On the reflex norm

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March 11, 2005

1 The reflex field

Let F denote a CM field, $[F:\mathbf{Q}]=2g$. Denote by F^+ the maximal totally real subfield of F. Let C be a CM field which is Galois over $\mathbf{Q}$ and splits F. We denote by R the maximal totally real subfield of C. To ease notation, every time we write $A\otimes B$, for two $\mathbf{Q}$-algebras A and B, we mean $A\otimes_{\mathbf{Q}} B$. There is a canonical map of $F\otimes C$-algebras

$$F\otimes C \xrightarrow{\sim} \prod_{\phi\in \mathrm{Hom}_{\mathbf{Q}}(F,C)} C_\phi \tag{1}$$

given by $c\otimes f\mapsto (c\phi(f))_\phi$, which is an isomorphism; the notation C_ϕ means C made into an F-algebra via ϕ. We have a similar decomposition for $F\otimes R$. Indeed, $F\otimes R$ is canonically isomorphic to the (finite) product $\prod_{\mathfrak{p}\in\mathrm{Spec}(F\otimes R)}(F\otimes R)/\mathfrak{p}$, where each of the quotients is non-canonically R-isomorphic to C. We naturally have $(F\otimes R)/\mathfrak{p}\cong (F\otimes R)_{\mathfrak{p}}$. We let $F_{\mathfrak{p}}$ denote $(F\otimes R)_{\mathfrak{p}}$ and P denote $\mathrm{Spec}(F\otimes R)$. We also write $F^+_{\mathfrak{p}}$ for the corresponding factor field of $F^+\otimes R$. We have a natural identification:

$$(\mathrm{Res}_{F/\mathbf{Q}}\,\mathbf{G}_m)_R \xrightarrow{\sim} \prod_{\mathfrak{p}\in P}\mathrm{Res}_{F_{\mathfrak{p}}/R}\,\mathbf{G}_m$$

which for any R-algebra A is given by

$$(F\otimes A)^\times \xrightarrow{\sim} (F\otimes R\otimes_R A)^\times \xrightarrow{\sim} \prod_{\mathfrak{p}\in P}(F_{\mathfrak{p}}\otimes_R A)^\times.$$

Let (F,Φ) be a CM type (over C, cf. Brian's talk), i.e., $\Phi\subset \mathrm{Hom}_{\mathbf{Q}}(F,C)$ is a set of representatives for $\mathrm{Hom}_{\mathbf{Q}}(F,C)$ mod $\mathrm{Gal}(F/F^+)$. Every $\phi\in\mathrm{Hom}_{\mathbf{Q}}(F,C)$ uniquely extends to an R-algebra map $\tilde{\phi}:F\otimes R\to C$ ($f\otimes r\mapsto \phi(f)r$) which must factor through exactly one of the factor fields $F_{\mathfrak{p}}$, yielding an R-algebra isomorphism $\phi_{\mathfrak{p}}:F_{\mathfrak{p}}\xrightarrow{\sim} C$. Hence, we can think of Φ as the set $\{\phi_{\mathfrak{p}}:F_{\mathfrak{p}}\xrightarrow{\sim} C\mid \mathfrak{p}\in P\}$. If $\phi_{\mathfrak{p}}\in\Phi$ we will denote the other R-algebra isomorphism $F_{\mathfrak{p}}\xrightarrow{\sim} C$ by $\overline{\phi}_{\mathfrak{p}}$ since it is related to $\phi_{\mathfrak{p}}$ via $\mathrm{Gal}(C/R)$..

Let $\mathbf{S}:=\mathrm{Res}_{C/R}\,\mathbf{G}_m$. We define a map $h_\Phi:\mathbf{S}\to(\mathrm{Res}_{F/\mathbf{Q}}\,\mathbf{G}_m)_R\cong\prod_{\mathfrak{p}\in P}\mathrm{Res}_{F_{\mathfrak{p}}/R}\,\mathbf{G}_m$, which for every R-algebra A is given by $h_\Phi(A)=\left(\prod_{\phi_{\mathfrak{p}}\in\Phi}\phi_{\mathfrak{p}}^{-1}\right)\otimes 1:(C\otimes_R A)^\times\to(F\otimes A)^\times\cong\prod_{\mathfrak{p}\in P}F_{\mathfrak{p}}\otimes_R A$.

Example 1.1. On R-points, the map h_Φ is the map $C^\times\to\prod_{\mathfrak{p}\in P}F_{\mathfrak{p}}^\times$ given by $z\mapsto(\phi_{\mathfrak{p}}^{-1}(z))_{\mathfrak{p}}$. In fact, since $\mathbf{S}$ is a torus and R is infinite, $\mathbf{S}(R)$ is Zariski-dense in $\mathbf{S}$ and hence knowing h_Φ on R-points determines it completely.

