)/F) \cong \Gamma \times \Delta$ with $\Gamma \cong \boldsymbol{Z}_p$ and Δ a finite group.

Theorem 5.12. *Assume $p \nmid \#\Delta$. If*

$$\#(\mathcal{O}/L^{\mathrm{int}}(0, \phi)) = p^{[\mathcal{O}:\boldsymbol{Z}_p]},$$

then ρ_0 does not admit any Σ-minimal upper-triangular deformation to $\mathrm{GL}_2(\mathcal{O}/\varpi^2\mathcal{O})$. *In particular, $R_{\Sigma,\mathcal{O}}$ is a discrete valuation ring.*

Remark 5.13. Let $\mathcal{O}'$ be the ring of integers in any finite extension of $\boldsymbol{Q}_p$ containing $L^{\mathrm{int}}(0, \phi)$. Note that the L-value condition in Theorem 5.12 is equivalent to $\#(\mathcal{O}'/L^{\mathrm{int}}(0, \phi)) = p^{[\mathcal{O}':\boldsymbol{Z}_p]}$.

Proof. Put $\tilde{\chi}_0 = \Psi|_\Delta$. Write X_∞ for $\mathrm{Gal}(M(F(\Psi))/F(\Psi))$ with $M(F(\Psi))$ the maximal abelian pro-p-extension of $F(\Psi)$ unramified away from the primes lying over $\mathfrak{p}$ and $(X_\infty \otimes \mathcal{O})^{\tilde{\chi}_0^{-1}}$ the $\tilde{\chi}_0^{-1}$-part of $X_\infty \otimes \mathcal{O}$. Moreover, write $M(F(\Psi_2))_\Psi$ for the maximal abelian pro-p-extension of $F(\Psi_2)$ unramified away from $\mathfrak{p}$ on which $\mathrm{Gal}(F(\Psi_2)/F)$ acts via Ψ^{-1}. We will use the following two lemmas.

Lemma 5.14. *We have*

$$\#((X_\infty \otimes \mathcal{O})^{\tilde{\chi}_0^{-1}}/(\gamma - \Psi^{-1}(\gamma))(X_\infty \otimes \mathcal{O})^{\tilde{\chi}_0^{-1}}) \leqslant \#(\mathcal{O}/L^{\mathrm{int}}(0, \phi)).$$

Lemma 5.15. *We have*

$$\#(\mathrm{Gal}(M(F(\Psi_2))_\Psi/F(\Psi_2))\otimes\mathcal{O})^{\tilde{\chi}_0^{-1}} \leqslant \#((X_\infty\otimes\mathcal{O})^{\tilde{\chi}_0^{-1}}/(\gamma-\Psi^{-1}(\gamma))(X_\infty\otimes\mathcal{O})^{\tilde{\chi}_0^{-1}}). \quad (5.1)$$

We first show how Theorem 5.12 follows from these lemmas. Suppose that L as in Remark 5.9 existed. Then one would have $L \subset M(F(\Psi_2))_\Psi$. One also has $F(\Psi_2)F(\rho_0) \subset M(F(\Psi_2))_\Psi$, hence

$$\#(\mathrm{Gal}(F(\Psi_2)F(\rho_0)/F(\Psi_2)) \otimes \mathcal{O})^{\tilde{\chi}_0^{-1}} \leqslant \#(\mathrm{Gal}(M(F(\Psi_2))_\Psi/F(\Psi_2)) \otimes \mathcal{O})^{\tilde{\chi}_0{}^{-1}}, \quad (5.2)$$

but $F(\Psi_2)F(\rho_0) \neq L$, because $\mathrm{Gal}(F(\Psi_2)/F)$ does not act faithfully on the group $\mathrm{Gal}(F(\Psi_2)F(\rho_0)/F(\Psi_2))$. It is easy to see that the quantity on the left-hand side of (5.2) is $p^{[\mathcal{O}:\boldsymbol{Z}_p]}$. Hence, if the conditions of Theorem 5.12 are satisfied, the inequalities in Lemmas 5.14 and 5.15 become equalities and this easily implies that $F(\Psi_2)F(\rho_0) = M(F(\Psi_2))_\Psi$. Thus L cannot exist. $\square$

