YOSHIDA LIFTS AND THE BLOCH-KATO CONJECTURE FOR THE CONVOLUTION L-FUNCTION

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ABSTRACT. Let f_1 (resp. f_2) denote two (elliptic) newforms of prime level N, trivial character and weight 2 (resp. $k+2$, where $k \in \{8, 12\}$). We provide evidence for the Bloch-Kato conjecture for the motive $M = \rho_{f_1} \otimes \rho_{f_2}(-k/2-1)$ by proving that under some assumptions the p-valuation of the order of the Bloch-Kato Selmer group of M is bounded from below by the p-valuation of the relevant L-value (a special value of the convolution L-function of f_1 and f_2). We achieve this by constructing congruences between the Yoshida lift $Y(f_1 \otimes f_2)$ of f_1 and f_2 and Siegel modular forms whose p-adic Galois representations are irreducible. Our result is conditional upon a conjectural formula for the Petersson norm of $Y(f_1 \otimes f_2)$.

1. Introduction

The Bloch-Kato conjecture [6] is one of the central open problems in algebraic number theory. Loosely speaking it asserts that the order of the (Bloch-Kato) Selmer group associated with a motive M should be controlled by a special value of the corresponding L-function divided by some canonically defined period. Let $p > 12$ be a prime. This article provides evidence for this conjecture for the motive

$$M = \mathrm{Hom}(\rho_{f_2}, \rho_{f_1}(k/2)) \cong \rho_{f_1} \otimes \rho_{f_2}(-k/2-1),$$

where f_1 (respectively f_2) are classical (elliptic) cuspidal newforms of weight 2 (resp. $k+2$ for $k = 8$ or 12) and prime level N and we denote by ρ_f the p-adic Galois representation attached to a modular form f. More specifically, let E denote a sufficiently large finite extension of $\mathbf{Q}_p$ (so that in particular M is defined over E), $\mathcal{O} \subset E$ its ring of integers and ϖ a choice of a uniformizer. Let $H^1_f(\mathbf{Q}, M \otimes E/\mathcal{O})$ denote the Bloch-Kato Selmer group (for a precise definition see section 8.2, and especially Remark 8.10) and we write $L^{\mathrm{alg}}(2+k/2, f_1 \times f_2)$ for the algebraic part of the value at $2+k/2$ of the convolution L-function of f_1 and f_2. Then we prove under some assumptions that

$$\mathrm{val}_p(\#H^1_f(\mathbf{Q}, M \otimes E/\mathcal{O})) \geq \mathrm{val}_p(\#\mathcal{O}/L^{\mathrm{alg}}(2+k/2, f_1 \times f_2)). \tag{1.1}$$

Roughly speaking our result thus provides 'one-half' of the Bloch-Kato conjecture for this motive, i.e., one inequality (see Remark 8.10 for a discussion of how our result relates to the Bloch-Kato conjecture).

Our proof proceeds via constructing extensions of ρ_{f_2} by $\rho_{f_1}(k/2)$ over some Artinian rings whose existence on the other hand is deduced from the existence of congruences between some Siegel modular forms. Construction of these congruences comprises the heart of this paper. This general approach is now standard and has been applied by many authors ([29, 3, 4, 11, 25]). In our case we study congruences

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between the Yoshida lift $Y(f_1 \otimes f_2)$ of f_1 and f_2 and Siegel modular forms which have irreducible p-adic Galois representations. The "amount" of these congruences is measured by the order of the quotient of a certain Hecke algebra $\mathbf{T}^Y_{\mathcal{O}}$ by what we call a "Yoshida ideal" I_{f_1,f_2}. Under some assumptions we prove the following inequality

$$\mathrm{val}_p(\#\mathbf{T}^Y_{\mathcal{O}}/I_{f_1,f_2}) \geq \mathrm{val}_p(\#\mathcal{O}/L^{\mathrm{alg}}(2+k/2, f_1 \times f_2)), \tag{1.2}$$

thus giving a lower bound on the "amount" of congruences between $Y(f_1 \otimes f_2)$ (which necessarily has a reducible Galois representation) and forms with irreducible Galois representations. This result may in fact be of independent interest (it is used, for example, by T. Berger and the second author to prove modularity of some 4-dimensional p-adic Galois representations [5]).

Let us now briefly elaborate on our assumptions. Our approach of relating the L-value to congruences is similar to that of [11, 25] in that one looks for a Siegel modular form, say $\mathcal{E}$ which has $\mathcal{O}$-integral Fourier coefficients, is not an eigenform and is not orthogonal to $Y(f_1 \otimes f_2)$. Then one uses the equality

$$\mathcal{E} = \frac{\langle \mathcal{E}, Y(f_1 \otimes f_2)\rangle}{\langle Y(f_1 \otimes f_2), Y(f_1 \otimes f_2)\rangle} Y(f_1 \otimes f_2) + G', \tag{1.3}$$

where G' is a Siegel modular form orthogonal to $Y(f_1 \otimes f_2)$. One expresses the inner products by L-values. The denominator is related to $L(2 + k/2, f_1 \times f_2)$, but the precise formula is known only up to a constant, so one of our assumptions concerns the p-adic valuation of that unknown constant. In our case $\mathcal{E}$ is related to a pullback of a certain Eisenstein series on GSp_8 and its analyticity, cuspidality and $\mathcal{O}$-integrality properties are either known or can be deduced from existing results. In fact our $\mathcal{E}$ is dependent on a certain Hecke character which we use as a parameter and choose appropriately to make the inner product in the numerator a p-adic unit. Using integrality of Fourier coefficients of $Y(f_1 \otimes f_2)$ due to Jia [24] (here we need to impose some assumptions that are already present in the work of Jia) we can then deduce that whenever ϖ divides $L^{\mathrm{alg}}(2+k/2, f_1 \times f_2)$, the Yoshida lift $Y(f_1 \otimes f_2)$ is congruent to some Siegel modular form G which is orthogonal to $Y(f_1 \otimes f_2)$. A large portion of the article is then devoted to proving that G can be chosen to be an eigenform with irreducible Galois representation. We achieve this by constructing a certain Hecke operator T^S which has the property that it kills the eigenforms F_i in the expansion of $G = \sum F_i$ which have a reducible Galois representation. Similar Hecke operator was constructed by Brown [11] and the second author [25], but in the current case we are confronted with some technical difficulties resulting from the fact that the Hecke algebra descent $\Phi : \mathbf{T}^S \to \mathbf{T} \otimes \mathbf{T}$ from the Siegel modular Hecke algebra to the tensor product of the elliptic Hecke algebras acting on the spaces containing f_1 and f_2 induced by the Yoshida lifting is not a priori surjective. Working with completed Hecke algebras we use Galois representations and a modularity result due to Diamond, Flach and Guo [16] to circumvent this difficulty. This is where we need the restriction on the weight $k \in \{8, 12\}$ and the assumption that N is a prime. See Assumption 6.1 for a complete list of assumptions that we make.

Independently of us Böcherer, Dummigan and Schulze-Pillot had a similar idea to provide evidence for the Bloch-Kato conjecture via Yoshida lifts [7]. To the best of our knowledge however, their method and scope would differ substantially from ours. In particular they work with any even k, but assume at the outset that the forms f_1 and f_2 are not congruent to any other cusp forms. Our approach (while more restrictive) allows us to avoid this assumption and instead "kill" the possible congruences by applying the Hecke operator T^S discussed above. Also to construct congruences Böcherer et al. would use the approach of Katsurada rather than the method applied in [11, 25], and hence their L-value conditions guaranteeing the existence of congruences should differ from ours.

Let us now briefly outline the organization of the paper. In section 2 we collect the notation that is used throughout the paper. In section 3 we gather some basic facts concerning modular forms on quaternion algebras and the Jacquet-Langlands correspondence. We also define an integral structure on the space of these modular forms that is necessary for the construction of an *integral* Yoshida lift. The lifting procedure due to Yoshida [40] defines $Y(f_1 \otimes f_2)$ only up to a complex constant. Since we are interested in the arithmetic of this lift, we need to choose an appropriate integral structure and specify the lift itself up to a p-adic unit. The latter is carried out in section 5.1. In all this we closely follow [24]. As a consequence one obtains $\mathcal{O}$-integrality of the Fourier coefficients of the lift $Y(f_1 \otimes f_2)$ (see Theorem 5.4). This result is due to Jia [24]. In section 5.2 we study the Hecke algebra descent $\Phi : \mathbf{T}^S \to \mathbf{T} \otimes \mathbf{T}$, and in section 5.3 we relate the Petersson norm of $Y(f_1 \otimes f_2)$ to $L(2 + k/2, f_1 \times f_2)$. In section 6 we construct the asserted congruence between $Y(f_1 \otimes f_2)$ and a Siegel cusp form $G = \sum_i F_i$ which is a linear combination of eigenforms F_i with irreducible Galois representations (Theorem 6.5) and prove the bound (1.2) on $\mathbf{T}^Y_{\mathcal{O}}/I_{f_1,f_2}$ (Corollary 6.10). To do this we need the form $\mathcal{E}$ as in (1.3). This form is constructed in section 4, where we also compute the inner product $\langle \mathcal{E}, Y(f_1 \otimes f_2)\rangle$. We also need the Hecke operator T^S "killing" all the forms F_i, which a priori might have had a reducible Galois representation. The construction of this operator is carried out in section 7. Finally in section 8 we deduce (1.1) from (1.2).

2. Notation and definitions

2.1. **Number fields and Hecke characters.** Throughout this paper ℓ will always denote an odd prime. We write i for $\sqrt{-1}$.

Let L be a number field with ring of integers $\mathcal{O}_L$. For a place v of L, denote by L_v the completion of L at v and by $\mathcal{O}_{L,v}$ the valuation ring of L_v. For a prime p, let val_p denote the p-adic valuation on $\mathbf{Q}_p$. For notational convenience we also define $\mathrm{val}_p(\infty) := \infty$. If $\alpha \in \mathbf{Q}_p$, then $|\alpha|_{\mathbf{Q}_p} := p^{-\mathrm{val}_p(\alpha)}$ denotes the p-adic norm of α. For $p = \infty$, $|\cdot|_{\mathbf{Q}_\infty} = |\cdot|_{\mathbf{R}} = |\cdot|$ is the usual absolute value on $\mathbf{Q}_\infty = \mathbf{R}$.

In this paper we fix once and for all an algebraic closure $\overline{\mathbf{Q}}$ of the rationals and algebraic closures $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$, as well as compatible embeddings $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$ for all finite places p of $\mathbf{Q}$. We extend val_p to a function from $\overline{\mathbf{Q}}_p$ into $\mathbf{Q}$. We will write $\mathbf{C}_p$ for the completion (with respect to the extended val_p) of $\overline{\mathbf{Q}}_p$ and $\mathcal{O}_{\mathbf{C}_p}$ for its ring of integers. Let L be a number field. We write G_L for $\mathrm{Gal}(\overline{L}/L)$. If $\mathfrak{p}$ is a prime of L, we also write $D_\mathfrak{p} \subset G_L$ for the decomposition group of $\mathfrak{p}$ and $I_\mathfrak{p} \subset D_\mathfrak{p}$ for the inertia group of $\mathfrak{p}$. The chosen embeddings allow us to identify $D_\mathfrak{p}$ with $\mathrm{Gal}(\overline{L}_\mathfrak{p}/L_\mathfrak{p})$.

For a number field L let $\mathbf{A}_L$ denote the ring of adeles of L and put $\mathbf{A} := \mathbf{A}_\mathbf{Q}$. Write $\mathbf{A}_{L,\infty}$ and $\mathbf{A}_{L,f}$ for the infinite part and the finite part of $\mathbf{A}_L$ respectively. For $\alpha = (\alpha_p) \in \mathbf{A}$ set $|\alpha|_\mathbf{A} := \prod_p |\alpha|_{\mathbf{Q}_p}$. By a *Hecke character* of $\mathbf{A}_L^\times$ (or of L, for short) we mean a continuous homomorphism

$$\psi : L^\times \setminus \mathbf{A}_L^\times \to \mathbf{C}^\times$$

whose image is contained inside $\{z \in \mathbf{C} \mid |z| = 1\}$. The trivial Hecke character will be denoted by **1**. The character ψ factors into a product of local characters $\psi = \prod_v \psi_v$, where v runs over all places of L. If $\mathfrak{n}$ is the ideal of the ring of integers $\mathcal{O}_L$ of L such that

- $\psi_v(x_v) = 1$ if v is a finite place of L, $x_v \in \mathcal{O}_{L,v}^\times$ and $x - 1 \in \mathfrak{n}\mathcal{O}_{L,v}$
- no ideal $\mathfrak{m}$ strictly containing $\mathfrak{n}$ has the above property,

then $\mathfrak{n}$ will be called the *conductor of* ψ.

Finally if $z \in \mathbf{C}$ we will sometimes write $e(z)$ for $e^{2\pi i z}$.

2.2. The symplectic group.

Let

$$H_n := \mathrm{GSp}_{2n} := \{g \in \mathrm{GL}_{2n} \mid g^{\mathrm{t}} w_n g = \mu(g) w_n, \mu(g) \in \mathrm{GL}_1\}$$

be the similitude group scheme (over $\mathbf{Z}$) of the alternating form

$$(v, w) \mapsto v^{\mathrm{t}} w_n w$$

for v, w two vectors in $\mathbf{G}_a^{2n}$ and $w_n = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}$. We will write

$$H_n^1 := \mathrm{Sp}_{2n} = \ker \mu.$$

Also set $\mathbf{i}_n := i I_n$. When $n = 2$, we drop it from notation.

2.3. The group of quaternions.

Let L be a number field. Let D_0 be a quaternion algebra over L, i.e., a central simple division algebra of degree 4 over L (cf. e.g., [27], p.199). The algebra D_0 comes equipped with an involution $x \mapsto x^\iota$. We set $\mathrm{tr}\,(x) := x + x^\iota$ and $n(x) := x x^\iota$. We call these *trace* and *norm* respectively.

For any L-algebra A, we set

$$D(A) := D_0 \otimes_L A.$$

Also set

$$D^\times(A) := (D_0 \otimes_L A)^\times.$$

The functors D and $D^\times$ are algebraic groups over $\mathbf{Q}$. We say that D is *split* (resp. *ramified*) at v if $D(L_v) \cong M_2(L_v)$ (resp. $D(L_v)$ is a division algebra). The number of places where D is ramified is finite and even ([20], p. 229). Set $\mathrm{disc}(D)$ to be the product of the finite primes at which D is ramified. We say that D is *definite* if it is ramified at the infinite places. We will always assume that our division algebras are definite.

Remark 2.1. Note that the trace map can be extended to a morphism $\mathrm{tr} : D \to \mathbf{G}_a$ of groups schemes over $\mathbf{Q}$. Similarly we can extend the norm map to a morphism of group schemes $n : D^\times \to \mathbf{G}_m$.

From now on let $L = \mathbf{Q}$. Let $m(X) = X^2 + bX + c$ be an irreducible monic polynomial with coefficients in $\mathbf{Q}$. Denote by $\Delta_m = b^2 - 4c$ the discriminant of $m(X)$. Choose $\delta \in D$ so that $\mathbf{Q}(\sqrt{\Delta_m}) = \mathbf{Q}(\delta)$. Then one has an orthogonal decomposition

$$D = \mathbf{Q}(\delta) \oplus \mathbf{Q}(\delta)^\perp$$

with respect to the bilinear form $(x, y) = \mathrm{tr}\,(xy^{\iota})$. We also choose $j \in D$ so that $\mathbf{Q}(\delta)^{\perp} = j\mathbf{Q}(\delta)$. Such a set consisting of δ and j is sometimes called a *basis* of D. We fix such a basis once and for all in what follows.

Let $F = \mathbf{Q}(\sqrt{n(j)})$ and write K_j for $\mathbf{Q}(\delta)^{\perp} \otimes F$. Note that $\mathbf{Q}(\delta)$ is imaginary quadratic (cf. [24], p.18) while F is real quadratic, so K_j is a field. The choice of a basis determines an injective homomorphism (cf. [24], p.20):

$$\epsilon : D^{\times} \hookrightarrow \mathrm{Res}_{K_j/\mathbf{Q}}(\mathrm{GL}_{2/K_j}), \quad \alpha + j\beta \mapsto \begin{bmatrix} \alpha & -\sqrt{n(j)} \cdot \beta^{\iota} \\ \sqrt{n(j)} \cdot \beta & \alpha^{\iota} \end{bmatrix} \tag{2.1}$$

for $\alpha + j\beta \in D = \mathbf{Q}(\delta) \oplus \mathbf{Q}(\delta)^{\perp}$.

Let $\mathbf{H}$ be the Hamilton quaternion algebra, i.e., $\mathbf{H} = \mathbf{R} + \mathbf{R}I + \mathbf{R}J + \mathbf{R}K$ with relations $I^2 = J^2 = K^2 = -1$ and $IJ = -JI = K$, $IK = -KI = J$, $JK = -KJ = I$. Then $\mathbf{H} \cong D(\mathbf{R})$.

We fix once and for all a maximal order $R_{\max}$ of $D(\mathbf{Q})$. Set $R_{p,\max} := R_{\max} \otimes_{\mathbf{Z}} \mathbf{Z}_p$. Let R be an order contained in $R_{\max}$. Set $R_p = R \otimes_{\mathbf{Z}} \mathbf{Z}_p$. For the order R_p we define $R_p^{\vee}$ to be the dual $\mathbf{Z}_p$-lattice in $D(\mathbf{Q}_p)$, i.e.,

$$R_p^{\vee} := \{x \in D(\mathbf{Q}_p) \mid \mathrm{tr}\,(xy^{\iota}) \in \mathbf{Z}_p\}.$$

Let $p^{-n_p}\mathbf{Z}_p$, $n_p \geq 0$ be the fractional ideal of $\mathbf{Z}_p$ generated by all the reduced norms of the elements of $R_p^{\vee}$. The integer n_p is called *the level of* R_p ([40], p. 203).