Let T be the kernel of the composite $\mathrm{Res}_{F/\mathbf{Q}}\,\mathbf{G}_m\to\mathrm{Res}_{F^+/\mathbf{Q}}\,\mathbf{G}_m\to\left(\mathrm{Res}_{F^+/\mathbf{Q}}\,\mathbf{G}_m\right)/\mathbf{G}_m$, where the first arrow is the norm map N_{F/F^+} and the second is the natural projection. In Nick's talk it was shown that the kernel of the first arrow is a torus, so connectedness of $\mathbf{G}_m$ implies connectedness of T, hence $T\subset\mathrm{Res}_{F/\mathbf{Q}}\,\mathbf{G}_m$ is a $\mathbf{Q}$-torus.

Lemma 1.2. *The map h_Φ factors through T_R.*

Proof. We want to show that $h_\Phi^{-1}(T_R) = \mathbf{S}$. As rational points are Zariski-dense in any torus over an infinite field, it is enough to check the equality holds on R-points, i.e., we want to show that $h_\Phi(\mathbf{S}(R)) \subset T(R)$. By Example 1.1, this says that the map

$$C^\times \xrightarrow{\prod_{\phi_\mathfrak{p}\in\Phi}\phi_\mathfrak{p}^{-1}} \prod_{\phi_\mathfrak{p}\in\Phi} F_\mathfrak{p}^\times \xrightarrow{\prod N_{F_\mathfrak{p}/F_\mathfrak{p}^+}(\cdot)} \prod_{\mathfrak{p}\in P}(F_\mathfrak{p}^+)^\times$$

lands in $R^\times$ diagonally embedded in $\prod_{\mathfrak{p}\in P}(F_\mathfrak{p}^+)^\times$. However this is clear as the canonical maps $R^\times \to (F_\mathfrak{p}^+)^\times$ are isomorphisms and for every $\mathfrak{p}\in P$ we have that $\phi_\mathfrak{p}\left(N_{F_\mathfrak{p}/(F^+)_\mathfrak{p}}(\phi_\mathfrak{p}^{-1}(z))\right) = N_{C/R}(z) \in R$ is independent of $\mathfrak{p}$. $\square$

Choose a totally negative element $\alpha \in R^\times$ for which there exists $\delta \in C^\times$ such that $\delta^2 = \alpha$. (The existence of such α and δ was shown in Tong's talk via weak approximation.) We define a map

$$\mu_\Phi : \mathbf{G}_{m/C} \hookrightarrow \mathbf{S}_C \xrightarrow{(h_\Phi)_C} T_C,$$

where the first arrow $j : \mathbf{G}_{m/C} \hookrightarrow \mathbf{S}_C$ is defined on A-points (for any C-algebra A) by

$$j(a) = \frac{1}{2} \otimes (a+1) + \frac{\delta \otimes \delta(a-1)}{2\alpha}.$$

It is easy to see a priori (using that R is the maximal totally real subfield of the CM field C) that j is independent of α and δ and that $j(aa') = j(a)j(a')$, so j really is unit-valued and is a group morphism. More concretely, if we identify $\mathbf{S}_C$ with $\mathbf{G}_m \times c^*(\mathbf{G}_m)$ (with c denoting the non-trivial element of $\mathrm{Gal}(C/R)$) then j is given (on Yoneda points) by $z \mapsto (z, 1)$.

Definition 1.3. The subfield $E_{(F,\Phi)}$ of C which is the field of definition of μ_Φ is called the *reflex field* of (F, Φ).

We will denote the descent of μ_Φ to $E := E_{(F,\Phi)}$ by $\mu_{\Phi,E}$, or simply by μ_Φ when it cannot cause confusion. We have $\mathrm{Gal}(C/E_{(F,\Phi)}) = \{\sigma \in \mathrm{Gal}(C/\mathbf{Q}) \mid \sigma^*(\mu_\Phi) = \mu_\Phi\}$.

Remark 1.4. In Eiji's talk the reflex field F^* was defined to be the subfield of C generated by elements $\sum_{\phi\in\Phi}\phi(a)$, $a \in F$. He also proved that F^* is the fixed field of the group $\{\sigma \in \mathrm{Gal}(C/\mathbf{Q}) \mid \sigma\Phi = \Phi\}$. The next two lemmas imply that the notion of the reflex field in the sense of Definition 1.3 coincides with that defined in Eiji's talk.