Proof of Lemma 5.14. For any Galois character $\tau : G_F \to \mathcal{O}^\times$ put $A_\tau = E/\mathcal{O}(\tau)$ and set $G := \mathrm{Gal}(F(\Psi)/F)$. It follows from Proposition 5.21 (see § 5.5) that the module $\mathrm{Hom}_G(X_\infty, A_{\Psi^{-1}})$ is finite. By [**15**, Propositions 2.2(i) and 2.3],

$$\mathrm{Hom}_G(X_\infty, A_{\Psi^{-1}}) \cong \mathrm{S}^{\mathrm{str}}_{A_{\Psi^{-1}}}(F),$$

where $\mathrm{S}^{\mathrm{str}}_{A_{\Psi^{-1}}}(F) \subset H^1(\mathrm{G_F}, A_{\Psi^{-1}})$ denotes the strict Selmer group defined by Greenberg (see [**15**, § 1] for a definition). Note that the class number restriction in [**15**] is not required for these results.

It is clear that

$$\mathrm{S}^{\mathrm{str}}_{A_{\Psi^{-1}}}(F) \cong \mathrm{S}^{\mathrm{str}}_{A_{(\Psi^{-1})^{\mathrm{c}}}}(F) = \mathrm{S}^{\mathrm{str}}_{A_\Psi}(F).$$

The duality result of [**16**, Theorem 2] implies an isomorphism

$$\mathrm{S}^{\mathrm{str}}_{A_\Psi}(F) \cong \mathrm{S}^{\mathrm{str}}_{A_{\Psi^{-1}\epsilon}}(F)$$

if both Selmer groups are finite. By the observation at the beginning of the proof we know that $\mathrm{S}^{\mathrm{str}}_{A_\Psi}(F)$ is finite. For the Selmer group of the dual character the arguments of the proof of Proposition 2.2 of [**15**] imply that

$$\mathrm{S}^{\mathrm{str}}_{A_{\Psi^{-1}\epsilon}}(F) \hookrightarrow \mathrm{Hom}_G(X_\infty, A_{\Psi^{-1}\epsilon}).$$

By applying the main conjecture of Iwasawa theory, Wiles [**44**, p. 532] proves that

$$\#\,\mathrm{Hom}_G(X_\infty, A_{\Psi^{-1}\epsilon}) \leqslant \#(\mathcal{O}/L^{\mathrm{int}}(0,\phi)).$$

(For similar results towards the Bloch–Kato conjecture see also [**15**] who treats imaginary quadratic fields of class number one but Hecke characters of general infinity types.) Finally, it is easy to see that

$$\#\,\mathrm{Hom}_G(X_\infty, A_{\Psi^{-1}}) = \#(X_\infty \otimes \mathcal{O})^{\tilde{\chi}_0^{-1}}/(\gamma - \Psi^{-1}(\gamma))(X_\infty \otimes \mathcal{O})^{\tilde{\chi}_0^{-1}}.$$

$\square$

Proof of Lemma 5.15. The restriction provides a surjective $\mathcal{O}$-linear homomorphism

$$(X_\infty \otimes \mathcal{O})^{\tilde{\chi}_0^{-1}} \twoheadrightarrow (\mathrm{Gal}(M(F(\Psi_2))/F(\Psi_2)) \otimes \mathcal{O})^{\tilde{\chi}_0^{-1}}.$$

Since $\mathrm{Gal}(F(\Psi_2)/F)$ acts on $\mathrm{Gal}(M(F(\Psi_2))_\Psi/F(\Psi_2))$ via Ψ^{-1} the composite

$$(X_\infty \otimes \mathcal{O})^{\tilde{\chi}_0^{-1}} \twoheadrightarrow (\mathrm{Gal}(M(F(\Psi_2))/F(\Psi_2)) \otimes \mathcal{O})^{\tilde{\chi}_0^{-1}} \twoheadrightarrow (\mathrm{Gal}(M(F(\Psi_2))_\Psi/F(\Psi_2)) \otimes \mathcal{O})^{\tilde{\chi}_0^{-1}} \tag{5.3}$$

clearly factors through

$$(X_\infty \otimes \mathcal{O})^{\tilde{\chi}_0^{-1}}/(\gamma - \Psi^{-1}(\gamma))(X_\infty \otimes \mathcal{O})^{\tilde{\chi}_0^{-1}}.$$

$\square$

5.4. Modularity theorem

In this section we state a modularity theorem which is a consequence of the results of the previous sections. To make its statement self-contained, we explicitly include all the assumptions we have made so far.