Proposition 2.2. *The integer $n_p = 0$ if D is split at p and $R_p = R_{p,\max}$. The integer $n_p = 1$ if D ramifies at p and $R_p = R_{p,\max}$.*

Proof. This is easy. ([40], p. 203). □

Let R be an order in $D(\mathbf{Q})$. We say R is an *Eichler order* if $R_p := R \otimes_{\mathbf{Z}} \mathbf{Z}_p = R_{p,\max}$ for all p at which D is split. See also [8], p.60 for a more general definition. By Proposition 2.2 it makes sense to make the following definition.

Definition 2.3. Let R be an Eichler order. The *level* of R is $\prod_p p^{n_p}$.

Let R be an Eichler order of level N. Set

$$K := \mathbf{H}^{\times} \times \prod_p R_p^{\times}.$$

Then we can write

$$D^{\times}(\mathbf{A}) = \bigsqcup_{i=1}^{h(D)} D^{\times}(\mathbf{Q}) y_i K$$

for some elements $y_i \in D^{\times}(\mathbf{A})$ that represent different left-ideal classes in R. We can always choose y_i to have norm 1.

3. Modular forms on the quaternion algebras

3.1. **Definitions.** Let D be a definite quaternion algebra over $\mathbf{Q}$ of discriminant N. Fix ν an even positive integer. Define the symmetric tensor representation of degree ν to be the representation

$$Sym^{\nu} : \mathrm{GL}_2(\mathbf{C}) \to \mathrm{GL}_{\nu+1}(\mathbf{C})$$

which sends $g \in \mathrm{GL}_2(\mathbf{C})$ to the automorphism of the $(\nu+1)$-dimensional vector space $Sym^{\nu}\mathbf{C}$ given by sending the symmetric combination of the $(\nu+1)$-tensors $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_{\nu+1}}$ to the corresponding symmetric combination of $ge_{i_1} \otimes ge_{i_2} \otimes \cdots \otimes$

$ge_{i_{\nu+1}}$, where $i_j \in \{1,2\}$ and e_1, e_2 is the canonical basis for $\mathbf{C}^2$. Let ι_0 be a fixed embedding of $\mathbf{H}^\times$ into $\mathrm{GL}_2(\mathbf{C})$. For $g \in \mathbf{H}^\times$ write $\sigma_\nu(g) = Sym^\nu(\iota_0(g))n(g)^{-\nu/2}$, where n is the reduced norm of $\mathbf{H}$. We will write V for the representation space of σ_ν.

Let R be an order contained in $R_{\max}$. Let Z_D be the center of $D^\times$ and let ω be a character of $Z_D(\mathbf{A})/Z_D(\mathbf{Q})$.

Definition 3.1. An *automorphic form* on $D^\times(\mathbf{A})$ *of type* (R, ν, ω) is a function $\phi : D^\times(\mathbf{A}) \to V$ such that

- ϕ is left-invariant under $D^\times(\mathbf{Q})$;
- $\phi(gk) = \sigma_\nu(k)\phi(g)$ for $k \in \mathbf{H}^\times$ and $g \in D^\times(\mathbf{A})$;
- ϕ is right-$\prod_{p\nmid\infty} R_p$-invariant;
- $\phi(zg) = \omega(z)\phi(g)$ for $z \in Z_D(\mathbf{A})$ and $g \in D^\times(\mathbf{A})$.

The $\mathbf{C}$-space of such forms will be denoted by $\mathcal{A}_\nu^D(R, \omega)$ or $\mathcal{A}_\nu^D(R)$ if $\omega = 1$. We will write $S_\nu^D(R, \omega)$ and $S_\nu^D(R)$ for the corresponding subspaces of cuspforms.

Remark 3.2. Note that the usual growth condition one imposes on an automorphic form is automatically satisfied in this case (see [20], p.233). Indeed, the quotient $D^\times(\mathbf{Q})Z_D(\mathbf{R}) \setminus D^\times(\mathbf{A})$ is compact ([20], p. 227), hence an automorphic form on

$$D^\times(\mathbf{A}) = \bigsqcup_{i=1}^{h(D)} D^\times(\mathbf{Q}) y_i K_f \mathbf{H}^\times \tag{3.1}$$

is determined by its values on the set of y_i's and on

$$\mathbf{H}^\times = Z_D(\mathbf{R}) K_\infty, \tag{3.2}$$

where K_∞ is the maximal compact subgroup of $\mathbf{H}^\times$. So the continuity of the automorphic form implies the growth condition. Note the contrast to the GL_2-situation, where the infinite component modulo its center is not compact.

3.2. **Jacquet-Langlands correspondence.**

Theorem 3.3 ([20], Theorem 10.2)**.** *Let F be a global field. Let S denote the finite set of places v in F such that $D(F_v)$ is a division algebra. Then there is a one-to-one correspondence $\pi' \mapsto \pi = \bigotimes_v \pi_v$ between the collection of irreducible unitary representations π' of $D^\times(\mathbf{A}_F)$ and the collection of cuspidal automorphic representations π of $\mathrm{GL}_2(\mathbf{A}_F)$ with π_v square-integrable for each $v \in S$.*

Remark 3.4. The correspondence in Theorem 3.3 preserves the Hecke eigenvalues at the primes where the automorphic representations are unramified in the following sense. Let l be such a prime. Let π be an automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ and π' the representation of $D^\times(\mathbf{A})$ attached to π via the Jacquet-Langlands correspondence. Assume that the central character of π' restricted to the infinite component of the center is trivial. Then π'_∞ descends to a representation of the projective group $PD^\times(\mathbf{R})$ and as such it is equivalent to (σ_ν, V) for some even integer $\nu \geq 0$ (cf. [24], section 2.2.6). Call this integer the *weight* of π'. If φ is an automorphic form of type (R, ν, ω) with $\omega|_{Z_D(\mathbf{R})} = 1$, which is an eigenform for the local Hecke algebras, then the corresponding automorphic representation π' has weight ν. Let T_l^D be the standard Hecke operator at l (denoted by $T(l)$ in [40]). Let λ_l^D be the eigenvalue of that operator corresponding to π'. Let T_l be the standard (elliptic) Hecke operator at l and write λ_l for its eigenvalue corresponding to the representation π. Then one has (cf. [40], Proposition 7.1)

$$\lambda_l^D = l^{\nu/2}\lambda_l. \tag{3.3}$$

3.3. **Arithmetic forms on $D^\times$.** The Jacquet-Langlands correspondence as stated in Theorem 3.3 is a correspondence between automorphic representations on GL_2 and on $D^\times$. Our goal in the end is to study a certain theta lift (called Yoshida lift) from $D^\times$ to H, which composed with the Jacquet-Langlands correspondence will give us a way to associate a Siegel modular form Y to a pair of elliptic modular newforms f_1 and f_2. The forms f_1 and f_2 have nice arithmetic properties and we would also like Y to have similar properties. This is why we will make a specific choice of a vector inside the automorphic representation of $D^\times(\mathbf{A})$ corresponding to the automorphic representation associated to an elliptic newform via the Jacquet-Langlands correspondence. Arithmeticity properties of the Yoshida lift were studied in detail by Jia [24]. The contents of this section and large parts of section 5.1 are essentially taken from [24], and we refer the reader to [loc. cit.] for details as well as proper justification for the choices and definitions we will make in what follows.

We begin by putting a certain integral structure on the space V as above. In this we follow [13] and [24]. Let $\mathcal{V}_\nu$ be the $\mathbf{Z}$-submodule of the polynomial ring $\mathbf{Z}[X]$ consisting of polynomials of degree not exceeding ν. The group $\mathrm{GL}_2(\mathbf{Z})$ acts on $\mathcal{V}_\nu$ in two ways:

$$\sigma_\nu(g)\cdot f(X) = \det g^{-\nu/2}(bX+d)^\nu f\left(\frac{aX+c}{bX+d}\right),$$

$$\sigma_\nu^\vee(g)\cdot f(X) = \det g^{-\nu/2}(-cX+a)^\nu f\left(\frac{dX-b}{-cX+a}\right),$$

where $g = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$. We denote these representations by $(\mathcal{V}_\nu, \sigma_\nu)$ and $(\mathcal{V}_\nu, \sigma_\nu^\vee)$, respectively. The monomials

$$\mathbf{t}_i := X^{\nu/2+i} \quad i = -\nu/2, -\nu/2+1, \dots, \nu/2$$

give a basis of $\mathcal{V}_\nu$. We define a pairing on $\mathcal{V}_\nu \times \mathcal{V}_\nu$ given by

$$\left\langle \sum_{i=-\nu/2}^{\nu/2} a_i\mathbf{t}_i, \sum_{i=-\nu/2}^{\nu/2} b_i\mathbf{t}_i \right\rangle_\nu := \sum_{i=-\nu/2}^{\nu/2} (-1)^i \frac{(\nu/2+i)!(\nu/2-i)!}{((\nu/2)!)^2} a_i b_i.$$

This pairing has the property that

$$\langle \sigma_\nu(g)v, \sigma_\nu^\vee(g)w\rangle_\nu = \langle v, w\rangle_\nu \quad \text{for all } g \in \mathrm{GL}_2(\mathbf{Z}),\ v,w\in\mathcal{V}_\nu.$$

This pairing establishes a duality between $\sigma_\nu \otimes \mathbf{Z}\left[\frac{1}{(\nu/2)!}\right]$ and $\sigma_\nu^\vee \otimes \mathbf{Z}\left[\frac{1}{(\nu/2)!}\right]$. It is not hard to see that after extending the scalars to $\mathbf{C}$ the first representation recovers the representation (V, σ_ν) considered in section 3.1.

Let $\phi = \sum_{j=-\nu/2}^{\nu/2} \phi_j\mathbf{t}_j$ and $\psi = \sum_{j=-\nu/2}^{\nu/2} \psi_j\mathbf{t}_j$ be automorphic forms on $D^\times(\mathbf{A})$ and assume that the infinite part of the central character is trivial. Since the quotient $D^\times(\mathbf{Q})Z_D(\mathbf{R}) \setminus D^\times(\mathbf{A})$ is compact one can define the (Petersson) inner product by

$$\langle \phi, \psi\rangle_D := \sum_j \int_{D^\times(\mathbf{Q})Z_D(\mathbf{R})\setminus D^\times(\mathbf{A})} \phi_j(x)\overline{\psi_j(x)}d^\times x,$$

where $d^\times x$ is the multiplicative measure defined by

$$d^\times x_p = \begin{cases} (1-p^{-1})^{-1} \cdot \frac{dx_p}{|n(x_p)|^2} & \text{if } p \nmid \infty \\ \frac{dx_p}{|n(x_p)|^2} & \text{if } p = \infty, \end{cases}$$

and dx_p is the Haar measure for the group $D(\mathbf{Q}_p)$ giving $R_{p,\max}$ volume $1/p$ (if $p \mid \operatorname{disc}(D)$), 1 (if $p \nmid \operatorname{disc}(D) \cdot \infty$) and equals $4 \cdot dx_1 \cdot dx_2 \cdot dx_3 \cdot dx_4$ if $p = \infty$.

If one takes y_i's as in Remark 3.2, then it follows from (3.1) and (3.2) that (3.4)

$$\langle \phi, \psi \rangle_D = \sum_{j=0}^{\nu} \sum_{i=1}^{h(D)} \phi_j(y_i)\overline{\psi_j(y_i)} \operatorname{vol}(K_\infty K_f) = \frac{3}{\operatorname{disc}(D)} \prod_{l \mid \operatorname{disc}(D)} \frac{1}{1-\frac{1}{l}} \cdot \sum_{i,j} \phi_j(y_i)\overline{\psi_j(y_i)}$$

(for the volume calculation see [24], p.37).

Composing the representations σ_ν and $\sigma_\nu^\vee$ with the homomorphism ϵ from (2.1), we obtain two representations of $D^\times(\mathbf{Q})$ which we will denote in the same way. Let ℓ be a prime. Write $\mathcal{V}_{\nu,\ell}$ for $\mathcal{V}_\nu \otimes_{\mathbf{Z}} \mathbf{C}_\ell = \mathbf{C}_\ell[T]_\nu$ and denote the corresponding representations by $\sigma_{\nu,\ell}$ and $\sigma_{\nu,\ell}^\vee$. Denote by $\mathcal{M}_{\nu,\ell}$ the $\mathcal{O}_{\mathbf{C}_\ell}$-lattice in $\mathcal{V}_{\nu,\ell}$ generated by

$$\sigma_{\nu,\ell}^\vee(\mathrm{GL}_2(\mathbf{Z}_\ell)) \cdot \mathbf{t}_0.$$

From now on assume $R = R_{\max}$. Let ϕ be an automorphic form on $D^\times(\mathbf{A})$ of type (R, ν, ω). We say that ϕ is *algebraic* if $\phi(D^\times(\mathbf{A}_{\mathrm{f}})) \subset \mathcal{V}_\nu \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}$.

The following fact is immediate from the definition of an algebraic automorphic form and equation (3.4).

Lemma 3.5. *Let ϕ and ψ be two algebraic automorphic forms on $D^\times(\mathbf{A})$. Then $\langle \phi, \psi \rangle_D \in \overline{\mathbf{Q}}$.*

Let ϕ be an algebraic automorphic form as above. Let δ and j be as in section 2.3. Define (cf. [24], p.33)

$$C = \begin{cases} \begin{bmatrix} n(j)^{-1/2} & 0 \\ 0 & 1 \end{bmatrix} & \text{if } \ell \text{ splits in } \mathbf{Q}(j) \\ \begin{bmatrix} \Delta_\delta^{1/2}/2 & 1 \\ (-\Delta_\delta)^{1/2}/2 & -i \end{bmatrix} & \text{if } \ell \text{ does not split in } \mathbf{Q}(j). \end{cases}$$

Definition 3.6 ([24], p.47)**.** An algebraic automorphic form ϕ is called *ℓ-integral* if

$$\phi(x) \in \sigma_{\nu,\ell}^\vee(x_\ell) \cdot \sigma_{\nu,\ell}^\vee(C) \cdot \mathcal{M}_{\nu,\ell} \quad \text{for all } x \in D^\times(\mathbf{A}_{\mathrm{f}})$$

Definition 3.7. An ℓ-integral automorphic form ϕ is called *non-Eisenstein* if $\langle \phi, \mathbf{t}_k \rangle_\nu$ is a non-constant function modulo ℓ.

Remark 3.8. Let π be an automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ and π' the automorphic representation of $D^\times(\mathbf{A})$ corresponding to π via the Jacquet-Langlands correspondence. Assume that π' is unramified away from the primes dividing the discriminant of D. Write $V(\pi')$ for the one-dimensional vector subspace of $(\pi' \otimes \mathcal{V})$ consisting of vectors fixed by $\prod_{p \nmid \infty} R_p^\times \times D^\times(\mathbf{R})$ (cf. [24], p.42). There exists a non-zero vector $\phi(\pi) \in V(\pi')$ which is algebraic in the above sense ([24], p.46). Moreover, it follows from the proof of [24], Proposition 2.2 that when the ℓ-adic Galois representation associated to π is residually irreducible, then the vector $\phi(\pi)$ can be chosen to be ℓ-integral and non-Eisenstein.

Definition 3.9. Let D be as above and write N for the discriminant of D. Let ℓ be a rational prime with $\ell \nmid N$. Let $f \in S_k(N)$ be an elliptic newform and write π for the automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ associated with f. Assume that the ℓ-adic Galois representation ρ_f is residually irreducible. Let π' be the automorphic representation of $D^\times(\mathbf{A})$ corresponding to π via the Jacquet-Langlands correspondence. Then we will denote by $JL(f)$ an ℓ-integral (non-Eisenstein) vector $\phi(\pi)$ defined as above.

Lemma 3.10. *If ϕ and ϕ' are two ℓ-integral non-Eisenstein automorphic forms on $D^\times(\mathbf{A})$, then*

$$\mathrm{val}_\ell(\langle \phi, \phi \rangle_D) = \mathrm{val}_\ell(\langle \phi', \phi' \rangle_D).$$

Proof. Since $\phi, \phi' \in V(\pi')$ and $V(\pi')$ is one-dimensional, there exists $\alpha \in \overline{\mathbf{Q}}$ such that $\phi' = \alpha\phi$. Note that $\mathrm{val}_\ell(\alpha) = 0$ since otherwise ϕ' couldn't be integral and non-Eisenstein at the same time when ϕ is. Hence $\langle \phi', \phi' \rangle_D = |\alpha|^2 \langle \phi, \phi \rangle_D$. □

4. Siegel modular forms

4.1. **Definitions.** For any commutative ring R we let $M_n(R)$ denote the set of $n \times n$ matrices with entries in R. For $g \in M_{2n}(R)$ let $A_g, B_g, C_g, D_g \in M_n(R)$ be defined by

$$g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}$$

where we will drop the subscript g when it is clear from the context.