Lemma 1.5. $\sigma^*(\mu_\Phi) = \mu_{\sigma\Phi}$

Proof. Since μ_Φ is a map between connected algebraic groups in characteristic zero, it is uniquely determined by the induced map on the Lie algebras. For any affine group scheme $G_{/k}$, where k is a field, the Lie algebra $\mathrm{Lie}(G)$ is in canonical bijection with the points in $G(k[\epsilon])$ mapping to $1 \in G(k)$ (here $\epsilon^2 = 0$). The map $\mu_\Phi = (h_\Phi)_C \circ j$ induces a map on $C[\epsilon]$-points

$$C[\epsilon]^\times \longrightarrow (C \otimes_R C[\epsilon])^\times \longrightarrow (F \otimes C[\epsilon])^\times = \prod_{\mathfrak{p}\in P}(F_\mathfrak{p} \otimes_R C[\epsilon])^\times \tag{2}$$

where the first arrow satisfies $1 + c\epsilon \mapsto \frac{1}{2} \otimes (2 + c\epsilon) + \frac{\delta\otimes\delta c}{2\alpha}\epsilon$ and the second arrow satisfies

$$1 + (c \otimes c')(1 \otimes \epsilon) \mapsto (1 + (\phi_\mathfrak{p}^{-1}(c) \otimes c')(1 \otimes \epsilon))_\mathfrak{p}.$$

We now investigate the effect of the σ^*-operation on the C-linear map $\mathrm{Lie}\,\mu_\Phi$. We consider the following situation. Let G, H be two affine group schemes of finite type over $\mathbf{Q}$ and let V and W be the tangent spaces at identity of G and H respectively. A C-group map $G_C \xrightarrow{\psi} H_C$ induces a C-linear transformation $C \otimes V \xrightarrow{T_\psi} C \otimes W$ that is $\mathrm{Lie}(\psi)$. On the other hand, for $\sigma \in \mathrm{Gal}(C/\mathbf{Q})$ the map $\sigma^*\psi$ induces the Lie algebra map $C \otimes V \xrightarrow{\sigma^* T_\psi} C \otimes W$ that is the scalar extension of T_ψ via σ, so we get a commutative diagram of C-vector spaces

$$\begin{tikzcd} C_\sigma \otimes_C C \otimes V \arrow[r, "\mathrm{id} \otimes T_\psi"] & C_\sigma \otimes_C C \otimes W \\ C \otimes V \arrow[u, "c\otimes v \mapsto c \otimes 1 \otimes v", "\wr"'] \arrow[r, "\sigma^* T_\psi"] & C \otimes W \arrow[u, "\wr", "c\otimes w \mapsto c \otimes 1 \otimes w"'] \end{tikzcd} \tag{3}$$

Hence, if $v \in V$ and $T_\psi(v) = \sum c_i \otimes w_i \in C \otimes W$, then $(\sigma^* T_\psi)(v) = \sum \sigma(c_i) \otimes w_i = (\sigma \otimes 1) \circ T_\psi(v)$. So in our case, $\sigma^*(\mu_\Phi)(1+\epsilon) = (1 \otimes \sigma)(\mu_\Phi(1+\epsilon))$ for $v \in \mathbf{Q}[\epsilon]^\times \subset C[\epsilon]^\times$, where $1 \otimes \sigma : (F \otimes C)^\times \to (F \otimes C)^\times$ is induced from $1 \otimes \sigma : F \otimes C \to F \otimes C$. We have the following commutative diagram of rings:

$$\begin{tikzcd} F \otimes C \arrow[rr, "1\otimes\sigma"'] \arrow[d, "\wr"', "f\otimes z \mapsto (f_\mathfrak{p} \otimes z)_v"] & & F \otimes C \arrow[d, "\wr"'] \\ \prod_{\mathfrak{p}\in P} F_\mathfrak{p} \otimes_R C \arrow[rr, "(f_\mathfrak{p}\otimes z_\mathfrak{p})_\mathfrak{p} \mapsto (f_\mathfrak{p} \otimes \sigma(z_\mathfrak{p}))_\mathfrak{p}"] \arrow[d, "\wr"', "(f_\mathfrak{p}\otimes z_\mathfrak{p})_\mathfrak{p} \mapsto (\phi_\mathfrak{p}(f_\mathfrak{p})z_\mathfrak{p}, \overline{\phi_\mathfrak{p}}(f_\mathfrak{p})z_\mathfrak{p})"] & & \prod_{\mathfrak{p}\in P} F_\mathfrak{p} \otimes_R C \arrow[dd, "\wr"', "(f_\mathfrak{p}\otimes z_\mathfrak{p})_\mathfrak{p} \mapsto (\tau(f_\tau)z_\tau)_\tau"] \\ \prod_{\mathfrak{p}\in P} C_{\phi_\mathfrak{p}} \times C_{\overline{\phi}_\mathfrak{p}} \arrow[d, "="] & & \\ \prod_{\tau : F \hookrightarrow C} C_\tau \arrow[rr, "(z_\tau)_\tau \mapsto (\sigma(z_{\sigma^{-1}\tau}))_\tau"] & & \prod_{\tau : F \hookrightarrow C} C_\tau \end{tikzcd} \tag{4}$$

where $f_\tau := f_\mathfrak{p}$ and $z_\tau := z_\mathfrak{p}$ if τ and $\overline{\tau}$ are the two ways to identify $F_\mathfrak{p}$ with C as R-algebras.