Theorem 5.16. *Let ϕ_1, ϕ_2 be Hecke characters of F with split conductors and of infinity type z and z^{-1} respectively such that $\phi := \phi_1/\phi_2$ is unramified. Assume that the conductor $\mathfrak{M}_1$ of ϕ_1 is coprime to (p) and that $p \nmid \#(\mathcal{O}_F/\mathfrak{M}_1)^\times$. Moreover, assume that* $\mathrm{val}_p(L^{\mathrm{int}}(0,\phi)) > 0$.

Let $\rho : G_\Sigma \to \mathrm{GL}_2(E)$ be a continuous irreducible representation that is ordinary at all places $v \mid p$. Suppose $\bar{\rho}^{\mathrm{ss}} \cong \chi_1 \oplus \chi_2$ with $\chi_1 = \overline{\phi_{1,\mathfrak{p}}\epsilon}$, $\chi_2 = \overline{\phi_{2,\mathfrak{p}}}$. Set $\chi_0 := \chi_1\chi_2^{-1}$. If all of the following conditions are satisfied:

1. $\Sigma \supset \{v \mid pd_F\mathfrak{M}_1\mathfrak{M}_1^{\mathrm{c}}\}$,
2. *the representation $\bar{\rho} \otimes \chi_2^{-1}$ admits no upper-triangular Σ-minimal deformation to* $\mathrm{GL}_2(\mathcal{O}/\varpi^2\mathcal{O})$,
3. χ_0 *is Σ-admissible,*
4. $\det(\rho) = \phi_1\phi_2\epsilon$,
5. $\rho \otimes \phi_{2,\mathfrak{p}}^{-1}$ *is Σ-minimal,*

then ρ is modular in the sense of Definition 2.5.

Remark 5.17. Write $\mathrm{Gal}(F(\Psi)/F) = \Gamma \times \Delta$ with $\Gamma \cong \boldsymbol{Z}_p$. If $p \nmid \#\Delta$ then by Theorem 5.12, Condition (2) in Theorem 5.16 can be replaced by $\#(\mathcal{O}/L^{\mathrm{int}}(0,\phi)) = p^{[\mathcal{O}:\boldsymbol{Z}_p]}$.

Remark 5.18. Theorem 4.4 and Remark 4.5 show that the conditions for the conductor and infinity type of ϕ can be relaxed if one imposes a condition on the torsion freeness of a cohomology group.

Example 5.19. We now turn to a numerical example in which we can verify the conditions of Theorem 5.16 (under an additional assumption which we discuss below). Let $F = \boldsymbol{Q}(\sqrt{-51})$ and $p = 5$ (which splits in F). Since the class number is 2, there are two unramified Hecke characters of infinity type z^2. For each of them the functional equation relates the L-value at 0 to the L-value at 0 of a Hecke character of infinity type $\bar{z}/z$. The latter one in turn is equal (by the Weil lifting; see, for example, [**25**, Theorem 4.8.2] or [**18**, Theorem 12.5]) to the L-value at 1 of a weight 3 modular form of level 51 and character the Kronecker symbol $(\frac{-51}{\cdot})$. Let ϕ be the Hecke character of infinity type z^2 corresponding to the modular form with q-expansion starting with $q + 3q^3 + \dots$. Using Magma [**10**] one calculates (see Remark 5.20) that

$$\mathrm{val}_5(L^{\mathrm{int}}(0,\phi)) \geqslant 1.$$

Assuming that the 5-valuation is exactly 1 (see Remark 5.20 explaining the computational issues involved) this is enough to satisfy Condition (2) of Theorem 5.16 (cf. Remarks 5.13 and 5.17). The character $\chi_0 = \overline{\phi_{\mathfrak{p}}\epsilon}$ is Σ-admissible for appropriate sets Σ (i.e. they satisfy Conditions (1), (3), (4) and (5) of Definition 3.2) because the ray class field of conductor 5 (a degree 16 extension over F) has class number 3 (as calculated by Magma assuming the Generalized Riemann Hypothesis). Here we use that the splitting field $F(\chi_0)$ is contained in the ray class field of F of conductor 5.