Let $H_n := \mathrm{GSp}_{2n}$. Set $\mathbf{H}_n = \{z \in M_n(\mathbf{C}) | z^t = z, \mathrm{Im}(z) > 0\}$ to be the Siegel upper half space and we let $H_n^+(\mathbf{R}) = \{\gamma \in H_n(\mathbf{R}) | \mu_n(\gamma) > 0\}$. Then $H_n^+(\mathbf{R})$ acts on $\mathbf{H}_n$ via

$$\gamma(z) = (a_\gamma z + b_\gamma)(c_\gamma z + d_\gamma)^{-1}, \qquad \gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in H_n^+(\mathbf{R}).$$

Let

$$\Gamma_{0,n}^S(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbf{Z}) | c \equiv 0 \mod N\},$$

be a congruence subgroup of $\mathrm{Sp}_{2n}(\mathbf{Z})$. For k a positive integer and $\gamma \in H_n^+(\mathbf{R})$ we define the slash operator by:

$$(F|_k\gamma)(z) := \mu(\gamma)^{nk/2} j(\gamma, z)^{-\kappa} F(\gamma z) \qquad \text{for} \qquad z \in \mathbf{H}_n$$

where $j(\gamma, z) = \det(c_\gamma z + d_\gamma)$. We say that $F : \mathbf{H}_n \to \mathbf{C}$ is a holomorphic Siegel modular form (of genus n) of weight k with level N and character χ if F is holomorphic on $\mathbf{H}_n$ and

$$F|_k\gamma = \chi(\det(d_\gamma))F \qquad \text{for } \gamma \in \Gamma_{0,n}^S(N).$$

We denote the space of holomorphic Siegel modular forms (of genus n) of weight k, and level $\Gamma_{0,n}^S(N)$ and character χ by $M_{n,k}^S(N, \chi)$. If $\chi = \mathbf{1}$ then we will usually write $M_{n,k}^S(N)$ for $M_{n,k}^S(N, \chi)$. If $F \in M_{n,k}^S(N, \chi)$ then F has a Fourier expansion given by

$$F(z) = \sum_{T \in S_n^{\geq 0}(\mathbf{Z})} a(T, F) e^{2\pi i \mathrm{tr}(Tz)}$$

where $S_n^{\geq 0}(\mathbf{Z})$ is the semigroup of symmetric, positive semi-definite, $n \times n$ semi-integral matrices. We call F a Siegel cusp form if for all $\alpha \in H_n^+(\mathbf{R})$ one has

$a(T, F|_k\alpha) = 0$ for every T such that $\det T = 0$. We denote by $S^S_{n,k}(N,\chi)$ the vector space of Siegel cusp forms of weight k and level $\Gamma^S_0(N)$ and character χ. If $\chi = \mathbf{1}$ then we omit any mention of the character. For A a subalgebra of $\mathbf{C}$, we define $M^S_{n,k}(N,\chi,A)$(resp. $S^S_{n,k}(N,\chi,A)$) as the space of Siegel modular (resp. cusp) forms with Fourier coefficients in A. Let F^c be the Siegel modular form given by

$$F^c(z) = \sum_{T \in S_n^{\geq 0}(\mathbf{Z})} \overline{a(T,F)} e^{2\pi i \mathrm{tr}(Tz)}$$

For F and G two Siegel modular forms of weight k, level $\Gamma^S_{n,0}(N)$ and either of them being a cusp form we define the Petersson inner product

$$\langle F, G\rangle = \int_{\Gamma^S_{n,0}(N)\backslash \mathbf{H}_n} F(z)\overline{G(z)}(\det y)^k d\mu z$$

where

$$d\mu z = (\det y)^{-(n+1)} \Pi_{\alpha\leq\beta} dx_{\alpha,\beta} \Pi_{\alpha\leq\beta} dy_{\alpha,\beta},$$

$z = x + iy$ and $z = (x_{\alpha,\beta}) + i(y_{\alpha,\beta})$ and $dx_{\alpha,\beta}$ and $dy_{\alpha,\beta}$ are the usual Euclidean measures on $\mathbf{R}$. In all of the above if $n = 2$ we usually drop it from notation.

4.2. **Eisenstein series - setup.** We now define some subgroups of $H_n(\mathbf{A})$. Let N be an integer. For a finite place $\ell \mid N$, define

$$K_{0,\ell} = \{g \in H_n(\mathbf{Q}_\ell) | A_g, B_g, D_g \in M_n(\mathbf{Z}_\ell), C_g \in M_n(N\mathbf{Z}_\ell)\}$$

and for $\ell \nmid N$ define

$$K_{0,\ell} = \{g \in H_n(\mathbf{Q}_\ell) | A_g, B_g, C_g, D_g \in M_n(\mathbf{Z}_\ell)\}$$

and put

$$K_{0,\mathbf{f}}(N) = \Pi_{\ell\nmid\infty} K_{0,\ell}(N).$$

Let

$$K_\infty = \{g \in \mathrm{Sp}_{2n}(\mathbf{R}) | g\mathbf{i}_{2n} = \mathbf{i}_{2n}\}$$

and

$$K_0(N) = K_\infty K_{0,\mathbf{f}}(N).$$

The Siegel parabolic $Q_n \subset H_n$ is defined by

$$Q_n = \{g \in H_n | C_g = 0\}.$$

The parabolic Q_n has a Levi decomposition given by $Q_n = N_{Q_n} M_{Q_n}$ where N_{Q_n} is the unipotent radical and M_{Q_n} is the Levi subgroup. More precisely N_{Q_n} and M_{Q_n} are given by

$$M_{Q_n} = \left\{ \begin{pmatrix} g & \\ & \alpha(g^t)^{-1} \end{pmatrix} | g \in \mathrm{GL}_n, \alpha \in \mathrm{GL}_1 \right\}$$

and

$$N_{Q_n} = \left\{ \begin{pmatrix} I_n & s \\ & I_n \end{pmatrix} | s = s^t, s \in M_n \right\}$$

4.3. **Eisenstein series on H_n.** In this section, we will define a Siegel Eisenstein series on H_n attached to a Hecke character of $\mathbf{Q}$. Assume $N > 1$. Let k be a positive integer such that $k > \max\{3, n+1\}$ and let $\tau : \mathbf{A}^\times \to \mathbf{C}^\times$ be a Hecke character such that for any finite place l,

$$\tau_l(x) = 1$$

for $x \in \mathbf{Z}_l^\times$ with $N \mid x - 1$ and has infinity type

$$\tau_\infty(x) = (\mathrm{sgn}(x))^k.$$

Define $\epsilon(g, s; k, N, \tau)$ on $H_n(\mathbf{A}) \times \mathbf{C}$ by

$$\epsilon(g, s; k, N, \tau) = 0 \text{ if } g \notin Q_n(\mathbf{A})K_0(N)$$

and

$$\epsilon(g, s; k, N, \tau) = \epsilon_\infty(g, s; k, \tau) \prod_{l \nmid N} \epsilon_l(g, s; k, \tau) \prod_{l \mid N} \epsilon_l(g, s; k, N, \tau)$$

where for $g = q\theta \in Q_n(\mathbf{A})K_0(N), q \in Q_n(\mathbf{A})$ and $\theta \in K_0(N)$ we let

$$\epsilon_\infty(g, s; k, \tau) = \tau_\infty(\det A_{g,\infty})|\det(A_{g,\infty})|^{2s} j(\theta, \mathbf{i}_n)^{-k}$$

$$\epsilon_l(g, s; k, \tau) = \tau_l(\det A_{g,l})|\det(A_{g,l})|_{\mathbf{Q}_l}{}^{2s}$$

$$\epsilon_l(g, s; k, N, \tau) = \tau_l(\det A_{g,l})\tau_l(\det(d_\theta))^{-1}|\det(A_{g,l})|_{\mathbf{Q}_l}{}^{2s}.$$

For the section $\epsilon(g, s; k, N, \tau)$, we define the Siegel Eisenstein series associated to it by

$$E(g, s) := \sum_{\gamma \in Q_n(\mathbf{Q}) \backslash H_n(\mathbf{Q})} \epsilon(\gamma g, s; k, N, \tau).$$

The Siegel Eisenstein series converges absolutely and uniformly for (g, s) on compact subsets of $H_n(\mathbf{A}) \times \{s \in \mathbf{C} \mid \mathrm{Re}(s) > (n+1)/2\}$. It defines an automorphic form on H_n and a holomorphic function on $\{s \in \mathbf{C} \mid \mathrm{Re}(s) > (n+1)/2\}$ which has a meromorphic continuation in s to all $\mathbf{C}$ with at most finitely many poles. This Eisenstein series has a functional equation relating $E(g, (n+1)/2 - s)$ and $E(g, s)$ [26].

We can associate a classical Eisenstein series $E(z, s)$ to the Siegel Eisenstein series by

$$E(z, s) = (j(g_\infty, \mathbf{i}_n))^k E(g, s)$$

where $z = g_\infty(\mathbf{i}_n)$ and $g = g_{\mathbf{Q}} g_\infty \theta_f \in H_n(\mathbf{Q})H_n(\mathbf{R})K_{0,\mathbf{f}}(N)$. By [34] the Eisenstein series $E(z, (n+1)/2 - k/2)$ is a holomorphic Siegel modular form of weight k and level N. Following Shimura [33] let

$$E^*(g, s) = E(g\iota_{\mathbf{f}}^{-1}, s)$$

where $\iota_{\mathbf{f}} \in H_n(\mathbf{A})$ is the matrix whose all finite components are $\begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}$ and the infinite component equals I_{2n}. Let $E^*(z, s)$ be the corresponding classical Eisenstein series. Then $E^*(z, s)$ has a Fourier expansion given by

$$E^*(z, s) = \sum_{h \in S_n(\mathbf{Q})} a(h, y, s) e(\mathrm{tr}\,(hx))$$

for $z = x + \mathbf{i}y \in \mathbf{H}_n$.

Now we define a normalized Eisenstein series

$$D_{E^*}(z,s) := \pi^{-n(n+2)/4} L^N(2s,\tau) \prod_{j=1}^{[n/2]} L^N(4s-2j,\tau^2) E^*(z,s).$$

For a definition of the L-factors see section 5.3.

Theorem 4.1. *Let ℓ be an odd prime such that $\ell > n$ and $(\ell, N)=1$. Then*

$$D_{E^*}(z,(n+1)/2-k/2) \in M^S_{n,k}(N, \mathbf{Z}_p[\tau, i^{nk}]).$$

Proof. See e.g., [11] or [1]. □

Consider an embedding

$$\mathbf{H}_2 \times \mathbf{H}_2 \to \mathbf{H}_4$$

given by

$$z \times w \mapsto \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} = \mathrm{diag}[z,w].$$

Then by a classical interpretation of the pullback formula of Garrett and Shimura we have the following theorem:

Theorem 4.2 (Brown [11], Theorem 4.5)**.** *Let $n=4$. Let $F \in S^S_k(N)$ be a (genus 2) Siegel cusp eigenform then*

$$\left\langle D_{E^*}(\mathrm{diag}[z,w],(5-k)/2)) \mid_{1\times \iota_{\mathbf{f}}^{-1}}, F^c(w) \right\rangle = \pi^{-3} \mathcal{A}_{k,N} L^N_{\mathrm{st}}(5-k, F^c, \tau) F(z)$$

where $\mathcal{A}_{k,N} = \frac{(-1)^k 2^{2k-3} v_N}{3[Sp_4(\mathbf{Z}):\Gamma_0^S(N)]}$, $v_N = \pm 1$ and $L^N_{\mathrm{st}}(5-k,F,\tau)$ is the standard L-function of F (cf. Definition 5.15).

5. Yoshida lifts

5.1. Definition and integrality of the Fourier coefficients of Yoshida lift.

Yoshida lifting is a procedure which associates a Siegel modular form to a pair of elliptic modular forms. In this section we mainly follow [40], section 2 and [24].

Let $R = R_{\max}$ be the maximal Eichler order in $D^\times(\mathbf{Q})$. Let ν_1, ν_2 be two non-negative integers. Later we will specialize them by taking $\nu_1 = 0$ and $\nu_2 \in \{2,4,6,8,12\}$. These restrictions are in fact not necessary for defining the Yoshida lift, however we will only use the lift for the weights in these ranges and for these weights the exposition becomes much easier. The space D can be regarded as a 4-dimensional quadratic space over $\mathbf{Q}$ with respect to the reduced norm n. Set

$$(x,x)_D = \mathrm{tr}\,(xx^\iota) = 2n(x).$$

Let

$$\mathrm{GO}(D) := \{h \in \mathrm{GL}(D) \mid n(h\cdot x) = \lambda(h)n(x) \text{ for all } x \in D\}$$

denote the corresponding group of orthogonal similitudes, where λ is the similitude character. Let $D^\times \times D^\times$ act on D by

$$(a,b)\cdot x = axb^{-1}.$$

This gives a homomorphism $D^\times \times D^\times \to \mathrm{GO}(D)$ with kernel the center $Z(D^\times) = \mathbf{G}_m$ diagonally embedded inside $D^\times \times D^\times$. The image is the connected component $\mathrm{GSO}(D)$ of $\mathrm{GO}(D)$ defined by the condition $\det h = \lambda(h)^2$. Set

$$\mathfrak{D} := (D^\times \times D^\times)/Z(D^\times) \cong \mathrm{GSO}(D).$$

Let φ_i, $i = 1, 2$ be automorphic forms on $D^\times(\mathbf{A})$ of type $(R, \nu_i, 1)$. In section 3.3 we defined for every even integer ν, a free $\mathbf{Z}$-module $\mathcal{V}_\nu$ of rank $\nu + 1$. In this section we will write $\mathcal{V}_1$ instead of $\mathcal{V}_{\nu_1}$ and $\mathcal{V}_2$ instead of $\mathcal{V}_{\nu_2}$ and similarly for the representations σ also defined in section 3.3. We hope this will cause no confusion. Put

$$\mathcal{V} = \mathcal{V}_1 \otimes \mathcal{V}_2 \quad \text{and } V = \mathcal{V} \otimes_{\mathbf{Z}} \mathbf{C}.$$

(Note that if $\nu_1 = 0$, then $\mathcal{V} = \mathcal{V}_2$). In section 3.3 we also defined a pairing $\langle \cdot, \cdot \rangle_{\nu_i}$ on each of the $\mathcal{V}_{\nu_i}$, which induces a pairing on $\mathcal{V} \times \mathcal{V}$ by

$$\langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle_{1,2} := \langle v_1, w_1 \rangle_{\nu_1} \langle w_1, w_2 \rangle_{\nu_2}.$$

Consider $\varphi := \varphi_1 \otimes \varphi_2 \in \mathcal{A}^D_{\nu_1}(R) \otimes \mathcal{A}^D_{\nu_2}(R)$. The element φ descends to a function $\varphi : \mathfrak{D}(\mathbf{A}) \to V$ defined by $\varphi(x_1, x_2) = \varphi_1(x_1) \otimes \varphi_2(x_2)$. This function is an automorphic form on $\mathfrak{D}$ (cf. [24], p.55).

We will now associate a Siegel modular form to φ. We follow the exposition in [24], chapter 4. Because we are interested in a lift that has very particular arithmetic properties we will not work in complete generality, but instead we will make very specific choices to ensure certain integrality properties of the resulting Siegel modular form. Write $\mathbf{w}_1$ and $\mathbf{w}_2$ for the canonical basis of the free $\mathbf{Z}$-module $L := \mathbf{Z} \oplus \mathbf{Z}$ and write $\mathbf{w}_1^\vee$, $\mathbf{w}_2^\vee$ for the canonical basis of its dual $L^\vee$. If we identify GL_4 with the group of automorphisms of the rank 4 free $\mathbf{Z}$-module $W := L \oplus L^\vee$, then under the embedding $\mathrm{GSp}_4 \hookrightarrow \mathrm{GL}_4$ the group GSp_4 becomes the similitude group of the alternating form on W defined by

$$((x_1, y_1), (x_2, y_2))_W := y_1(x_2) - y_2(x_1),$$

where $x_i \in L$ and $y_i \in L^\vee$. We denote the similitude factor (as before) by μ.

We first study the dual pair $(\mathrm{Sp}_4, O(D))$, where $O(D)$ is the kernel of λ. Set $\mathbf{W} = W \otimes D$ to be the $\mathbf{Q}$-vector space with the alternating form $(\cdot,\cdot)_{\mathbf{W}} := (\cdot,\cdot)_W \otimes (\cdot,\cdot)_D$. It has $\mathbf{X} := L^\vee \otimes D$ as a maximal totally isotropic subspace and a complete polarization given by $\mathbf{W} \cong (L \otimes D) \oplus (L^\vee \otimes D)$. The subgroups Sp_4 and $O(D)$ form a dual reductive pair. Denote by ω the Weil representation on $\mathrm{Sp}_4(\mathbf{A}) \times O(D)(\mathbf{A})$ (which a priori depends on a choice of a character, but we will suppress it from notation). For all this see [24], p. 80.

We now proceed to the dual pair $(\mathrm{GSp}_4, \mathfrak{D})$. First note that it is possible to extend ω to a representation of a subgroup H' of $\mathrm{GSp}_4 \times \mathfrak{D}$ defined as follows:

$$H' = \{(g, h) \in \mathrm{GSp}_4 \times \mathfrak{D} \mid \mu(g) = \lambda(g)\}.$$

Locally at every place v for a local Schartz-Bruhat function f_v on $\mathbf{X}_v$ we define

$$(\omega_v(g, h) f_v)(x) = |\lambda(h)|^{-2} (\omega(g_1) f_v)(h^{-1} \cdot x),$$

where

$$g_1 := g \begin{bmatrix} I_2 & 0 \\ 0 & \mu(g) I_2 \end{bmatrix}^{-1} \in \mathrm{Sp}_4(\mathbf{Q}_v).$$

From now on take $\nu_1 = 0$ and $\nu_2 \in \{2, 4, 6, 8, 12\}$. We will write k for ν_2. Then $\mathcal{V} = \mathcal{V}_2$. Let $\mathbf{t}_i$ be as in section 3.3. Fix a smooth vector $f = \sum_{i=-k/2}^{k/2} f_i \otimes \mathbf{t}_i \in \mathcal{S}(\mathbf{X}_{\mathbf{A}}) \otimes V$ (see [24], page 83 and (5.1.10) on p.94), where $\mathcal{S}(\mathbf{X}_{\mathbf{A}})$ denotes the space of the Schwartz-Bruhat functions on $\mathbf{X}_{\mathbf{A}}$, and define the V-valued *theta kernel* Θ_f

to be the function on $H'(\mathbf{A})$ given by

$$\Theta_f(g,h) := \sum_i \Theta_{f_i}(g,h)\otimes \mathbf{t}_i = \sum_{x\in\mathbf{X}(\mathbf{Q})}\sum_i (\omega(g,h)f_i)(x)\otimes \mathbf{t}_i.$$

We define the *theta lift* of φ (with respect to f) to be

$$\theta_f(\varphi)(g) := \frac{1}{\mathrm{vol}(H^1(\mathbf{R}))}\cdot \int_{H^1(\mathbf{Q})\backslash H^1(\mathbf{A})} \langle \Theta_f(g,hh_g),\varphi(hh_g)\rangle_{1,2}\, d^1h,$$

where h_g is any element of $\mathfrak{D}$ with $\lambda(h_g)=\mu(g)$- see [24] (5.1.10) on p.94.