Let $v = 1 + \epsilon \in \mathbf{Q}[\epsilon]^\times$. We first calculate the image of $\mu_\Phi(1+\epsilon) \in (F \otimes C[\epsilon])^\times$ inside $(\prod_{\tau:F\hookrightarrow C} C_\tau[\epsilon])^\times$. By the definition of maps (2), we have $\mu_\Phi(1+\epsilon) = \left(1 + \frac{1}{2}\left(1 + \frac{\phi_\mathfrak{p}^{-1}(\delta)\otimes\delta}{\alpha}\right)(1 \otimes \epsilon)\right)_\mathfrak{p} \in \prod_{\mathfrak{p}\in P} (F_\mathfrak{p} \otimes C)_\mathfrak{p}^\times$ which gives the element $(1+\epsilon, 1) \in \prod_{\mathfrak{p}\in P} C_{\phi_\mathfrak{p}}^\times \times C_{\overline{\phi}_\mathfrak{p}}^\times$, i.e., $\mu_\Phi(1+\epsilon) = (\beta_\tau)_\tau$, where $\beta_\tau = 1+\epsilon$ if $\tau \in \Phi$ and $\beta_\tau = 1$ otherwise.

Now, $\sigma^*(\mu_\Phi)(1+\epsilon) = (1 \otimes \sigma)(\mu_\Phi(1+\epsilon))$ viewed as an element of $(\prod_{F\hookrightarrow C} C_\tau[\epsilon])^\times$ is identified with $\Psi_\sigma((\beta_\tau)_\tau)$ by commutativity of diagram (4), where Ψ_σ is the right vertical arrow in diagram (4). We have $\Psi_\sigma((\beta_\tau)_\tau) = (\sigma(\beta_{\sigma^{-1}\tau}))_\tau$, so by the definition of β_τ we have $\sigma(\beta_{\sigma^{-1}\tau}) = 1+\epsilon$ if $\sigma^{-1}\tau \in \Phi$ and $\sigma(\beta_{\sigma^{-1}\tau}) = 1$ otherwise. Hence $\sigma^*(\mu_\Phi) = \mu_{\sigma\Phi}$.

$\square$

Lemma 1.6. *$\mu_\Phi = \mu_{\Phi'}$ if and only if $\Phi = \Phi'$.*

Proof. One implication is trvial. The other implication follows immediately from the description of μ_Φ in the proof of Lemma 1.5. Indeed, we calculated there that $\mu_\Phi(1+\epsilon) = (\beta_\tau)_\tau \in (\prod_{\tau:F\hookrightarrow C} C_\tau)^\times$, where $\beta_\tau = 1+\epsilon$ if $\tau \in \Phi$ and $\beta_\tau = 1$ otherwise. Hence if $\mu_\Phi = \mu_{\Phi'}$, we must have $\Phi = \Phi'$.

$\square$

2 The reflex norm

We will now define the notion of the reflex norm. Let L'/L be a finite separable field extension (not necessarily Galois), and X a commutative affine L-group of finite type. Fix a separable closure L^{sep}/L. There is a canonical "trace" map

$$\mathrm{Res}_{L'/L} X_{L'} \to X, \tag{5}$$

which for an L-algebra A is given by

$$X_{L'}(A \otimes_L L') = X(A \otimes_L L') \to X(A \otimes_L L^{\mathrm{sep}})^{\mathrm{Gal}(L^{\mathrm{sep}}/L)} = X(A)$$

$$a' \mapsto \prod_{\sigma \in \mathrm{Hom}_L(L', L^{\mathrm{sep}})} \sigma(a').$$

This is independent of the choice of L^{sep} and is natural in X. Setting $L = \mathbf{Q}$, $L' = E = E_{(F,\Phi)}$, and $X = T$, we obtain a canonical map

$$\psi : \mathrm{Res}_{E/\mathbf{Q}} T_E \to T.$$

Remark 2.1. Such a map can also be obtained with $\mathrm{Res}_{F/\mathbf{Q}} \mathbf{G}_m$ replacing T, and the two are compatible via the inclusion $\iota : T \hookrightarrow \mathrm{Res}_{F/\mathbf{Q}} \mathbf{G}_m$ since (5) is natural in X. We will use this fact in the proof of Proposition 2.5. On C-points with $X = \mathrm{Res}_{F/\mathbf{Q}} \mathbf{G}_m$, the "trace" $\mathrm{Res}_{L'/\mathbf{Q}} X_{L'} \to X$ is the map $(F \otimes L' \otimes C)^\times \to (F \otimes C)^\times$ induced by the product of $F \otimes C$-algebra maps $x \otimes y \otimes z \mapsto x \otimes \sigma(y) z$ over all $\sigma \in \mathrm{Hom}_{\mathbf{Q}}(L', C)$, i.e., this is the usual ring-theoretic norm on units.