Remark 5.20. In our calculation above we use an operation in Magma called LRatio which calculates a rational normalization of the L-value of a modular form using modular symbols. This calculation gives 5-valuation equal to 1. Because of the different period used by Magma we can only confirm that this provides a lower bound on the 5-valuation of $L^{\mathrm{int}}(0,\phi) = L(0,\phi)/\Omega^2$, for Ω the Neron period of a suitable elliptic curve with complex multiplication by F (see, for example, [**13**, p. 768]). This follows from the following relations between the different periods.

(1) The proof of Lemma 7.1 of [**12**] shows that the period used by Magma (RealVolume) is an integral multiple of the canonical period $\Omega(f)^+$ defined by Vatsal [**41**] (up to divisors of $Nk!$ for the level $N = 51$ and weight $k = 3$ of the modular form).

(2) Vatsal [**41**] proves that one can find a Dirichlet character χ such that

$$\tau(\bar{\chi})\frac{L(1,f,\chi)}{(-2\pi \mathrm{i})\Omega(f)^{\pm}}$$

(with $\chi(-1) = (-1)^{\pm}$) is a 5-unit. Note that Vatsal's condition that $\bar{\rho}_f$ is absolutely irreducible is satisfied in our case and $\Omega(f)^- \sim \Omega(f)^+$ because f has complex multiplication. Here we write '$\sim$' to indicate equivalence up to 5-unit. Because $\pi L(1,f,\chi) \sim L(0,\phi\cdot \mathrm{res}_F^{\boldsymbol{Q}}(\bar{\chi}))$ this implies that $\pi^2\cdot\Omega(f)^+$ is a 5-integral multiple of Ω^2.

5.5. A reducible deformation of ρ_0

Let $\Psi = \phi_{\mathfrak{p}}\epsilon$. Then $\chi_0 = \bar{\Psi}$. For a finite set of primes S of F, let $L_\Psi(S)$ denote the maximal abelian pro-p extension of $F(\Psi)$ unramified outside S and such that $\mathrm{Gal}(F(\Psi)/F)$ acts on $\mathrm{Gal}(L_\Psi(S)/F(\Psi))$ via Ψ^{-1}.

Proposition 5.21. *The group* $\mathrm{Gal}(L_\Psi(\Sigma \setminus \{\bar{\mathfrak{p}}\})/F(\Psi))$ *is a torsion* $\boldsymbol{Z}_p$*-module.*

Proof. The Σ-admissibility of χ_0 implies that the extension $L_\Psi(\Sigma\setminus\{\bar{\mathfrak{p}}\})/F(\Psi)$ is unramified away from the primes lying over $\mathfrak{p}$. Then the claim follows from the anticyclotomic main conjecture of Iwasawa theory for imaginary quadratic fields (see [**23**, **28**, **36**]) after noting that $L(0,\phi) \neq 0$. □

Corollary 5.22. *There does not exist a* Σ*-minimal reducible deformation of* ρ_0 *into* $\mathrm{GL}_2(A)$ *if* A *is not a torsion* $\mathcal{O}$*-algebra.*

Proof. As in Proposition 5.4 such a deformation would have to be of the form

$$\rho = \begin{bmatrix} 1 & * \\ & \Psi \end{bmatrix}. \tag{5.4}$$

By ordinarity, one must also have

$$\rho|_{I_{\bar{\mathfrak{p}}}} \cong \begin{bmatrix} 1 & \\ & \Psi|_{I_{\bar{\mathfrak{p}}}} \end{bmatrix},$$

which implies that the upper shoulder $*$ in (5.4) corresponds to an extension $L/F(\Psi)$ which is unramified away from primes lying over $\mathfrak{p}$. Since A is not a torsion $\boldsymbol{Z}_p$-module, this would contradict Proposition 5.21. □