It remains to choose the vector f. This choice is crucial to ensure integrality of the Fourier coefficients of the Yoshida lift. In this we again follow closely [24] (section 4.4). We begin by noting that the function f will depend on the choice of a "basis" $\{\delta,j\}$ which we fixed once and for all in section 2.3. However, in fact the Yoshida lift will be independent of that choice (see [24] section 4.5.2 (p.94)).

For every place v of $\mathbf{Q}$, let $\mathcal{S}(\mathbf{X}_v)$ be the space of Schwarz-Bruhat functions on $\mathbf{X}_v$. Write

$$f=\prod_{v\nmid\infty} f_v\cdot \sum_i f_i\otimes \mathbf{t}_i \in \mathcal{S}(\mathbf{X}_{\mathbf{A}})\otimes V$$

for an element satisfying the following two conditions:

- For a rational prime p, f_p is the characteristic function of $R_p\oplus R_p\subset D(\mathbf{Q}_p)\oplus D(\mathbf{Q}_p)=\mathbf{X}_p$
- $f_\infty\in\mathcal{S}(\mathbf{X}_\infty)\otimes V$.

We will now make a choice of f_∞. We take $f_\infty=\sum_{i=-k/2}^{k/2} f_i\otimes\mathbf{t}_i$ as above and set

$$f_i(x)=\mathbf{P}_i((x_1,x_2))e(-2\pi(n(x_1)+n(x_2))),$$

where $\mathbf{P}_i$ is a harmonic polynomial on $\mathbf{X}_{\mathbf{R}}$, which we now describe. First of all, $\mathbf{P}_i((x_1,x_2))=\tilde{\mathbf{P}}_i(x_0)$, where $x_0=\frac{1}{2}(x_1x_2^\iota - x_2x_1^\iota)$, and $\tilde{\mathbf{P}}_i$ is a harmonic polynomial on the trace zero elements on D_∞, defined in the following way (cf. [24], section 1.2.5):

$$\tilde{\mathbf{P}}_i(x)=(-1)^i\frac{((k/2)!)^2}{(k/2+i)!(k/2-i)!}P_i(\epsilon_{\mathbf{R}}(x)),$$

where the map $\epsilon_{\mathbf{R}}: D^\times(\mathbf{R})\hookrightarrow \mathrm{GL}_{2,K_j}(\mathbf{R})$ is the map induced by the homomorphism ϵ in (2.1) and the polynomial P_i is defined recursively in the following way. First, for integers l,m,n set

$$M_{l,m,n}\left(\begin{bmatrix} a & b\\ c & -a\end{bmatrix}\right)=a^l\left(\frac{b}{2}\right)^m\left(\frac{c}{2}\right)^n.$$

We define an action of the "Lie algebra" operators Y^+ and Y^- on the set of these monomials by

- $Y^+\cdot M_{l,m,n}=m\cdot M_{l+1,m-1,n}-2l\cdot M_{l-1,m,n+1}$;
- $Y^-\cdot M_{l,m,n}=2l\cdot M_{l-1,m+1,n}-n\cdot M_{l+1,m,n-1}$.

Set $\mu=k/4$ if $4\mid k$ and $\mu=(k/2-1)/2$ if $4\nmid k$. Define

$$P_0=\sum_{i=0}^{\mu}(-1)^i\binom{k/2}{2i}\binom{2i}{i}M_{k-2i,i,i}$$

and

$$P_i = \begin{cases} \frac{(k/2+i)!}{(k/2)!}(-Y^+)^{-i}P_0 & \text{if } i < 0, \\ \frac{(k/2-i)!}{(k/2)!}(-Y^-)^{i}P_0 & \text{if } i > 0. \end{cases}$$

Theorem 5.1. *Let f be chosen as above. Let $\mathcal{A}^S$ denote the space of automorphic forms on $H(\mathbf{A})$ with trivial central character. Then the assignment*

$$\varphi = \varphi_1 \otimes \varphi_2 \mapsto \theta_f(\varphi)$$

defines a $\mathbf{C}$-linear map from $\mathcal{A}_0^D(R) \otimes \mathcal{A}_k^D(R)$ to $\mathcal{A}^S$.

Proof. The continuity and left $H(\mathbf{Q})$-invariance of $\theta_f(\varphi)$ follow from Proposition 2.1 in [40]. Let $K = K_\infty K_f$ be the maximal compact subgroup of $H(\mathbf{A})$ defined by

$$K_\infty = \{M \in H(\mathbf{R}) \mid MM^t = I_4\} = \left\{ M = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in H(\mathbf{R}) \mid A, B \in M_2(\mathbf{R}) \right\}$$

and

$$K_f = \prod_p K_p = \prod_p H(\mathbf{Z}_p).$$

Then the right K_f-finiteness of $\theta_f(\varphi)$ follows from the fact that the Weil representations ω_p are finite-dimensional when restricted to K_p and one-dimensional for all but finitely many p ([40], Proposition 2.5). Also, as discussed above, K_∞ acts on $\theta_f(\varphi)$ via character. Finally $\mathfrak{z}$-finiteness and moderate growth condition follow from holomorphicity of the corresponding function (denoted below by $Y_f(\varphi_1 \otimes \varphi_2)$) defined on the Siegel upper half-space as discussed on pages 203-204 in [40]. Here $\mathfrak{z}$ denotes the center of the universal enveloping algebra of $H(\mathbf{R})$. □

We will now make a translation to the language of Siegel modular forms. Let $\mathbf{H}_2$ denote (as before) the Siegel upper half-space. From now on we fix the choice of f as above. For $z \in \mathbf{H}_2$, there exists $g = (g_\infty, I_4) \in H(\mathbf{R})H(\mathbf{A}_\mathrm{f})$ such that $z = g_\infty \mathbf{i}$. Set

$$Y(\varphi_1 \otimes \varphi_2)(z) = Y_f(\varphi_1 \otimes \varphi_2)(z) := \theta_f(\varphi_1 \otimes \varphi_2)(g) j(g_\infty, \mathbf{i})^{k/2+2}.$$

This function is well-defined since for $\kappa = \left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right] \in K_\infty$, we have

$$\theta_f(\varphi)(g\kappa) = \det(A - B\mathbf{i})^{-k/2-2} \theta_f(\varphi)(g).$$

Theorem 5.2. *Let k, R, φ_1 and φ_2 be as above. The assignment*

$$\varphi_1 \otimes \varphi_2 \mapsto Y(\varphi_1 \otimes \varphi_2)$$

defines a $\mathbf{C}$-linear map from $\mathcal{A}_0^D(R) \otimes \mathcal{A}_k^D(R)$ to the space of Siegel modular forms of weight $k/2+2$, level N and trivial character.

Proof. Cf. [40], Theorem 2.7. □

Definition 5.3. Let k, R and f be as above. The function $Y(\varphi_1 \otimes \varphi_2)$ will be called the *Yoshida lift of* $\varphi_1 \otimes \varphi_2$. It is an element in $S^S_{k/2+2}(N)$ (by Theorem 5.2).

We will now state a result of Jia which guarantees the integrality of the Fourier coefficients of $Y(\varphi_1 \otimes \varphi_2)$ for an appropriate choice of φ_1 and φ_2.

Theorem 5.4 (Jia, [24], Theorem 4.10 and 4.13)**.** *Let ℓ be an odd prime, $\ell \nmid N$, $\ell > k$. Suppose φ_1, φ_2 are ℓ-integral (in the sense of Definition 3.6). Then every Fourier coefficient of $Y(\varphi_1 \otimes \varphi_2)$ (with the vector f chosen as above) lies in a finite extension of $\mathbf{Q}$ and is ℓ-integral, i.e., viewed as an element of $\overline{\mathbf{Q}}_\ell$ under our* *fixed choice of embedding* $\overline{\mathbf{Q}}_\ell \hookrightarrow \mathbf{C}$ *has a non-negative* ℓ*-adic valuation. Moreover, assuming the Artin's conjecture on primitive roots (for details see* [24]*, Theorem 4.13) and that the forms* φ_1 *and* φ_2 *are non-Eisenstein (cf. Definition 3.7) there exists a Fourier coefficient of* $Y(\varphi_1 \otimes \varphi_2)$ *which is an* ℓ*-adic unit.*

5.2. **The Hecke action.** We let $\mathbf{T}^S$ denote the standard Hecke algebra acting on the space of Siegel cusp forms $S^S_{k/2+2}(N)$. At every place l it is generated by the operators

$$T(l^{d_1}, l^{d_2}, l^{d_3}, l^{d_4}) := \Gamma_0^S(N) \begin{bmatrix} l^{d_1} & & & \\ & l^{d_2} & & \\ & & l^{d_3} & \\ & & & l^{d_4} \end{bmatrix} \Gamma_0^S(N),$$

where the d_i are non-negative integers.

Theorem 5.5 (Yoshida [40], section 6)**.** *Let* $l \nmid N$ *be a prime. Let* $\varphi_1 \in S_0^D(N)$*,* $\varphi_2 \in S_k^D(N)$ *be eigenfunctions of* T_l^D *with corresponding eigenvalues* $\lambda_{l,1}$*,* $\lambda_{l,2}$*. Then the Yoshida lift* $Y(\varphi_1 \otimes \varphi_2)$ *of* $\varphi_1 \otimes \varphi_2$ *is a common eigenfunction of the all the Hecke operators* $T(l^{d_1}, l^{d_2}, l^{d_3}, l^{d_4})$*, i.e., it is a common eigenfunction of the entire Hecke algebra at* l*. Moreover, one has*

$$\begin{aligned} T(1,1,l,l)Y(\varphi_1 \otimes \varphi_2) &= l^{k/2}(\overline{\lambda_{l,1}} + \overline{\lambda_{l,2}})Y(\varphi_1 \otimes \varphi_2), \\ T(1,l,l,l^2)Y(\varphi_1 \otimes \varphi_2) &= (l^{k-2}(l^2 - 1 + l\overline{\lambda_{l,1}})Y(\varphi_1 \otimes \varphi_2). \end{aligned} \tag{5.1}$$

Remark 5.6. Note that the assumption that $R_q^\times$ contains an element of reduced norm l for every rational prime $q \neq l$ appearing in the statement of the corresponding theorem in [40], section 6 is always satisfied in our case.

From now on let N be a prime. In Definition 3.9 we defined a lift

$$JL : S_{n+2}(N) \to S_n^D(N),$$

which has the property that if $f \in S_{n+2}(N)$ is an eigenform for T_l, $l \nmid N$, then $JL(f) \in S_n^D(N)$ is an eigenform for the corresponding Hecke operatos T_l^D, $l \nmid N$ acting on $S_n^D(N)$. The composition

$$S_2(N) \otimes S_{k+2}(N) \xrightarrow{JL} S_0^D(N) \otimes S_k^D(N) \xrightarrow{Y} S^S_{k/2+2}(N)$$

is a $\mathbf{C}$-linear map which is Hecke-equivariant away from N. We will denote this composite also by Y. Note also that for a prime $\ell > k$, $\ell \nmid N$ by the construction carried out in section 3.3 and by Theorem 5.4, the composite Y has the property that it takes normalized newforms $f_1 \in S_2(N)$ and $f_2 \in S_{k+2}(N)$ to a Siegel modular form with ℓ-integral Fourier coefficients.

Let l be a prime not dividing N. Let $f_1 \in S_2(N)$ and $f_2 \in S_{k+2}(N)$ be eigenforms for the operator T_l with eigenvalues λ_1, λ_2 respectively. Then $JL(f_1) \in S_0^D(N)$ and $JL(f_2) \in S_k^D(N)$ are eigenforms of T_l^D with corresponding eigenvalues $\lambda_1^D = \lambda_1$ and $\lambda_2^D = l^{k/2}\lambda_2$ respectively (see (3.3)).

Remark 5.7. For $n \in \{2, 4, 6, 8, 10, 14\}$ and N prime the space $S_n(N)$ has a basis of newforms ([27], p. 153). Moreover the Hecke eigenvalues of any newform in $S_n(N)$ are real ([27], formula (4.6.17) and Theorem 4.6.17(2)).

Corollary 5.8. *Let N be a prime and assume $k \in \{2,4,6,8,12\}$. Let $l \nmid N$ be another prime and let $f_1 \in S_2(N)$ (resp. $f_2 \in S_{k+2}(N)$) be eigenforms for the operators T_l with eigenvalues λ_1 (resp. λ_2). Then one has*

$$\begin{aligned} T(1,1,l,l)Y(f_1 \otimes f_2) &= (l^{k/2}\lambda_1 + \lambda_2)Y(f_1 \otimes f_2) \\ T(1,l,l,l^2)Y(f_1 \otimes f_2) &= (l^k - l^{k-2} + l^{k/2-1}\lambda_1\lambda_2)Y(f_1 \otimes f_2). \end{aligned} \tag{5.2}$$

Proof. This follows from (3.3), Theorem 5.5 and Remark 5.7. □

Since the operators $T(1,1,l,l)$ and $T(1,l,l,l^2)$ generate the full local Hecke algebra at l, we get the following theorem.

Theorem 5.9. *Assume $k \in \{2,4,6,8,12\}$. Let $Y : S_2(N) \otimes S_{k+2}(N) \to S^S_{k/2+2}(N)$ denote the Yoshida lift. Let $\mathbf{T}^S$ denote the $\mathbf{C}$-Hecke algebra acting on $S^S_{k/2+2}(N)$ generated by all the local Hecke algebras away from N. There exists a homomorphism of Hecke algebras $\Phi : \mathbf{T}^S \to \mathbf{T} \otimes \mathbf{T}$ such that for every $T \in \mathbf{T}^S$ the following diagram commutes:*

$$\begin{array}{ccc} S^S_{k/2+2}(N) & \xrightarrow{\ T\ } & S^S_{k/2+2}(N) \\ {\scriptstyle Y}\uparrow & & \uparrow{\scriptstyle Y} \\ S_2(N) \otimes S_{k+2}(N) & \xrightarrow{\ \Phi(T)\ } & S_2(N) \otimes S_{k+2}(N) \end{array}$$

For a prime $l \neq N$ the map Φ is given explicitly by

$$\begin{aligned} \Phi(T(1,1,l,l)) &= l^{k/2}T_l \otimes 1 + 1 \otimes T_l \\ \Phi(T(1,l,l,l^2)) &= l^k - l^{k-2} + l^{k/2-1}T_l \otimes T_l \end{aligned} \tag{5.3}$$

Note our slight abuse of notation. We use the same symbol T_l (resp. $\mathbf{T}$) even though we sometimes mean the Hecke operator (resp. Hecke algebra) acting on $S_2(N)$ and sometimes on $S_{k+2}(N)$.

One can also show that in fact $Y(f_1 \otimes f_2)$ is an eigenform for all the local Hecke algebras (including at the prime N). This follows from Lemma 7.3 in [8] (for $k = 2$) and from section 4 of [9] (for $k > 2$). We summarize it in the following proposition.

Proposition 5.10 (Böcherer-Schulze-Pillot)**.** *The Yoshida lift $Y(f_1 \otimes f_2)$ is an eigenform for the local Hecke algebra at N.*

5.3. **The Petersson norm of a Yoshida lift.**

5.3.1. *L-functions and Satake isomorphism.* The goal of this section is to express the Petersson norm of a Yoshida lift by L-functions. Unfortunately the resulting formula involves a constant that we are unable to compute explicitly. We begin by defining the Dirichlet L-functions and then L-functions of elliptic and Siegel modular forms.

Let Σ be a finite set of rational primes and N a positive integer whose all prime divisors are in Σ. Let M be a positive integer and $\chi : (\mathbf{Z}/M)^\times \to \mathbf{C}^\times$ be a Dirichlet character. Define the *Dirichlet L-function* associated to χ to be

$$L^\Sigma(s,\chi) = \prod_{l \not\in \Sigma}(1 - \chi(l)l^{-s})^{-1},$$

where we set $\chi(l) = 0$ if $l \mid M$. The properties of these L-functions are well-known, see for example [28].