Definition 2.2. The map $r_\Phi = \psi \circ \mathrm{Res}_{E/\mathbf{Q}} \mu_\Phi : \mathrm{Res}_{E/\mathbf{Q}} \mathbf{G}_m \to T$ is the *reflex norm.*

We will now give an alternative definition of the reflex norm and show that it is equivalent to the one given above.

Let A be an abelian variety over C, with $i : F \hookrightarrow \mathrm{End}^\circ(A)$ of type (F, Φ). Let E be the reflex field of (F, Φ). The abelian variety (A, i) typically does not descend to E. However the tangent space at identity $t_{A/C}$ is isomorphic $\prod_{\phi \in \Phi} C_\phi$ as an $F \otimes C$-module and this concrete model $\prod_{\phi \in \Phi} C_\phi$ has a natural C-semilinear action of $\mathrm{Gal}(C/E)$ via $\sigma((a_\phi)_\phi) = (\sigma(a_{\sigma^{-1}\phi}))_\phi$, so it descends to E in the following sense.

Lemma 2.3. *Up to isomorphism, there exists a unique $F \otimes E$-module t_Φ such that $t_\Phi \otimes_E C \cong \prod_{\phi \in \Phi} C_\phi$ as $F \otimes C$-modules compatible with the natural $\mathrm{Gal}(C/E)$-actions on both sides.*

Proof. See Eiji's talk. We only note here that explicitly t_Φ is defined to be the $F \otimes E$-submodule of $\mathrm{Gal}(C/E)$-invariants in $\prod_{\phi \in \Phi} C_\phi$. ☐

We now define a map $N_\Phi : \mathrm{Res}_{E/\mathbf{Q}} \mathbf{G}_m \to \mathrm{Res}_{F/\mathbf{Q}} \mathbf{G}_m$ which for any $\mathbf{Q}$-algebra A is given by

$$N_\Phi(A) : (E \otimes A)^\times \to (F \otimes A)^\times,$$

via $s \mapsto \det_{F \otimes_{\mathbf{Q}} A} (s : t_\Phi \otimes_{\mathbf{Q}} A \to t_\Phi \otimes_{\mathbf{Q}} A)$.

Example 2.4. Let F be a quadratic imaginary field. Then $\Phi = \{\phi : F \hookrightarrow C\}$ is a one-element set, hence $E = \phi(F)$. In this case t_Φ is a one-dimensional F-vector space and $N_\Phi(\mathbf{Q}) : E^\times \to F^\times$ is given by ϕ^{-1}.

The following proposition shows that N_Φ coincides with the reflex norm in the sense of Definition 2.2.

Proposition 2.5. *Let $\iota : T \hookrightarrow \mathrm{Res}_{F/\mathbf{Q}} \mathbf{G}_m$ be the inclusion map. Then $\iota \circ r_\Phi = N_\Phi$.*

Remark 2.6. It is important that the target group of the reflex norm is the torus T, and not merely the full group $\mathrm{Res}_{F/\mathbf{Q}} \mathbf{G}_m$. The reason which makes this fact significant for us is (as will be proved later, using results in Nick's talk) that the $\mathbf{Q}$-points of T are discrete in $T(\mathbf{A}_f)$, where $\mathbf{A}_f$ denotes the finite adeles of $\mathbf{Q}$. This makes the quotient space $T(\mathbf{Q}) \backslash T(\mathbf{A}_f)$ Hausdorff. This is not the case if we replace T with $\mathrm{Res}_{F/\mathbf{Q}} \mathbf{G}_m$ unless the group of intgral units of F is finite, which is to say F is imaginary quadratic.

Proof of Proposition 2.5. First note that $E \subset C$ by definition. To ease notation we will denote by $\mathcal{E}$ the set $\mathrm{Hom}_{\mathbf{Q}}(E, C)$. Also for each $\sigma \in \mathcal{E}$ we choose $\tilde{\sigma} \in \mathrm{Gal}(C/\mathbf{Q})$ satisfying $\tilde{\sigma}|_E = \sigma$. Then $\tilde{\sigma}\phi$ makes sense for every $\phi \in \Phi$.