Remark 5.23. In [**29**] Skinner and Wiles prove an $R = T$ theorem for deformations of a certain class of reducible (non-semisimple) residual representations of $G_{\boldsymbol{Q}}$ of the form $\left[\begin{smallmatrix} 1 & * \\ & \chi \end{smallmatrix}\right]$ for $\chi : G_{\boldsymbol{Q}} \to \bar{\boldsymbol{F}}_p^\times$ a continuous character. They apply the numerical criterion of Wiles and Lenstra [**20**, **44**] by first relating the size of the relevant universal deformation ring to a special value of the L-function of χ. They achieve this by studying the Galois cohomology of $\mathrm{ad}\,\rho$ for a Σ-minimal reducible deformation ρ with values in a characteristic zero $\boldsymbol{Z}_p$-algebra $\mathcal{O}$. Here Σ is a finite set of primes of $\boldsymbol{Q}$ satisfying similar conditions to the ones we imposed on our sets Σ. Corollary 5.22 means that their method cannot be applied in our case.

Even though no Σ-minimal characteristic zero deformations of ρ_0 exist, we now show that if one drops the ordinarity condition at $\bar{\mathfrak{p}}$, it is possible to construct a reducible (non-ordinary) deformation of ρ_0 into $\mathrm{GL}_2(\mathcal{O})$.

Proposition 5.24. *There exists a unique deformation* $\rho : G_\Sigma \to \mathrm{GL}_2(\mathcal{O})$ *of* ρ_0 *of the form*

$$\rho \cong \begin{bmatrix} 1 & * \\ & \Psi \end{bmatrix}.$$

The extension $F(\rho)/F(\Psi)$ *is unramified away from* $\{\mathfrak{p}, \bar{\mathfrak{p}}\}$.

Proof. To show that a deformation of the desired form exists (and is unique up to strict equivalence) it is enough to show that

$$H^1(G_\Sigma, \mathcal{O}(\Psi^{-1})) \cong \mathcal{O} \tag{5.5}$$

as $\mathcal{O}$-modules. Noting the equality of the Euler–Poincaré characteristics

$$\sum_i (-1)^i \operatorname{rk}_{\mathcal{O}}(H^i(G_\Sigma, \mathcal{O}(\Psi^{-1}))) = \sum_i (-1)^i \dim_{\boldsymbol{F}}(H^i(G_\Sigma, \boldsymbol{F}(\chi_0^{-1})))$$

(see, for example, [**26**, Lemma A.1.8]), it follows from the global Euler characteristic formula [**24**, Theorem 5.1] that

$$\operatorname{rk}_{\mathcal{O}}(H^1(G_\Sigma, \mathcal{O}(\Psi^{-1}))) \geqslant 1. \tag{5.6}$$

On the other hand, uniqueness of ρ_0 (Corollary 3.7) implies that

$$\dim_{\boldsymbol{F}}(H^1(G_\Sigma, \mathcal{O}(\Psi^{-1}))/\varpi) \leqslant \dim_{\boldsymbol{F}}(H^1(G_\Sigma, \boldsymbol{F}(\Psi^{-1}))) = 1.$$

This, together with (5.6) and Nakayama's lemma gives (5.5). Finally, note that Σ-admissibility of χ_0 forces $F(\rho)/F(\chi_0)$ to be unramified away from $\{\mathfrak{p}, \bar{\mathfrak{p}}\}$. □

Remark 5.25. The representation ρ in Proposition 5.24 is not ordinary. Indeed, if it were ordinary the representation $\rho|_{D_{\bar{\mathfrak{p}}}}$ would have an unramified quotient. Since it clearly has an unramified submodule, it would be split and thus the upper shoulder $*$ would correspond to a non-$\boldsymbol{Z}_p$-torsion extension of $F(\Psi)$ unramified away from $\mathfrak{p}$, which does not exist by Proposition 5.21. On the other hand, ρ is *nearly ordinary* in the sense of Tilouine (see, for example, Definition 3.1 of [**43**]) with respect to the upper-triangular Borels at $\mathfrak{p}$ and $\bar{\mathfrak{p}}$. Since one has

$$\rho|_{I_{\bar{\mathfrak{p}}}} \cong \begin{bmatrix} 1 & * \\ & \epsilon|_{I_{\bar{\mathfrak{p}}}} \end{bmatrix},$$

the representation ρ is, however, not de Rham.

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