Let $f \in S_n(N)$ be a normalized common eigenform of all T_l, $l \notin \Sigma$. Write

$$f(z) = \sum_{n=1}^{\infty} a_n q^n.$$

Then if l is a prime not in Σ, one has $T_l f = a_l f$. For such an l, let $\alpha_{l,1}$ and $\alpha_{l,2}$ be the l-Satake parameters of f, i.e., the unique complex numbers such that $\alpha_{l,1} + \alpha_{l,2} = a_l$ and $\alpha_{l,1}\alpha_{l,2} = l^{n-1}$. Define the *standard L-function* of f to be

$$L^{\Sigma}(s, f) = \prod_{l \notin \Sigma} (1 - \alpha_{l,1} l^{-s})^{-1} (1 - \alpha_{l,2} l^{-s})^{-1}.$$

Let $g \in S_m(N)$ be another common eigenform for all T_l with $l \notin \Sigma$. Write $\beta_{l,1}, \beta_{l,2}$ for its l-Satake parameters. Let χ be a Dirichlet character as above. We define the *convolution L-function* of f and g twisted by χ to be

$$\begin{aligned} L^{\Sigma}(s, f \times g, \chi) = \prod_{l \notin \Sigma} \{ &(1 - \chi(l)\alpha_{l,1}\beta_{l,1} l^{-s})(1 - \chi(l)\alpha_{l,1}\beta_{l,2} l^{-s}) \times \\ &\times (1 - \chi(l)\alpha_{l,2}\beta_{l,1} l^{-s})(1 - \chi(l)\alpha_{l,2}\beta_{l,2} l^{-s}) \}^{-1}. \end{aligned} \tag{5.4}$$

To ease notation we set

$$L^{\Sigma}(s, f \times g) = L^{\Sigma}(s, f \times g, \mathbf{1}).$$

For the well-known properties of this function we refer the reader to any of the following sources [35, 21, 23]. We will now define L-functions associated to a Siegel modular form. Let $F \in S_n^S(N)$ be a common eigenform for all the local Hecke algebras away from Σ. Let $\mathbf{T}^{S,\Sigma} \subset \mathrm{End}_{\mathbf{C}}(S^S_{k/2+2}(N))$ denote the $\mathbf{C}$-subalgebra generated by the local Hecke algebras at all primes $l \notin \Sigma$. Then F defines a $\mathbf{C}$-algebra homomorphism $\lambda_F : \mathbf{T}^{S,\Sigma} \to \mathbf{C}$ sending T to its F-eigenvalue.

Definition 5.11. Let Σ and F be as above. Set $t_0 = \lambda_F(T(1,1,l,l))$, $t_1 = \lambda_F(T(1,l,l,l^2))$ and $t_2 = \lambda_F(T(l,l,l,l))$. The product

$$L^{\Sigma}_{\mathrm{spin}}(s, F) := \prod_{l \notin \Sigma} (1 - t_0 l^{-s} + \{l t_1 + l(l^2+1)t_2\} l^{-2s} - l^3 t_0 t_2 l^{-3s} + l^6 t_2^2 l^{-4s})^{-1} \tag{5.5}$$

is called the *spin L-function* associated to F.

The spin L-function can be given an alternative definition in terms of the Satake parameters. Let l be a prime not dividing N. Set

$$\Delta_l^S(N) = \left\{ g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in H(\mathbf{Q})^+ \cap \mathrm{GL}_4(\mathbf{Z}[l^{-1}]) \mid \mu(g) \in \mathbf{Z}[l^{-1}],\ C \equiv 0 \pmod{N} \right\}.$$

Let $\mathcal{L}_l^S(N)$ be the $\mathbf{Q}$-algebra generated by the double cosets $\Gamma_0^S(N) g \Gamma_0^S(N)$ for $g \in \Delta_l^S(N)$ subject to the usual law of multiplication (see [14], p.51). Let W denote the Weyl group of GSp_4. If $t = \mathrm{diag}(t_0 t_1^{-1}, t_0 t_2^{-1}, t_1, t_2)$ is an element of the maximal torus of T of GSp_4, then W can be identified with the subgroup of the group S of permutations of the entries of t consisting of those $\sigma \in S$ for which $\sigma(t) \in T$. This group is generated by σ_0, where

$$\sigma_0(t_0) = t_0, \quad \sigma_0(t_1) = t_2, \quad \sigma_0(t_2) = t_1$$

and σ_1 and σ_2, where

$$\sigma_j(t_0) = t_0 t_j, \quad \sigma_j(t_i) = t_i, \quad \sigma_j(t_j) = t_j^{-1}, \quad i \neq j.$$

Let x_0, x_1, x_2 be indeterminates. The group W acts on the ring of polynomials $R := \mathbf{Q}[x_0, x_0^{-1}, x_1, x_1^{-1}, x_2, x_2^{-1}]$ in the way defined above (replace t by x). Define a $\mathbf{Q}$-linear homomorphism Sat : $\mathcal{L}_l^S(N) \to R$ in the following way. For every $g \in \Delta_l^S(N)$ we can write

$$\Gamma_0^S(N) g \Gamma_0^S(N) = \bigsqcup_i \Gamma_0^S(N) g_i$$

with $g_i \in \Delta_l^S(N)$ upper-triangular with diagonal entries of the form $l^{e_0} l^{-e_1}$, $l^{e_0} l^{-e_2}$, l^{e_1}, l^{e_2} for some $e_0, e_1, e_2 \in \mathbf{Z}$. Set

$$\mathrm{Sat}(\Gamma_0^S(N) g \Gamma_0^S(N)) = \sum_i x_0^{e_0} (x_1 l^{-1})^{e_1} (x_2 l^{-2})^{e_2},$$

(see [2], p. 140-141 and p. 118).

Theorem 5.12 ([2], p. 141)**.** *The map* Sat : $\mathcal{L}_l^S(N) \to R$ *defines a* $\mathbf{Q}$*-linear isomorphism of* $\mathcal{L}_l^S(N)$ *with the subring* R^W *of* R *of all* W*-invariant polynomials in* x_0, x_1, x_2*.*

Let F and λ_F be as above. Let $l \notin \Sigma$ be a prime. Then the restriction of λ_F to the local $\mathbf{Q}$-Hecke algebra at l can be extended to a $\mathbf{Q}$-algebra homomorphism from $\mathcal{L}_l^S(N)$ to $\mathbf{C}$. We will denote this extension also by λ_F. The composite $\lambda_F \circ \mathrm{Sat}^{-1} : R^W \to \mathbf{C}$ determines complex numbers $\lambda_{l,j}$, $j = 0, 1, 2$ such that for any polynomial $P(x_0, x_0^{-1}, x_1, x_1^{-1}, x_2, x_2^{-1}) \in R^W$ one has $\lambda_F(P) = P(\lambda_{l,0}, \lambda_{l,0}^{-1}, \lambda_{l,1}, \lambda_{l,1}^{-1}, \lambda_{l,2}, \lambda_{l,2}^{-1})$. The element $\lambda_{l,0}$ is determined uniquely, while $\lambda_{l,1}$ and $\lambda_{l,2}$ are determined up to permutation.

Definition 5.13. For F as above, and a prime $l \nmid N$, the complex numbers $\lambda_{l,0}, \lambda_{l,1}, \lambda_{l,2}$ are called the *l-Satake parameters of F*.

The l-Satake parameters of $F \in S_{k/2+2}^S(N)$ satisfy the following relation ([14], formula (2.41))

$$\lambda_{l,0}^2 \lambda_{l,1} \lambda_{l,2} = l^{k+1}.$$

Proposition 5.14 ([14], section 2.1.6)**.** *Let* $\chi : (\mathbf{Z}/M)^\times \to \mathbf{C}^\times$ *be a Dirichlet character. One has*

$$L_{\mathrm{spin}}^\Sigma(s, F) = \prod_{l \notin \Sigma} \{(1 - \lambda_{l,0} l^{-s})(1 - \lambda_{l,0}\lambda_{l,1} l^{-s})(1 - \lambda_{l,0}\lambda_{l,2} l^{-s})(1 - \lambda_{l,0}\lambda_{l,1}\lambda_{l,2} l^{-s})\}^{-1}.$$

We will also have a use for the standard L-function associated to F.

Definition 5.15. The *standard L-function* of F is given by the following product

$$L_{\mathrm{st}}^\Sigma(s, F, \chi) = \prod_{l \notin \Sigma} \left\{ (1 - \chi(l) l^{-s}) \prod_{j=1}^{2} (1 - \chi(l)\lambda_{l,j} l^{-s})(1 - \chi(l)\lambda_{l,j}^{-1} l^{-s}) \right\}^{-1}$$

Proposition 5.16. *Let* N *be a prime and set* $\Sigma = \{N\}$*. Let* $f_1 \in S_2(N)$ *and* $f_2 \in S_{k+2}(N)$ *be common eigenforms for all Hecke operators away from the primes* *in Σ. Fix a prime $l \notin \Sigma$. Write α_1, α_2 (resp. β_1, β_2) for the l-Satake parameters of f_1 (resp. f_2). Write λ_0, λ_1, λ_2 for the l-Satake parameters of $Y(f_1 \otimes f_2)$. Then*

$$\begin{aligned} \lambda_0 &= \alpha_1 l^{k/2} \\ \lambda_1 &= \alpha_1^{-1}\beta_1 l^{-k/2} = \alpha_2\beta_1 l^{-k/2-1} \\ \lambda_2 &= \alpha_1^{-1}\beta_1^{-1} l^{k/2+1} = \alpha_2\beta_2 l^{-k/2-1} \end{aligned} \quad . \tag{5.6}$$

Theorem 5.17. *Let N be a prime and set $\Sigma = \{N\}$. Let $f_1 \in S_2(N)$ and $f_2 \in S_{k+2}(N)$ be common eigenforms for all Hecke operators away from the primes in Σ. Let M be a positive integer and $\chi : (\mathbf{Z}/M)^\times \to \mathbf{C}^\times$ a Dirichlet character. Then one has*

$$L^{\Sigma}_{\rm spin}(s, Y(f_1 \otimes f_2)) = L^{\Sigma}(s - k/2, f_1) L^{\Sigma}(s, f_2) \tag{5.7}$$

and

$$L^{\Sigma}_{\rm st}(s, Y(f_1 \otimes f_2), \chi) = L^{\Sigma}(s, \chi) L^{\Sigma}(s + 1 + k/2, f_1 \times f_2, \chi). \tag{5.8}$$

Proof. Formula (5.7) was obtained by Yoshida ([40], Theorem 7.2), while formula (5.8) is an easy calculation using Propositions 5.14 and 5.16. □

Corollary 5.18. *With the assumptions as in Theorem 5.17 the standard L-function $L^{\Sigma}_{\rm st}(s, Y(f_1 \otimes f_2))$ of a Yoshida lift has a simple pole at $s = 1$ with residue equal to*

$$L^{\Sigma}(2 + k/2, f_1 \times f_2) \times \prod_{l \in \Sigma} (1 - l^{-1}).$$

Let N be a prime and k an even positive integer. Let $f_1 \in S_2(N)$, $f_2 \in S_{k+2}(N)$ be common eigenforms for all T_l, $l \neq N$.

Conjecture 5.19. *Let $\ell > k$ be a prime, $\ell \neq N$. One has*

(5.9)
$$\begin{aligned} \frac{\langle Y(f_1 \otimes f_2), Y(f_1 \otimes f_2)\rangle}{\langle \varphi_1, \varphi_1\rangle \langle \varphi_2, \varphi_2\rangle} &= \pi^{1-k} c_{\rm alg}(f_1 \otimes f_2) \mathrm{res}_{s=1} L^{(N)}_{\rm st}(s, Y(f_1 \otimes f_2)) \\ &= \pi^{1-k} c_{\rm alg}(f_1 \otimes f_2) L^{(N)}(2 + k/2, f_1 \times f_2) \cdot (1 - \frac{1}{N}), \end{aligned}$$

where $c_{\rm alg}(f_1 \otimes f_2)$ is an algebraic number which is an ℓ-adic unit.

Remark 5.20. The algebraicity of $c_{\rm alg}(f_1 \otimes f_2)$ will be proved in section 6. The second equality is a consequence of Corollary 5.18. When $k = 2$, equation (5.9) is proved in [8], Proposition 10.2, where the constant $c = c_{\rm alg}(f_1 \otimes f_2)$ is computed and it follows that it is in fact independent of f_1 and f_2. One also sees that $\mathrm{val}_\ell(c) = 0$.

6. The congruence

In this section we construct a congruence between the Yoshida lift $Y(f_1 \otimes f_2)$ (which has a reducible ℓ-adic Galois representation) and a cuspidal Siegel eigenform F with an irreducible Galois representation. To carry out the construction we will need a certain Hecke operator, whose existence is proved in section 7.

To make the statement of the main theorem (Theorem 6.5) self-contained let us gather here all the main assumptions which we need for Theorem 6.5.

Assumption 6.1. Consider the following set of assumptions:

(1) $k \in \{8, 12\}$;

(2) N a prime;
(3) ℓ a prime such that $\ell > k$, $\ell \neq N$, $\ell \nmid (N+1)$ (the last one is needed for Corollary 7.22);
(4) $\Sigma := \{N, \ell\}$;
(5) $f_1 \in S_2(N)$ and $f_2 \in S_{k+2}(N)$ common eigenforms for all Hecke operators ordinary at ℓ such that the residual (mod ϖ) Galois representations $\overline{\rho}_{f_i}$ have the property that $\overline{\rho}_{f_i}|_L$ are absolutely irreducible for $i = 1, 2$, where $L = \mathbf{Q}(\sqrt{(-1)^{(\ell+1)/2}\ell})$;
(6) Assume that the Artin's conjecture on primitive roots holds (see Theorem 5.4 and [24], Conjecture 6.6 for details);
(7) Assume Conjecture 5.19.

For the rest of the section assume that Assumption 6.1 holds. Set $\varphi_i = JL(f_i)$ for $i = 1, 2$ as in Definition 3.9. By Remark 3.8, the forms φ_i are non-Eisenstein. Let $Y(f_1 \otimes f_2)$ be the Yoshida lift associated to f_1 and f_2. Set

$$\mathcal{E}(z, w) := D_{E^*}(\operatorname{diag}[z, w], (5 - (k/2 + 2))/2)) \mid_{1 \times \iota_{\mathfrak{f}}^{-1}}$$

to be the holomorphic Siegel Eisenstein series introduced in section 4 with weight $k/2 + 2$ and level N. Let $F_0 := Y(f_1 \otimes f_2), F_1, F_2, ..., F_r$ be an orthogonal basis of $S^S_{k/2+2}(N)$ consisting of eigenforms for the Hecke operators away from the primes in Σ.

Let E be a sufficiently large finite extension of $\mathbf{Q}_\ell$, $\mathcal{O}$ its ring of integers and ϖ a uniformizer. By "sufficiently large" we mean that we will assume that it contains all the number fields that we will define below. In particular we require that it contains the number field $\mathbf{Q}[\tau]$ generated by the Fourier coefficients of the Eisenstein series $\mathcal{E}(z, w)$ (cf. Theorem 4.1) and the number field generated by the Hecke eigenvalues of the eigenforms $F_0, F_1, \dots, F_r$ (for the proof that these eigenvalues indeed generate a number field see e.g., [34] Theorem 10.7 and the proof of Theorem 28.5 in [loc. cit.]). Note that on the other hand Theorem 5.4 only guarantees that the Fourier coefficients of $Y(f_1 \otimes f_2)$ lie in the ring of integers $\overline{\mathcal{O}}$ of $\overline{\mathbf{Q}}_\ell$. Moreover, it follows from [34] Theorem 28.5 that we can scale the F_i's so that their Fourier coefficients all lie in $\overline{\mathbf{Q}}_\ell$. In what follows we scale the F_i's for $i > 0$ appropriately (leaving $Y(f_1 \otimes f_2)$ unchanged).

By the cuspidality result of Brown [12], section 3.2 for $\mathcal{E}(z, w)$ we can write

$$\mathcal{E}(z, w) = \sum_{i,j=0}^{r} c_{i,j} F_i(z) F_j^c(w) \tag{6.1}$$

with $c_{i,j} \in \overline{\mathbf{Q}}_\ell$.

Arguing as in Proposition 6.1 in [11] (using Theorem 4.2 and the fact that F_0 is an eigenform for all Hecke operators - see Corollary 5.8 and Proposition 5.10) we get that

$$\mathcal{E}(z, w) = c_{0,0} F_0(z) F_0^c(w) + \sum_{0 \leq i \leq r, 0 < j \leq r} c_{i,j} F_i(z) F_j^c(w) \tag{6.2}$$

and

$$c_{0,0} = \frac{\pi^{-3} \mathcal{A}_{k,N} L^N_{\mathrm{st}}(3 - k/2, F_0^c, \tau)}{\langle F_0^c(w), F_0^c(w) \rangle}$$

Our primary goal is to establish a congruence between a Siegel eigenform (here $F_0 = Y(f_1 \otimes f_2)$) whose associated Galois representation is reducible and another

Siegel eigenform G whose Galois representation is irreducible. To do this we will need a Hecke operator $T^S \in \mathbf{T}_{\mathcal{O}}^{\Sigma,S}$ such that $T^S F = \eta_1\eta_2 F$ for any $F \in Y_{f_1,f_2}$ (for notation see section 7) and $T^S F = 0$ for any eigenform orthogonal to Y_{f_1,f_2} whose associated Galois representation is reducible. Here η_1 and η_2 are the generators of the Hida congruence ideal for f_1(resp.f_2). The operator T^S will be constructed in the next section (see Corollary 7.22).

We now recall a theorem of Hida that gives a description of Hida's congruence invariant η.

Theorem 6.2 (Hida87). *If f is a newform ordinary at ℓ, then*

$$\eta = u\frac{\langle f, f\rangle}{\Omega_f^+\Omega_f^-},$$

where u is a ϖ-adic unit if $\ell \neq 2$ and Ω_f^+ and Ω_f^- are complex periods uniquely determined up to an $\mathcal{O}$-unit.