We will show that the following diagram commutes:

$$\begin{tikzcd}
(E \otimes C)^\times \arrow[r, "\mathrm{Res}_{E/\mathbf{Q}} \mu_{\Phi,E}"] \arrow[d, "E\otimes C\text{-action}"'] & (F \otimes E \otimes C)^\times \arrow[d, "\prod_{\sigma \in \mathcal{E}} [\sigma](\cdot)"] \\
\mathrm{Aut}_{F \otimes C}(t_\Phi \otimes C) \arrow[r, "\det_{F \otimes C}"] & (F \otimes C)^\times = \prod_{\tau : \hookrightarrow C} C_\tau^\times
\end{tikzcd} \tag{6}$$

where the top map is on C-points and $[\sigma](f \otimes e \otimes z) = f \otimes \sigma(e)z$ as an $F \otimes C$-algebra map from $F \otimes E \otimes C$ to $F \otimes C$. By Remark 2.1 the composite of the right vertical arrow with the top arrow is $\iota \circ r_\Phi(C)$, while the composite of the other two arrows is $N_\Phi(C)$. We will first describe the map $N_\Phi(C) = \det_{F \otimes C} \circ (E \otimes C)$-action. We have an $F \otimes E \otimes C$-linear decomposition

$$t_\Phi \otimes C = t_\Phi \otimes_E E \otimes C = \prod_{\sigma \in \mathcal{E}} t_\Phi \otimes_E C_\sigma = \prod_{\sigma \in \mathcal{E}} t_\Phi \otimes_E C \otimes_C C_\sigma = \prod_{\sigma \in \mathcal{E}} \prod_{\phi \in \Phi} C_\phi \otimes_C C_\sigma = \prod_{\sigma \in \mathcal{E}} \prod_{\phi \in \Phi} C_{\phi,\sigma}$$

where the second equality from the right follows from the definition of t_Φ in the proof of Lemma 2.3; the notation $C_{\phi,\sigma}$ means C made into an E-algebra and F-algebra via σ and $\tilde{\sigma}\phi$ respectively. Moreover $E \otimes C = \prod_{\sigma \in \mathcal{E}} C_\sigma$ acts on $t_\Phi \otimes C = \prod_{\sigma \in \mathcal{E}} \prod_{\phi \in \Phi} C_{\phi,\sigma}$ via the diagonal action of C_σ on $\prod_{\phi \in \Phi} C_{\phi,\sigma}$. To compute $\det_{F \otimes C}$ we need to rearrange the factors in $\prod_{\sigma \in \mathcal{E}} \prod_{\phi \in \Phi} C_{\phi,\sigma}$ according to the action of $F \otimes C$.

As $F \otimes C$-modules we have $t_\Phi \otimes C = \prod_{\tau : F \hookrightarrow C} V_\tau$, where $V_\tau = \prod_{(\phi,\sigma) \in \Sigma(\Phi,\tau)} C_{\phi,\sigma}$, with $\Sigma(\Phi,\tau) := \{(\phi,\sigma) \in \Phi \times \mathcal{E} \mid \tilde{\sigma}\phi = \tau\}$. For $\sigma \in \mathcal{E}$, let $\sigma' : E \otimes C \to C$ be the C-algebra map induced by $e \otimes z \mapsto \sigma(e)z$. On each V_τ an element $\xi \in E \otimes C$ acts by multiplication by the diagonal matrix $\operatorname{diag}(\sigma'(\xi))_{(\phi,\sigma) \in \Sigma(\Phi,\tau)}$, so the C_τ-determinant of this C-linear action equals $\prod_{(\phi,\sigma) \in \Sigma(\Phi,\tau)} \sigma'(\xi) \in C_\tau$ for each $\tau : F \hookrightarrow C$.
After we prove that diagram (6) commutes, this computation will provide a practical way to calculate the reflex norm (cf. Remark 2.7).

We will now describe the composite of the top and right maps in diagram (6). We first consider the top map; the difficulty is that μ_Φ over C is concrete but $\mu_{\Phi,E}$ is "abstract". Note that given two $\mathbf{Q}$-schemes X and Y, and an E-map $X_E \xrightarrow{f} Y_E$, with E a subfield of C, we have a commutative diagram:

$$\begin{tikzcd}
\left(\operatorname{Res}_{E/\mathbf{Q}} X_E\right)(C) = X_E(E \otimes C) \arrow[r, "\operatorname{Res}_{E/\mathbf{Q}} f"] \arrow[d, "\wr"] & \left(\operatorname{Res}_{E/\mathbf{Q}} Y_E\right)(C) = Y_E(E \otimes C) \arrow[d, "\wr"] \\
\left(\operatorname{Res}_{E \otimes C/C} X_{E \otimes C/C}\right)(C) \arrow[r, "\operatorname{Res}_{E \otimes C/C} f_C"] \arrow[d, "\wr"] & \left(\operatorname{Res}_{E \otimes C/C} Y_{E \otimes C/C}\right)(C) \arrow[d, "\wr"] \\
\prod_{\sigma : E \hookrightarrow C} \operatorname{Res}_{C_\sigma/C} X_{C_\sigma}(C) \arrow[r, "\prod_\sigma \operatorname{Res}_{C_\sigma/C} f_{C_\sigma}"] \arrow[d, "\wr"] & \prod_{\sigma : E \hookrightarrow C} \operatorname{Res}_{C_\sigma/C} Y_{C_\sigma}(C_\sigma) \arrow[d, "\wr"] \\
\prod_{\sigma : E \hookrightarrow C} X_{C_\sigma}(C_\sigma) \arrow[r, "\prod_\sigma f_{C_\sigma}"] & \prod_{\sigma : E \hookrightarrow C} Y_{C_\sigma}(C_\sigma)
\end{tikzcd} \tag{7}$$

where $f_{C_\sigma} = \tilde{\sigma}^*(f_C) : X_{C_\sigma} \to Y_{C_\sigma}$. Hence if $X = \mathbf{G}_m$, $Y = \operatorname{Res}_{F/\mathbf{Q}} \mathbf{G}_m$, and $f = \mu_{\Phi,E}$ then since $\mu_{\Phi/C_\sigma} = \tilde{\sigma}^*(\mu_\Phi) = \mu_{\tilde{\sigma}\Phi}$ (by Lemma 1.5), we have that $\operatorname{Res}_{E/\mathbf{Q}} \mu_{\Phi,E}$ on C-points is identified with the map

$$\prod_{\sigma : E \hookrightarrow C} C_\sigma^\times \xrightarrow{\prod_\sigma \mu_{\tilde{\sigma}\Phi}} \prod_{\sigma : E \hookrightarrow C} \left(\operatorname{Res}_{F/\mathbf{Q}} \mathbf{G}_m\right)_{C_\sigma}(C_\sigma) = \prod_{\sigma : E \hookrightarrow C} (F \otimes C_\sigma)^\times . \tag{8}$$

For any $\sigma : E \hookrightarrow C$, we have the following commutative diagram of rings

$$\begin{tikzcd}
C \otimes_R C_\sigma \arrow[r, "\prod_{\mathfrak{p} \in P} (\tilde{\sigma}\phi_\mathfrak{p})^{-1} \otimes 1"] \arrow[d, "\wr"] & F \otimes C_\sigma = \prod_{\mathfrak{p} \in P} F_\mathfrak{p} \otimes_R C_\sigma \arrow[d, "(f_\mathfrak{p} \otimes z_\sigma)_{\mathfrak{p},\sigma} \mapsto (\tilde{\sigma}\phi_\mathfrak{p}(f_{\tilde{\sigma}\phi_\mathfrak{p}}) z_\sigma, \tilde{\sigma}\overline{\phi_\mathfrak{p}(f_{\tilde{\sigma}\phi_\mathfrak{p}})} z_\sigma)_\mathfrak{p}"] \\
C_\sigma \times C_{\overline{\sigma}} \arrow[r, "(z_\sigma, z_{\overline{\sigma}}) \mapsto (z_\sigma, z_{\overline{\sigma}})_\mathfrak{p}"] & \prod_{\mathfrak{p} \in P} C_{\phi_\mathfrak{p},\sigma} \times C_{\overline{\phi}_\mathfrak{p},\sigma}
\end{tikzcd} \tag{9}$$

and a commutative diagram of unit groups

$$\begin{tikzcd}
C_\sigma^\times \arrow[r, "j_{C_\sigma}"] \arrow[dr, "z_\sigma \mapsto (z_\sigma, 1)"'] & (C \otimes_R C_\sigma)^\times \arrow[d, "\wr"] \\
& C_\sigma^\times \times C_{\overline{\sigma}}^\times
\end{tikzcd} \tag{10}$$

Note that the composite of the top map in diagram (9) with j_{C_σ} is the map $\mu_{\tilde{\sigma}\Phi}$. Hence if $z_\sigma \in C_\sigma^\times$, then $\mu_{\tilde{\sigma}\Phi}(z_\sigma) = (z_\sigma, 1)_{\mathfrak{p}}$ if we view it as an element of $\prod_{\mathfrak{p} \in P} C_{\phi_{\mathfrak{p}},\sigma}^\times \times C_{\overline{\phi}_{\mathfrak{p}},\sigma}^\times$ via the right vertical arrow in diagram (9). If we identify $\prod_{\mathfrak{p} \in P} C_{\phi_{\mathfrak{p}},\sigma} \times C_{\overline{\phi}_{\mathfrak{p}},\sigma}$ with $\prod_{\tau : F \hookrightarrow C} C_{\tau,\sigma}$, then $(z_\sigma, 1)_{\mathfrak{p}}$ is identified with the element $(z_{\tau,\sigma})_{\tau,\sigma}$, where $z_{\tau,\sigma} := z_\sigma$ if $\tau \in \Phi$ and $z_{\tau,\sigma} := 1$ otherwise.