Applying T^S (in the variable z - note that $\mathcal{E}(z,w)$ is cuspidal (as remarked above) when considered as only a function of z) to (6.2) we get

$$T^S\mathcal{E}(z,w) = \eta_1\eta_2 c_{0,0}F_0(z)F_0^c(w) + \sum_{0\leq i\leq r, 0<j\leq r}^{r} c_{i,j}T^S F_i(z)F_j^c(w). \tag{6.3}$$

Suppose $\mathrm{val}_\varpi(\eta_1\eta_2 c_{0,0}) = -M < 0$, that is, there exists $\beta \in \mathcal{O}^\times$ so that $\eta_1\eta_2 c_{0,0} = \omega^{-M}\beta$. By Proposition 7.1 any operator in $\mathbf{T}_{\mathcal{O}}^{\Sigma,S}$ preserves the ℓ-integrality of the Fourier coefficients of the forms it acts on. Hence $T^S\mathcal{E}(z,w)$ still has Fourier coefficients that lie in $\mathcal{O}$. By Theorem 5.4 there exists T_0 such that $\omega \nmid a(T_0, F_0^c)$, where $a(T_0,F_0^c)$ denotes the T_0-Fourier coefficient of F_0^c. Now it is easy to observe that $c_{i,j}T^S F_i(z)F_j^c(w) \neq 0$ for at least one pair $i,j\neq 0$ because otherwise

$$\omega^M T^S\mathcal{E}(z,w) = \beta F_0(z)F_0^c(w),$$

and this would imply that $F_0(z)F_0^c(w) \equiv 0 \pmod{\omega}$ which would lead to a contradiction by Theorem 5.4. Here and below if we write $F \equiv F' \pmod{\varpi^m}$ we mean that all of the Fourier coefficients of F are congruent to the corresponding Fourier coefficients of F' (i.e., their difference lies in $\varpi^m\overline{\mathcal{O}}$).

By expanding both sides of (6.3) in terms of w and comparing the coefficient of T_0 using the integrality of the coefficients of $\mathcal{E}(z,w)$ we have

$$F_0(z) \equiv -\frac{\omega^M}{a(T_0,F_0^c)\beta}\sum_{0\leq i\leq r, 0<j\leq r} c_{i,j}a(T_0,F_j^c)T^S F_i(z) \pmod{\omega^M}. \tag{6.4}$$

Let

$$G(z) = -\frac{\omega^M}{a(T_0,F_0^c)\beta}\sum_{0\leq i\leq r, 0<j\leq r} c_{i,j}a(T_0,F_j^c)T^S F_i(z).$$

Then $F_0(z) \equiv G \pmod{\omega^M}$. Clearly, $G(z) \neq 0$ due to the non-vanishing (mod ω) of F_0.

By Conjecture 5.19

$$\langle F_0^c(z), F_0^c(z)\rangle = \pi^{1-k}c_{\mathrm{alg}}(f_1\otimes f_2)L^N(2+k/2, f_1\times f_2)\cdot(1-\frac{1}{N})\cdot\langle\varphi_1,\varphi_1\rangle\cdot\langle\varphi_2,\varphi_2\rangle\,.$$

Hence by (5.8) noting that both f_1 and f_2 have real Hecke eigenvalues (cf. Remark 5.7), we get

$$\begin{aligned} c_{0,0} &= \frac{\pi^{-3}\mathcal{A}_{k,N}L^N_{\mathrm{st}}(3-k/2,F_0^c,\tau)}{\pi^{1-k}c_{\mathrm{alg}}(f_1\otimes f_2)L^N(2+k/2,f_1\times f_2)\cdot(1-\frac{1}{N})\cdot\langle\varphi_1,\varphi_1\rangle\cdot\langle\varphi_2,\varphi_2\rangle} \\ &= \frac{\pi^{-3}\mathcal{A}_{k,N}L^N(3-k/2,\tau)L^N(4,f_1\times f_2)}{\pi^{1-k}c_{\mathrm{alg}}(f_1\otimes f_2)L^N(2+k/2,f_1\times f_2)\cdot(1-\frac{1}{N})\cdot\langle\varphi_1,\varphi_1\rangle\cdot\langle\varphi_2,\varphi_2\rangle}. \end{aligned} \tag{6.5}$$

Theorem 6.3 (Shimura, [35], Theorem 4)**.** *Let $g_i \in S_{m_i}(N)$ be eigenforms, $i=1,2$. Assume $m_1 < m_2$. Let m be an integer such that $m_1 \leq m < m_2$. Then*

$$L^{N,\mathrm{alg}}(m,g_1\times g_2) := \frac{\pi^{m_1-2-2m}L^N(m,g_1\times g_2)}{\langle g_2,g_2\rangle} \in \overline{\mathbf{Q}}.$$

By Theorem 6.3 we have

$$c_{0,0} = \frac{\mathcal{A}_{k,N}L^N(3-k/2,\tau)L^{N,\mathrm{alg}}(4,f_1\times f_2)}{c_{\mathrm{alg}}(f_1\otimes f_2)L^{N,\mathrm{alg}}(2+k/2,f_1\times f_2)\cdot(1-\frac{1}{N})\cdot\langle\varphi_1,\varphi_1\rangle\cdot\langle\varphi_2,\varphi_2\rangle}.$$

Since $c_{0,0}\in\overline{\mathbf{Q}}_\ell$, we conclude that $c_{\mathrm{alg}}(f_1\otimes f_2)\in\overline{\mathbf{Q}}_\ell$ which proves this part of Conjecture 5.19. We enlarge E so it contains $c_{\mathrm{alg}}(f_1\otimes f_2)$. Also, by [30] (p.109) one has

$$\mathcal{A}_{k,N} = \frac{\pm 2^{2k-3}}{3(N^2+1)}.$$

Hence $\mathrm{val}_\varpi(\mathcal{A}_{k,N}) \leq 0$. Since by Proposition 7.23 the form F_0 spans Y_{f_1,f_2} we conclude that the eigenforms F_i in the sum $\sum_{0\leq i\leq r, 0<j\leq r} c_{i,j}a(T_0,F_0^c)T^SF_i(z)$ with $c_{i,j}T^SF_i\neq 0$ all have irreducible Galois representations. Define a period ratio

$$\Omega_{1,2} := \frac{\langle\varphi_1,\varphi_1\rangle_D\,\langle\varphi_2,\varphi_2\rangle_D}{\eta_1\eta_2}.$$

Using Theorem 6.2 and the fact that f_1 and f_2 are assumed ordinary we can also write (up to an ℓ-adic unit):

$$\Omega_{1,2} = \frac{\langle\varphi_1,\varphi_1\rangle_D\,\langle\varphi_2,\varphi_2\rangle_D\,\Omega^+_{f_1}\Omega^-_{f_1}\Omega^+_{f_2}\Omega^-_{f_2}}{\langle f_1,f_1\rangle\,\langle f_2,f_2\rangle}.$$

Remark 6.4. The authors do not know whether $\mathrm{val}_\ell(\Omega_{1,2})=0$ (note that this value is canonical by Lemma 3.10). Since η_i measures congruences between f_i and other cusp forms, while $\langle\varphi_i,\varphi_i\rangle_D$ is an (algebraic) period for the ℓ-integral automorphic form φ_i (thus should measure congruences between φ_i and other quaternionic modular forms, which should be "induced" from the f_i-congruences) it is perhaps reasonable to believe that $\Omega_{1,2}$ is an ℓ-adic unit.

We have proven the following theorem.

Theorem 6.5. *Let the notation and assumptions be as in Assumption 6.1. If for some Hecke character τ as above of conductor N the value*

$$\begin{aligned} M := {} & \mathrm{val}_\varpi(\Omega_{1,2}L^{N,\mathrm{alg}}(2+k/2,f_1\times f_2)) \\ & - \mathrm{val}_\varpi(L^N(3-k/2,\tau)L^{N,\mathrm{alg}}(4,f_1\times f_2)) \end{aligned} \tag{6.6}$$

is positive, then there exists a Siegel modular form G with Fourier coefficients in $\overline{\mathcal{O}}$ such that

$$Y(f_1\otimes f_2)\equiv G \pmod{\varpi^M},$$

where G is an E-linear combination of eigenforms whose associated Galois representations are irreducible.

Remark 6.6. Let the notation and assumptions be as in Assumption 6.1. Suppose it is possible to find τ such that the ratio $L^N(3-k/2,\tau)L^{N,\mathrm{alg}}(4,f_1\times f_2))/\Omega_{1,2}$ is an ℓ-adic unit. Then the congruence in Theorem 6.5 is modulo ϖ^M, where $M := \mathrm{val}_\varpi(L^{N,\mathrm{alg}}(2+k/2,f_1\times f_2))$.

Corollary 6.7. *Let the notation and assumptions be as in Assumption 6.1. Let M be as in Theorem 6.5. If $M>0$, then there exists a Siegel modular form G, which is an eigenform away from Σ with Fourier coefficients that lie in $\overline{\mathcal{O}}$ such that*

- *$Y(f_1\otimes f_2)\equiv G$ mod ϖ;*
- *the Galois representation associated to G is irreducible.*

Proof. Let S be the set of mutually orthogonal eigenforms (away from Σ) with Fourier coefficients in $\overline{\mathcal{O}}$ which are congruent to $Y(f_1\otimes f_2)$ mod ϖ. If none of the forms F_i $(i>0)$ in (6.4) is in S, then it follows from the decomposition (7.2) that there exists a Hecke operator $T_0\in\mathbf{T}^{\Sigma,S}_{\mathcal{O}}$ such that $T_0F_0=F_0$ and $TF_i=0$ for all $i>0$. Applying this operator to the congruence $F_0\equiv G$ (mod ϖ^n) with G as in Theorem 6.5 and keeping in mind that G is an E-linear combination of the F_i $(i>0)$, we get $F_0\equiv 0$ (mod ϖ), which yields a contradiction by Theorem 5.4. □

Note that it is not necessarily true that there exists an eigenform congruent to $Y(f_1\otimes f_2)$ mod ϖ^M. However, one can rephrase Theorem 6.5 in terms of congruences of Hecke eigenvalues rather than Fourier coefficients, as we do below. Write Y for the subspace of $S^S_{k/2+2}(N)$ spanned by common eigenforms F for $\mathbf{T}^{\Sigma,S}_{\mathcal{O}}$ such that

$$L^\Sigma_{\mathrm{spin}}(s,F)=L^\Sigma(s-k/2,f)L^\Sigma(s,g)$$

for some $f\in\mathcal{N}^{(2)}$ and $g\in\mathcal{N}^{(k+2)}$ (for notation see section 7).

Remark 6.8. By definition Y and its orthogonal component $Y^\perp$ are Hecke-stable subspaces. By Proposition 7.9 the eigenforms in Y are exactly those whose associated Galois representations are reducible.

Denote by $\mathbf{T}^Y_{\mathcal{O}}$ the image of $\mathbf{T}^{\Sigma,S}_{\mathcal{O}}$ inside $\mathrm{End}_{\mathcal{O}}(Y^\perp)$ and let $\phi:\mathbf{T}^{\Sigma,S}_{\mathcal{O}}\twoheadrightarrow\mathbf{T}^Y_{\mathcal{O}}$ be the canonical projection. Let $\mathrm{Ann}(Y(f_1\otimes f_2))\subset\mathbf{T}^{\Sigma,S}_{\mathcal{O}}$ denote the annihilator of $Y(f_1\otimes f_2)$. It is a prime ideal of $\mathbf{T}^{\Sigma,S}_{\mathcal{O}}$ and the map $\lambda_Y:\mathbf{T}^{\Sigma,S}_{\mathcal{O}}\twoheadrightarrow\mathcal{O}$ sending each operator T to its eigenvalue corresponding to $Y(f_1\otimes f_2)$ induces an $\mathcal{O}$-algebra isomorphism $\mathbf{T}^{\Sigma,S}_{\mathcal{O}}/\mathrm{Ann}(Y(f_1\otimes f_2))\xrightarrow{\sim}\mathcal{O}$.

Definition 6.9. As ϕ is surjective, $I_{f_1,f_2}:=\phi(\mathrm{Ann}(Y(f_1\otimes f_2)))$ is an ideal of $\mathbf{T}^Y_{\mathcal{O}}$. We will call it the *Yoshida ideal* associated to $Y(f_1\otimes f_2)$.

There exists a non-negative integer r for which the diagram

$$\begin{array}{ccc}
\mathbf{T}^{\Sigma,S}_{\mathcal{O}} & \xrightarrow{\phi} & \mathbf{T}^Y_{\mathcal{O}} \\
\downarrow & & \downarrow \\
\mathbf{T}^{\Sigma,S}_{\mathcal{O}}/\mathrm{Ann}(Y(f_1\otimes f_2)) & \xrightarrow{\phi} & \mathbf{T}^Y_{\mathcal{O}}/I_{f_1,f_2} \\
\lambda_Y\downarrow\wr & & \downarrow\wr \\
\mathcal{O} & \longrightarrow & \mathcal{O}/\varpi^r\mathcal{O}
\end{array} \tag{6.7}$$

all of whose arrows are $\mathcal{O}$-algebra epimorphisms, commutes.

Corollary 6.10. *Let the notation and assumptions be as in Assumption 6.1. If r is the integer from diagram (6.7), and M is as in Theorem 6.5, then $r \geq M$.*

Proof. Choose any $T \in \phi^{-1}(\varpi^r) \subset \mathbf{T}_{\mathcal{O}}^{\Sigma,S}$. Suppose that $r < M$, and let G be as in Theorem 6.5. We have

$$Y(f_1 \otimes f_2) \equiv G \pmod{\varpi^M}. \tag{6.8}$$

and $TG = \varpi^r G$. Hence applying T to both sides of (6.8), we obtain $0 \equiv \varpi^r G \pmod{\varpi^M}$, which leads to

$$G \equiv 0 \pmod{\varpi^{M-r}}. \tag{6.9}$$

Since $r < M$, (6.8) and (6.9) imply that $Y(f_1 \otimes f_2) \equiv 0 \pmod{\varpi}$, which gives a contradiction by Theorem 5.4. □

Remark 6.11. By Definition 6.9 and Remark 6.8 the Yoshida ideal measures Hecke-eigenvalue congruences (away from Σ) between the Yoshida lift $Y(f_1 \otimes f_2)$ and eigenforms whose associated Galois representations are irreducible. It can be thought of as an analogue of the classical Eisenstein ideal. See also [25] for a related notion of a CAP ideal.

7. The Hecke operator T^S

The goal of this section is to construct the Hecke operator T^S used in the previous section. In this section we fix an odd prime ℓ. Let E be a sufficiently large finite extension of $\mathbf{Q}_\ell$ and write $\mathcal{O}$ for its ring of integers. We fix a choice of a uniformizer $\varpi \in \mathcal{O}$. Let N be a prime, $k \in \{2, 4, 6, 8, 12\}$. Note that both $S_2(N)$ and $S_{k+2}(N)$ have bases consisting of newforms. These bases are unique and we denote them by $\mathcal{N}^{(2)}$ and $\mathcal{N}^{(k+2)}$ respectively. Let Σ be a finite set of rational primes. Write $\mathbf{T}_{\mathbf{Z}}^{\Sigma,(n)}$ for the $\mathbf{Z}$-subalgebra of $\mathrm{End}_{\mathbf{C}}(S_n(N))$ generated by $\{T_l \mid l \not\in \Sigma\}$. For any $\mathbf{Z}$-algebra A we set $\mathbf{T}_{A}^{\Sigma,(n)} := \mathbf{T}_{\mathbf{Z}}^{\Sigma,(n)} \otimes_{\mathbf{Z}} A$. It follows that $\mathbf{T}_{\mathcal{O}}^{\Sigma,(n)}$ is a semi-local complete finite $\mathcal{O}$-algebra. One has

$$\mathbf{T}_{\mathcal{O}}^{\Sigma,(n)} = \prod_{\mathfrak{m}} \mathbf{T}_{\mathfrak{m}}^{\Sigma,(n)},$$

where the product runs over all the maximal ideals of $\mathbf{T}_{\mathcal{O}}^{\Sigma,(n)}$ and $\mathbf{T}_{\mathfrak{m}}^{\Sigma,(n)}$ denotes the localization of $\mathbf{T}_{\mathcal{O}}^{\Sigma,(n)}$ at $\mathfrak{m}$. Moreover, one has

$$\mathbf{T}_{E}^{\Sigma,(n)} = \prod_{f \in \mathcal{N}^{(n)}} E. \tag{7.1}$$

Here and below we write n for 2 or $k+2$, where k is as above. Every $f \in \mathcal{N}^{(n)}$ defines an $\mathcal{O}$-algebra map $\lambda_f : \mathbf{T}_{\mathcal{O}}^{\Sigma,(n)} \to \mathcal{O}$ sending T to its eigenvalue corresponding to f. Fix such an f. Then the kernel of the map

$$\mathbf{T}_{\mathcal{O}}^{\Sigma,(n)} \to \mathcal{O} \twoheadrightarrow \mathbf{F}$$

is a maximal ideal, say $\mathfrak{m}_f$, of $\mathbf{T}_{\mathcal{O}}^{\Sigma,(n)}$. Set

$$\mathcal{N}_f^{(n)} := \{g \in \mathcal{N}^{(n)} \mid \mathfrak{m}_g = \mathfrak{m}_f\}.$$

In other words $\mathcal{N}_f^{(n)}$ consists of the newforms whose Hecke eigenvalues are congruent to those of f for all T_l, $l \not\in \Sigma$. We can identify $\mathbf{T}_{\mathfrak{m}_f}^{\Sigma,(n)}$ with the image of $\mathbf{T}_{\mathcal{O}}^{\Sigma,(n)}$ inside $\mathrm{End}_{\mathbf{C}}(\mathrm{span}\{\mathcal{N}_f^{(n)}\})$ (note that we have fixed an embedding $E \hookrightarrow \mathbf{C}$). In particular

$$\mathbf{T}_{\mathfrak{m}_f}^{\Sigma,(n)} \otimes_{\mathcal{O}} E = \prod_{f \in \mathcal{N}_f^{(n)}} E.$$

Similarly, write $\mathbf{T}_{\mathbf{Z}}^{\Sigma,S}$ for the $\mathbf{Z}$-subalgebra of $\mathrm{End}_{\mathbf{C}}(S_{k/2+2}^S(N))$ generated by

$$\mathcal{T}_\Sigma := \{T(1,1,l,l), T(1,l,l,l^2), T(l,l,l,l) \mid l \not\in \Sigma\}.$$

For any $\mathbf{Z}$-algebra A, set $\mathbf{T}_A^{\Sigma,S} = \mathbf{T}_{\mathbf{Z}}^{\Sigma,S} \otimes_{\mathbf{Z}} A$. It follows that $\mathbf{T}_{\mathcal{O}}^{\Sigma,S}$ is a semi-local complete finite $\mathcal{O}$-algebra. One has

$$\mathbf{T}_{\mathcal{O}}^{\Sigma,S} = \prod_{\mathfrak{m}} \mathbf{T}_{\mathfrak{m}}^{\Sigma,S}, \tag{7.2}$$

where the product runs over all the maximal ideals of $\mathbf{T}_{\mathcal{O}}^{\Sigma,S}$ and $\mathbf{T}_{\mathfrak{m}}^{\Sigma,S}$ denotes the localization of $\mathbf{T}_{\mathcal{O}}^{\Sigma,S}$ at $\mathfrak{m}$.

Proposition 7.1. *Let $a(h,F)$ be any Fourier coefficient of $F \in S_{k/2+2}^S(N)$. If $a(h,F) \in \mathcal{O}$, then for any $t \in \mathbf{T}_{\mathcal{O}}^{\Sigma,S}$, the corresponding Fourier coefficient $a(h,tF)$ of tF also lies in $\mathcal{O}$.*

Proof. This follows from explicit formulas for Fourier coefficients of a Siegel modular form acted upon by the operators in $\mathcal{T}_\Sigma$ - see for example, [10], Lemma II.10. □

Theorem 7.2. *Assume $N \in \Sigma$. The space $S_{k/2+2}^S(N)$ has an orthogonal basis consisting of common eigenfunctions of $\mathbf{T}_{\mathbf{C}}^{\Sigma,S}$.*

Proof. This is Theorem 1.9 on page 233 in [2]. The key fact is Proposition 6.14 in [23]. □

Theorem 7.3. *Let $F \in S_{k/2+2}^S(N)$ be a common eigenform for all the operators in $\mathbf{T}_{\mathcal{O}}^{\Sigma,S}$. If $T \in \mathbf{T}_{\mathcal{O}}^{\Sigma,S}$ and $\lambda \in \mathbf{C}$ is an eigenvalue of T associated to F, then $\lambda \in \mathcal{O}$.*

Proof. This may be seen for example from a theorem of Weissauer (see Theorem 8.1), because the eigenvalues of the Hecke operators away from Σ coincide with the eigenvalues of the Frobenii, which in turn are roots of their respective characteristic polynomials. Since (by compactness of $G_{\mathbf{Q}}$) one can conjugate the Galois representation to have image in $\mathrm{GL}_4(\mathcal{O})$, these coefficients must also lie in $\mathcal{O}$. □

Using Theorem 7.3, we can state similar results on the structure of the Hecke algebra as we did above for the elliptic modular forms. In particular, let $\mathcal{N}^S \subset S_{k/2+2}^S(N)$ be a basis consisting of common eigenforms of all the operators in $\mathbf{T}_{\mathcal{O}}^{\Sigma,S}$. As before if we write $\mathfrak{m}_F$ for the maximal ideal of $\mathbf{T}_{\mathcal{O}}^{\Sigma,S}$ corresponding to F, we can define $\mathcal{N}_F^S := \{G \in \mathcal{N}^S \mid \mathfrak{m}_G = \mathfrak{m}_F\}$. Then we can identify $\mathbf{T}_{\mathfrak{m}_F}^{\Sigma,S}$ with the image of $\mathbf{T}_{\mathcal{O}}^{\Sigma,S}$ inside $\mathrm{End}_{\mathbf{C}}(\mathrm{span}\{\mathcal{N}_F^S\})$.

Let $F \in S_{k/2+2}^S(N)$ be a common eigenform for all the operators in $\mathbf{T}_{\mathcal{O}}^{\Sigma,S}$.

Proposition 7.4. *There exists a continuous semi-simple representation*

$$\rho_F : G_{\mathbf{Q}} \to \mathrm{GL}_4(E)$$

unramified away from ℓ and N such that for a prime $l \notin \Sigma \cup \{\ell, N\}$ the characteristic polynomial $f(X)$ of $\rho_F(\mathrm{Frob}_l)$ coincides with the polynomial

$$1 - t_0 X + \{l t_1 + l(l^2+1)t_2\}X^2 - l^3 t_0 t_2 X^3 + l^6 t_2^2 X^4,$$

where $t_0 = \lambda_F(T(1,1,l,l))$, $t_1 = \lambda_F(T(1,l,l,l^2))$ and $t_2 = \lambda_F(T(l,l,l,l))$. Here λ_F is the map $\mathbf{T}_{\mathcal{O}}^{\Sigma,S} \to \mathcal{O}$ sending $t \in \mathbf{T}_{\mathcal{O}}^{\Sigma,S}$ to its eigenvalue corresponding to F.

Proof. This follows from a result of Weissauer (see Theorem 8.1), which assigns such a representation to a common eigenform of $\mathbf{T}_{\mathcal{O}}^{\emptyset,S}$ and Proposition 7.5 below. □

Proposition 7.5. *Let $F \in S_{k/2+2}^S(N)$ be a common eigenform for all the operators in $\mathbf{T}_{\mathcal{O}}^{\Sigma,S}$. There exists $F' \in S_{k/2+2}^S(N)$, which is a common eigenform for all the operators in $\mathbf{T}_{\mathcal{O}}^{\emptyset,S}$ such that for every $T \in \mathbf{T}_{\mathcal{O}}^{\Sigma,S}$, the eigenvalue of T corresponding to F' agrees with the eigenvalue of T corresponding to F.*

Proof. Proposition 7.5 follows immediately from the fact that $S_{k/2+2}^S(N)$ is finite-dimensional, $\mathbf{T}_{\mathcal{O}}^{\emptyset,S}$ is commutative ([33], Lemma 11.12(1)) and Lemma 7.6 below, where we take $\mathcal{T}$ to consist of the (finitely many) generators of the local Hecke algebras at primes in Σ. □

Lemma 7.6. *Let $\mathcal{R} = \mathcal{S} \sqcup \mathcal{T}$ be a family of commuting linear operators on a finite dimensional $\mathbf{C}$-vector space V. Assume $\#\mathcal{T} < \infty$. Let $v \in V$ be a common eigenvector for all operators in $\mathcal{S}$. Then there exists $w \in V$, which is a common eigenvector for all operators in $\mathcal{R}$ such that for every $S \in \mathcal{S}$, the eigenvalue of S corresponding to w agrees with the eigenvalue of S corresponding to v.*

Proof. Let v be as above. Note that it is enough to prove the lemma in the case when $\mathcal{T}$ consists of single operator. Indeed, then the general case can be proved by induction on $n := \#\mathcal{T}$. More precisely, if $\#\mathcal{T} = n+1$, $\mathcal{T} = \{T_1, \dots, T_{n+1}\}$, set $\mathcal{S}' := \mathcal{S} \cup \{T_1\}$ and $\mathcal{T}' := \{T_2, \dots, T_{n+1}\}$. First apply the inductive hypothesis to $\{T_1\}$, get $v' \in V$, a common eigenvector for $\mathcal{S}'$ and then apply it again to $\mathcal{T}'$. So, assume $\mathcal{T} = \{T\}$. Note that commutativity of $\mathcal{R}$ implies that for every $S \in \mathcal{S}$ and every integer $i \geq 0$, we have $ST^i v = T^i S v = T^i \lambda v = \lambda T^i v$, where λ is the eigenvalue of S corresponding to v. Hence every $w \in W := \mathrm{span}\{T^i v\}_{i=0}^{\infty}$ is a common eigenvector for all $S \in \mathcal{S}$ such that the eigenvalue of S corresponding to w agrees with the eigenvalue of S corresponding to v. Moreover, note that W is T-stable. Consider $T|_W : W \to W$. Since we are working over $\mathbf{C}$, the characteristic polynomial of $T|_W$ has a root, and thus T has an eigenvector $w \in W$. □

From now on fix $f_1 \in \mathcal{N}^{(2)}$ and $f_2 \in \mathcal{N}^{(k+2)}$. Let η_1 (resp. η_2) be a generator of the Hida congruence ideal for f_1 (resp. f_2).

Definition 7.7. Write Y for the subspace of $S_{k/2+2}^S(N)$ spanned by common eigenforms F for $\mathbf{T}_{\mathcal{O}}^{\Sigma,S}$ such that

$$L_{\mathrm{spin}}^{\Sigma}(s,F) = L^{\Sigma}(s-k/2,f)L^{\Sigma}(s,g)$$

for some $f \in \mathcal{N}^{(2)}$ and $g \in \mathcal{N}^{(k+2)}$. Write Y_{f_1,f_2} for the subspace of Y spanned by common eigenforms F for $\mathbf{T}_{\mathcal{O}}^{\Sigma,S}$ such that

$$L^{\Sigma}_{\rm spin}(s,F) = L^{\Sigma}(s-k/2, f_1)L^{\Sigma}(s,f_2).$$

Remark 7.8. Note that Y contains the image of the Yoshida lift. Also note that equivalently Y is the subspace of $S^S_{k/2+2}(N)$ spanned by common eigenforms F for $\mathbf{T}_{\mathcal{O}}^{\Sigma,S}$ such that $\rho_F = (\rho_f \otimes \epsilon^{k/2}) \oplus \rho_g$, where ρ_F is the 4-dimensional semi-simple ℓ-adic Galois representation attached to F as in Proposition 7.4, while ρ_f, (resp. ρ_g) denotes the 2-dimensional irreducible ℓ-adic Galois representation attached to f (resp. g) by Eichler-Shimura, Deligne. This follows from the Chebotarev Density Theorem and the Brauer-Nesbitt Theorem as if F is a common eigenform in Y then for all but finitely many primes l the characteristic polynomials of $\rho_F(\mathrm{Frob}_l)$ and of $(\rho_f(\mathrm{Frob}_l) \otimes \epsilon^{k/2}(\mathrm{Frob}_l)) \oplus \rho_g(\mathrm{Frob}_l)$ coincide for some f and g because of the L-function equality. Then the Brauer-Nesbitt Theorem implies that $\rho_F \cong (\rho_f \otimes \epsilon^{k/2}) \oplus \rho_g$ since ρ_F is semisimple, while f and g are cusp forms, so ρ_f and ρ_g are irreducible.

Proposition 7.9. *Suppose that f_1 and f_2 are ordinary at ℓ. Assume the ℓ-adic Galois representations attached to f_1 and f_2 are residually irreducible when restricted to $G_{\mathbf{Q}(\sqrt{(-1)^{(\ell-1)/2}\ell})}$. Let $F \in \mathcal{N}^S$ be such that the Hecke eigenvalues of F are congruent to those of $Y(f_1 \otimes f_2)$ for all the Hecke operators in $\mathbf{T}_{\mathcal{O}}^{\Sigma,S}$. Furthermore, assume that ρ_F is reducible. Then $F \in Y$.*

Proof. Let $\rho_F = \sigma_1 \oplus \sigma_2$ be as in the statement of the theorem. Note that there is no loss of generality in assuming that ρ_F has this form, as ρ_F is semi-simple by Proposition 7.4 and we do not assume that the representations σ_1, σ_2 are irreducible. Write $\overline{\rho}_F$ and $\overline{\rho}_{Y(f_1\otimes f_2)}$ for the reductions (mod ϖ) of the ℓ-adic Galois representations attached to F and $Y(f_1 \otimes f_2)$ respectively. Since the characteristic polynomials of $\overline{\rho}_F(\mathrm{Frob}_l)$ and of $\overline{\rho}_{Y(f_1\otimes f_2)}(\mathrm{Frob}_l)$ agree for $l \not\in \Sigma \cup \{\ell, N\}$, they agree on $G_{\mathbf{Q}}$ by the Chebotarev Density Theorem. Hence the Brauer-Nesbitt Theorem implies that the semisimplifications of $\overline{\rho}_F$ and $\overline{\rho}_{Y(f_1\otimes f_2)}$ are isomorphic. Both of the representations are semi-simple by Proposition 7.4. Hence both $\overline{\sigma}_1$ and $\overline{\sigma}_2$ must be 2-dimensional and irreducible as well and without loss of generality we can assume that $\overline{\sigma}_1 \cong \overline{\rho}_{f_1} \otimes \epsilon^{k/2}$ and $\overline{\sigma}_2 \cong \overline{\rho}_{f_2}$. Furthermore, by choosing the right bases we can take the isomorphisms to be equalities. This implies that σ_2 (and σ_1 after twisting) is a deformation of ρ_{f_2} (resp. ρ_{f_1}) unramified away from $N\ell$ and crystalline at ℓ as a subrepresentation of a crystalline representation ρ_F (see Theorem 8.1 (ii)). Then our assumptions imply that σ_1 and σ_2 are modular ([16], Theorem 0.3 - note that ρ_F is geometric in the sense of Fontaine [19]). Hence $\sigma_1 = \rho_{g_1} \otimes \epsilon^{k/2}$, $\sigma_2 = \rho_{g_2}$ for some modular forms g_1 and g_2 of correct weight and level. Thus $F \in Y$. □

Assumption 7.10. There exist $T^1 \in \mathbf{T}_{\mathcal{O}}^{\Sigma,(2)}$ and $T^2 \in \mathbf{T}_{\mathcal{O}}^{\Sigma,(k+2)}$ such that $T^1 f_1 = \eta_1 f_1$, $T^2 f_2 = \eta_2 f_2$ and $T^1 f = 0$ for all $f \in S_2(N)$ orthogonal to f_1 and $T^2 f = 0$ for all $f \in S_{k+2}(N)$ orthogonal to f_2.

Remark 7.11. Suppose that f_1 and f_2 are ordinary at ℓ or that $N > 4$. Then Assumption 7.10 is satisfied for $\Sigma = \emptyset$ by the definition of η_1 and η_2.

Proposition 7.12. *Suppose that f_1 and f_2 are ordinary at ℓ and that the ℓ-adic Galois representations attached to f_1 and f_2 are residually irreducible. Suppose that*

Assumption 7.10 is satisfied for a finite set of primes Σ. Then it is also satisfied for the set $\Sigma \cup \{\ell\}$.

Proof. Let's just show the statement for f_2. The proof for f_1 is the same. If $\ell \in \Sigma$, there is nothing to prove, so assume $\ell \not\in \Sigma$. Since Assumption 7.10 is satisfied for Σ, T^2 with desired properties exists in this case. Let S be any subset of the set of maximal ideals of $\mathbf{T}_{\mathcal{O}}^{\Sigma,(k+2)}$ containing the maximal ideal corresponding to f_2. Let $T_S^2 \in \mathbf{T}_{\mathcal{O}}^{\Sigma,(k+2)} = \prod_{\mathfrak{m}} \mathbf{T}_{\mathfrak{m}}^{\Sigma,(k+2)}$ be the operator $(0,0,\dots,0,T^2,T^2,\dots,T^2,0,\dots,0)$ where T^2 (or rather its image in $\mathbf{T}_{\mathfrak{m}}^{\Sigma,(k+2)}$) occurs at the places corresponding to $\mathfrak{m} \in S$, and zeroes for the other maximal ideals. Then T_S^2 also has the property of multiplying f_2 by η_2 and annihilating all the other newforms. We will regard T_S^2 as lying in $\prod_{\mathfrak{m}\in S} \mathbf{T}_{\mathfrak{m}}^{\Sigma,(k+2)}$.

From now on let $\Sigma' = \Sigma \cup \{\ell\}$. Let $\mathfrak{m}_{f_2}$ denote the maximal ideal of $\mathbf{T}_{\mathcal{O}}^{\Sigma',(k+2)}$ corresponding to f_2. The inclusion $\mathbf{T}_{\mathcal{O}}^{\Sigma',(k+2)} \hookrightarrow \mathbf{T}_{\mathcal{O}}^{\Sigma,(k+2)}$ descends to an inclusion

$$\mathbf{T}_{\mathfrak{m}_{f_2}}^{\Sigma',(k+2)} \hookrightarrow R := \prod_{\mathfrak{m}\in S} \mathbf{T}_{\mathfrak{m}}^{\Sigma,(k+2)},$$

where S denotes the subset of the set of maximal ideals $\mathfrak{m}$ of $\mathbf{T}_{\mathcal{O}}^{\Sigma,(k+2)}$ such that $\mathfrak{m} \cap \mathbf{T}_{\mathcal{O}}^{\Sigma',(k+2)} = \mathfrak{m}_{f_2}$. We will denote the image of T_ℓ in R also by T_ℓ. We need to show that T_S^2 (which a priori lies in R) lies in $\mathbf{T}_{\mathfrak{m}_{f_2}}^{\Sigma',(k+2)}$. Then extending T_S^2 by zeroes at the other maximal ideals of $\mathbf{T}_{\mathcal{O}}^{\Sigma',(k+2)}$ we get an operator in $\mathbf{T}_{\mathcal{O}}^{\Sigma',(k+2)}$ with the same properties as T^2. It is enough to show that T_ℓ (which a priori lies in R) lies in $\mathbf{T}_{\mathfrak{m}_{f_2}}^{\Sigma',(k+2)}$. Indeed, T^2 is polynomial in T_l with $l \not\in \Sigma$, so if we denote the image of T_l in R also by T_l, then T_S^2 is a polynomial in T_l with $l \not\in \Sigma$. However, $T_l \in \mathbf{T}_{\mathfrak{m}_{f_2}}^{\Sigma',(k+2)}$ for $l \not\in \Sigma$, $l \neq \ell$, so we just need to deal with T_ℓ.