Now, for every $\sigma_0 : E \hookrightarrow C$ we have the following commuative diagram of rings:

$$\begin{tikzcd}
E \otimes F \otimes C \arrow[r, "{[\sigma_0](\cdot)}"] \arrow[d, "\wr"] & F \otimes C \arrow[dd, "\wr"] \\
\prod_{\sigma : E \hookrightarrow C} F \otimes C_\sigma \arrow[d, "\wr"] & \\
\prod_{\sigma : E \hookrightarrow C} \prod_{\mathfrak{p} \in P} F_{\mathfrak{p}} \otimes_R C_\sigma \arrow[r, "{(f_{\mathfrak{p},\sigma} \otimes z_{\mathfrak{p},\sigma})_{\mathfrak{p},\sigma} \mapsto \left(f_{\mathfrak{p},\sigma_0} \otimes z_{\mathfrak{p},\sigma_0}\right)_{\mathfrak{p}}}"] \arrow[d, "\wr"] & \prod_{\mathfrak{p} \in P} F_{\mathfrak{p}} \otimes_R C \arrow[d, "\wr"] \\
\prod_{\sigma : E \hookrightarrow C} \prod_{\phi_{\mathfrak{p}} \in \Phi} C_{\phi_{\mathfrak{p}},\sigma} \times C_{\overline{\phi}_{\mathfrak{p}},\sigma} \arrow[d, "\wr"] & \prod_{\mathfrak{p} \in P} C_{\phi_{\mathfrak{p}}} \times C_{\overline{\phi}_{\mathfrak{p}}} \arrow[d, "\wr"] \\
\prod_{\sigma : E \hookrightarrow C} \prod_{\tau : F \hookrightarrow C} C_{\tau,\sigma} \arrow[r, "{(z_{\tau,\sigma})_{\tau,\sigma} \mapsto \left(z_{\tau_0,\sigma_0}\right)_{\tilde{\sigma}_0 \tau_0 = \tau}}"] & \prod_{\tau : F \hookrightarrow C} C_\tau
\end{tikzcd} \tag{11}$$

Our goal is to compute $\prod_{\sigma \in A} [\sigma] \left(\mathrm{Res}_{E/\mathbf{Q}}\, \mu_{\Phi,E}\right)(\xi)$, where $\xi \in \left(\mathrm{Res}_{E/\mathbf{Q}}\, \mathbf{G}_m\right)(C) = (E \otimes C)^\times$, but for $z_\sigma := \sigma'(\xi)$ by using diagrams (7) and (11) this is the same as first computing $\prod_{\sigma : E \hookrightarrow C} \mu_{\tilde{\sigma}\Phi}(z_\sigma)$, which is

$$(z_{\tau,\sigma})_{\tau,\sigma} \in \prod_{\sigma : E \hookrightarrow C} \prod_{\tau : F \hookrightarrow C} C_{\tau,\sigma}^\times,$$

then composing it with the bottom arrow in diagram (11) giving $(z_{\tau_0,\sigma_0})_{\tilde{\sigma}_0 \tau_0 = \tau} \in \prod_{\tau : F \hookrightarrow C} C_\tau^\times$, and finally taking the product over all $\sigma_0 : E \hookrightarrow C$, yielding

$$\left(\prod_{\substack{\tau_0, \sigma_0 \\ \tilde{\sigma}_0 \tau_0 = \tau}} z_{\tau_0,\sigma_0} \right)_\tau = \left(\prod_{\sigma \in \Sigma(\Phi,\tau)} z_\sigma \right)_\tau \in \prod_{\tau : F \hookrightarrow C} C_\tau^\times .$$

This is the same element we obtained when we calculated the composite of the other two maps in diagram (6).

$\square$

Remark 2.7. The proof of Proposition 2.5 gives a concrete expression for the reflex norm of an element $e \in E^\times$. Indeed, fix $\tau : F \hookrightarrow C$. Then the reflex norm $r_\Phi(e)$ is $\tau^{-1}\left(\prod_{\sigma \in \Sigma(\Phi,\tau)} \sigma(e)\right)$. Note that this is in fact independent of the choice of τ, as it should be. Moreover, we have $r_\Phi(e)\overline{r_\Phi(e)} = N_{E/\mathbf{Q}}(e)$.