Let $\mathcal{N}$ denote the subset of $\mathcal{N}^{(k+2)}$ consisting of newforms $g = \sum_{n=1}^\infty a_n(g)q^n$ whose Hecke eigenvalues $\lambda_g(T_l) = a_l(g)$ are congruent (mod ϖ) to the Hecke eigenvalues of f_2 for all T_l, $l \not\in \Sigma'$. First note that since f_2 is ordinary so is every such g. Indeed, let ρ_{f_2}, ρ_g denote the ℓ-adic Galois representations attached to f_2 and g respectively and write $\overline{\rho}_{f_2}$ and $\overline{\rho}_g$ for the residual representations. Since the Hecke eigenvalues of f_2 and g are congruent for all T_l, $l \not\in \Sigma'$ and Σ' is finite, we have $\operatorname{tr}\overline{\rho}_{f_2} = \operatorname{tr}\overline{\rho}_g$ by the Chebotarev Density Theorem. Since $\overline{\rho}_{f_2}$ is irreducible, Brauer-Nesbitt Theorem implies that $\overline{\rho}_{f_2} \cong \overline{\rho}_g$. Since f_2 is ordinary, $\overline{\rho}_{f_2}|_{D_\ell}$ is reducible. Hence $\overline{\rho}_g|_{D_\ell}$ must also be reducible, so by a theorem of Fontaine ([17], section 6) g must be ordinary as well.

By ordinarity

$$\rho_g|_{D_\ell} \cong \begin{bmatrix} \epsilon^{k+1}\chi_{1,g} & * \\ & \chi_{2,g} \end{bmatrix},$$

where $\chi_{1,g}$, $\chi_{2,g}$ are unramified and $\chi_{2,g}(\mathrm{Frob}_\ell) = a_\ell(g)$. We can identify $\mathbf{T}_{\mathfrak{m}_{f_2}}^{\Sigma',(k+2)}$ with R' the subalgebra of $\mathbf{T}_{\mathcal{O}}^{\Sigma',(k+2)}$ generated by $(a_l(g))_{g\in\mathcal{N}, l\not\in\Sigma'}$. Choose a basis of the Galois representation ρ_g for every such g so that $\rho_g|_{D_\ell}$ is of the above form. Then the product over all such g gives a representation $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\prod_{g\in\mathcal{N}} \mathcal{O})$. One has $(\chi_{2,g}(\mathrm{Frob}_\ell))_g = T_\ell$. We want to show that $T_\ell \in R'$. Let F denote any lift of Frob_ℓ to D_ℓ and choose $\sigma \in I_\ell$ to be such that $\epsilon^{k+1}(\sigma) = -1$. Since ϵ gives a surjection of I_ℓ onto $\mathbf{Z}_\ell^\times$ this is possible since $k+1$ is odd (i.e., if $\sigma \in I_\ell$ maps to -1 via ϵ, then $\epsilon^{k+1}(\sigma) = -1$). Then $T_\ell = \frac{1}{2}(\operatorname{tr}\rho(F\sigma) + \operatorname{tr}\rho(F))$. Now set τ to be either F or $F\sigma$. Since $G_{\mathbf{Q}}$ is generated by conjugates of Frobenii away from Σ' and ρ is continuous, we know that $\operatorname{tr}\rho(\tau) \in R$ is the limit of $\operatorname{tr}\rho(\mathrm{Frob}_l) \in R'$ for $l \notin \Sigma'$. But for $l \notin \Sigma'$, $\operatorname{tr}\rho(\mathrm{Frob}_l) = T_l \in R'$. By completeness of R', we get that $\operatorname{tr}\rho(\tau) \in R'$, so $T_\ell \in R'$. □

Proposition 7.13. *Suppose $\ell \nmid (N+1)$. Suppose that Assumption 7.10 is satisfied for a finite set Σ. Then it is also satisfied for the set $\Sigma \cup \{N\}$.*

Proof. Since N is prime, and the character of f_j, $j = 1, 2$ is trivial, we get by a result of Langlands (see for example, [15], Theorem 3.1(e) for weight 2 or [22], Theorem 3.26(3b) for an arbitrary weight) that for $j = 1, 2$,

$$\rho_{f_j}|_{D_N} \cong \begin{bmatrix} \chi\epsilon & * \\ & \chi \end{bmatrix},$$

where $\chi : D_N \to E^\times$ is the unique unramified character such that $\chi(\mathrm{Frob}_N) = a_{f_j}(N)$, where $a_{f_j}(N)$ is the eigenvalue of T_N corresponding to f_j. Thus for $j = 1, 2$, $\operatorname{tr}\rho_{f_j}(\mathrm{Frob}_N)$ is well-defined and one has

$$\operatorname{tr}\rho_{f_j}(\mathrm{Frob}_N) = (N+1)a_{f_j}(N).$$

This, as in the proof of Proposition 7.12, and the fact that $\ell \nmid (N+1)$ imply that $T_N \in \mathbf{T}^{\Sigma\cup\{N\},(2)}_{\mathfrak{m}_{f_j}}$, $j = 1, 2$. □

Corollary 7.14. *Assumption 7.10 is satisfied for any finite set of primes Σ provided we assume f_1 and f_2 are ordinary at ℓ and their Galois representations are residually irreducible, if $\ell \in \Sigma$ and $\ell \nmid (N+1)$, if $N \in \Sigma$.*

Proof. Let Σ be a finite set of primes and set $\Sigma' = \Sigma \setminus \{\ell, N\}$. Then as T_l is just the trace of Frobenius at l for $l \in \Sigma'$, a completeness argument as in the proof of Proposition 7.12 gives the result for Σ'. Hence the corollary follows from Propositions 7.12 and 7.13. □

Theorem 7.15. *Assume $N, \ell \in \Sigma$ and that $\ell > k$. Suppose Assumption 7.10 holds and that the residual Galois representations attached to f_1 and f_2 are irreducible. There exists $T \in \mathbf{T}^{\Sigma,S}_{\mathcal{O}}$ such that $TF = \eta_1\eta_2 F$ for all $F \in Y_{f_1,f_2}$ and $TF = 0$ for every $F \in Y$ orthogonal to Y_{f_1,f_2}.*

Proof. Consider the map $\mathbf{T}^S_{\mathbf{C}} \to \mathbf{T}^{(2)}_{\mathbf{C}} \otimes \mathbf{T}^{(k+2)}_{\mathbf{C}}$ as in Theorem 5.9. It descends to an $\mathcal{O}$-algebra homomorphism (which we also denote by Φ):

$$\Phi : \mathbf{T}^{\Sigma,S}_{\mathcal{O}} \to \mathbf{T}^{\Sigma,(2)}_{\mathcal{O}} \otimes \mathbf{T}^{\Sigma,(k+2)}_{\mathcal{O}}.$$

One has

$$\mathbf{T}^{\Sigma,(2)}_{\mathcal{O}} \otimes \mathbf{T}^{\Sigma,(k+2)}_{\mathcal{O}} = \prod_{\mathfrak{m}} \mathbf{T}^{\Sigma,(2)}_{\mathfrak{m}} \otimes \prod_{\mathfrak{n}} \mathbf{T}^{\Sigma,(k+2)}_{\mathfrak{n}} = \prod_{\mathfrak{m},\mathfrak{n}} \mathbf{T}^{\Sigma,(2)}_{\mathfrak{m}} \otimes \mathbf{T}^{\Sigma,(k+2)}_{\mathfrak{n}}, \tag{7.3}$$

where $\mathfrak{m}$ denotes a maximal ideal of $\mathbf{T}^{\Sigma,(2)}_{\mathcal{O}}$ and $\mathfrak{n}$ denotes a maximal ideal of $\mathbf{T}^{\Sigma,(k+2)}_{\mathcal{O}}$.

Proposition 7.16. *Suppose $\ell > k$. Let l be a prime such that $l \notin \Sigma$. There exists $S, T \in \mathbf{T}^{\Sigma,S}_{\mathcal{O}}$ such that $\Phi(S) = T_l \otimes 1$ and $\Phi(T) = 1 \otimes T_l$.*

Remark 7.17. Note that using the diagram from Theorem 5.9, Proposition 7.16 implies that $SY(f\otimes g)=Y(T_l f\otimes g)$ and $TY(f\otimes g)=Y(f\otimes T_l g)$ for the Yoshida lift $Y(f\otimes g)$ of $f\otimes g\in S_2(N)\otimes S_{k+2}(N)$ with f and g eigenforms. However, Y a priori contains more common eigenforms than those whose Hecke eigenvalues coincide with the Hecke eigenvalues of Yoshida lifts.

In other words the map $\Phi:\mathbf{T}^{\Sigma,S}_{\mathcal{O}}\to\mathbf{T}^{\Sigma,(2)}_{\mathcal{O}}\otimes\mathbf{T}^{\Sigma,(k+2)}_{\mathcal{O}}$ factors through the projection $\mathbf{T}^{\Sigma,S}_{\mathcal{O}}\twoheadrightarrow\mathbf{T}^{\Sigma,\mathrm{Yosh}}_{\mathcal{O}}$, where $\mathbf{T}^{\Sigma,\mathrm{Yosh}}_{\mathcal{O}}$ is the image of $\mathbf{T}^{\Sigma,S}_{\mathcal{O}}$ in the ring of $\mathbf{C}$-endomorphisms of the span of the Yoshida lifts inside $S^S_{k/2+2}(N)$. We will denote the resulting map $\mathbf{T}^{\Sigma,\mathrm{Yosh}}_{\mathcal{O}}\to\mathbf{T}^{\Sigma,(2)}_{\mathcal{O}}\otimes\mathbf{T}^{\Sigma,(k+2)}_{\mathcal{O}}$ also by Φ.

Before we prove Proposition 7.16 let us show how it implies Theorem 7.15. Let T^1, T^2 be as in Assumption 7.10. Then $T^1\otimes T^2\in\mathbf{T}^{\Sigma,(2)}_{\mathcal{O}}\otimes\mathbf{T}^{\Sigma,(k+2)}_{\mathcal{O}}$ is a polynomial in $T_l\otimes 1$, $1\otimes T_l$, $l\not\in\Sigma$ with coefficients in $\mathcal{O}$. Hence by Proposition 7.16, $T_1\otimes T_2$ is in the image of Φ. Choose $T\in\Phi^{-1}(T_1\otimes T_2)$. Then by commutativity of the diagram in Theorem 5.9, we see that $TY(f_1\otimes f_2)=\eta_1\eta_2 Y(f_1\otimes f_2)$, hence also $TF=\eta_1\eta_2F$ for any $F\in Y_{f_1,f_2}$ as $Y(f_1\otimes f_2)$ as such F has the same eigenvalues. On the other hand if $F\in Y$ is orthogonal to Y_{f_1,f_2}, then it is an E-linear combination of common eigenforms of $\mathbf{T}^{\Sigma,S}_{\mathcal{O}}$ orthogonal to Y_{f_1,f_2}. Fix such an eigenform F. Then $\rho_F=\rho_f\otimes\epsilon^{k/2}\oplus\rho_g$ for some $f\in\mathcal{N}^{(2)}$ and $g\in\mathcal{N}^{(k+2)}$.

Lemma 7.18. *One has $TF=0$.*

Proof. Because f, g and F are eigenforms for $\mathbf{T}^{\Sigma,(2)}_{\mathcal{O}}$, $\mathbf{T}^{\Sigma,(k+2)}_{\mathcal{O}}$ and $\mathbf{T}^{\Sigma,S}_{\mathcal{O}}$ respectively, each of them defines an $\mathcal{O}$-algebra homomorphism from its respective Hecke algebra to $\mathcal{O}$ sending an operator t to its eigenvalue. We will denote these homomorphisms by λ_f, λ_g and λ_F. The first two induce an $\mathcal{O}$-algebra homomorphism $\lambda_f\otimes\lambda_g:\mathbf{T}^{\Sigma,(2)}_{\mathcal{O}}\otimes\mathbf{T}^{\Sigma,(k+2)}_{\mathcal{O}}\to\mathcal{O}$, sending an operator $s\otimes t$ to $\lambda_f(s)\lambda_g(t)$. (Recall that the action of an element $s\otimes t\in\mathbf{T}^{\Sigma,(2)}_{\mathcal{O}}\otimes\mathbf{T}^{\Sigma,(k+2)}_{\mathcal{O}}$ on an element $h_1\otimes h_2\in S_2(N)\otimes S_{k+2}(N)$ is defined by $(s\otimes t)(h_1\otimes h_2)=(sh_1)\otimes(sh_2)$.) Let l be a prime, $l\not\in\Sigma$. We will show that if $s\in\{T(1,1,l,l),T(1,l,l,l^2),T(l,l,l,l)\}$ then $\lambda_F(s)=(\lambda_f\otimes\lambda_g)(\Phi(s))$. Since T is a polynomial in $T(1,1,l,l)$, $T(1,l,l,l^2)$, $T(l,l,l,l)$, $l\not\in\Sigma$, with coefficients in $\mathcal{O}$ and Φ is an $\mathcal{O}$-algebra map, that will clearly imply that

$$\lambda_F(T)=(\lambda_f\otimes\lambda_g)(\Phi(T))=(\lambda_f\otimes\lambda_g)(T_1\otimes T_2)=0.$$

Note that this would be obvious if we knew that $F=Y(f\otimes g)$, but we are not assuming this (cf. Remark 7.17). We have

$$\rho_F=\begin{bmatrix}\rho_f\otimes\epsilon^{k/2} & \\ & \rho_g\end{bmatrix}.$$

For $\sigma\in G_{\mathbf{Q}}$ write

$$f(\sigma)(X)=\sum_{n=0}^{4}a_n(\sigma)X^n=(1-(\epsilon^{k/2}(\sigma)\mathrm{tr}\,\rho_f(\sigma))X+(\epsilon^k(\sigma)\det\rho_f(\sigma))X^2)\times(1-(\mathrm{tr}\,\rho_g(\sigma))X+(\det\rho_g(\sigma))X^2) \tag{7.4}$$

for the characteristic polynomial of $\rho_F(\sigma)$. One has

$$f(\mathrm{Frob}_l)(X)=1-t_0X+\{lt_1+l(l^2+1)t_2\}X^2-l^3t_0t_2X^3+l^6t_2^2X^4,$$

where $t_0 = \lambda_F(T(1,1,l,l))$, $t_1 = \lambda_F(T(1,l,l,l^2))$ and $t_2 = \lambda_F(T(l,l,l,l))$. Hence we have

(7.5)
$$\lambda_F(T(1,1,l,l)) = -a_1 = \mathrm{tr}\,\rho_F(\mathrm{Frob}_l) = \epsilon^{k/2}(\mathrm{Frob}_l)\mathrm{tr}\,\rho_f(\mathrm{Frob}_l) + \mathrm{tr}\,\rho_g(\mathrm{Frob}_l) =$$
$$= l^{k/2}\lambda_f(T_l) + \lambda_g(T_l) = (\lambda_f \otimes \lambda_g)(l^{k/2}T_l \otimes 1 + 1 \otimes T_l) = (\lambda_f \otimes \lambda_g)(\Phi(T(1,1,l,l)).$$

Now note that
$$T(l,l,l,l)F = \mu(\mathrm{diag}(l,l,l,l))^{2(k/2+2)-3}\det(\mathrm{diag}(l,l))^{-(k/2+2)}F = l^{k-2}F$$
for any $F \in S^S_{k/2+2}(N)$ (see [10], p.27 or [2]). Hence (note that $l \neq \ell$)

$$(7.6)\quad \lambda_F(T(1,l,l,l^2)) = \lambda_F(l^{-1}a_2(\mathrm{Frob}_l) - (l^2+1)T(l,l,l,l)) =$$
$$= l^{-1}[\epsilon^k(\mathrm{Frob}_l)\det\rho_f(\mathrm{Frob}_l) + \det\rho_g(\mathrm{Frob}_l)] +$$
$$+ l^{-1}[(\epsilon^{k/2}(\mathrm{Frob}_l)\mathrm{tr}\,\rho_f(\mathrm{Frob}_l))(\mathrm{tr}\,\rho_g(\mathrm{Frob}_l))] - (l^2+1)l^{k-2} =$$
$$l^{-1}[l^{k+1} + l^{k+1} + l^{k/2}\lambda_f(T_l)\lambda_g(T_l)] - l^k - l^{k-2} =$$
$$= (\lambda_f \otimes \lambda_g)(l^k - l^{k-2} + T_l \otimes T_l) = (\lambda_f \otimes \lambda_g)(\Phi(T(1,l,l,l^2)).$$

Finally, the fact that
$$\lambda_F(T(l,l,l,l)) = (\lambda_f \otimes \lambda_g)(\Phi(T(l,l,l,l)))$$
follows directly from the fact that Φ is an $\mathcal{O}$-algebra homomorphism since $T(l,l,l,l)$ multiplies every Siegel modular form by a scalar. □

It remains to prove Proposition 7.16.

Proof of Proposition 7.16. We will just show the proof in case of S, the case of T being analogous. Since the Yoshida lifts are eigenforms for $\mathbf{T}^{\Sigma,S}_{\mathcal{O}}$, the span of all Yoshida lifts inside $S^S_{k/2+2}(N)$ is Hecke-invariant. Let $\mathbf{T}^{\Sigma,\mathrm{Yosh}}_{\mathcal{O}}$ denote the quotient of $\mathbf{T}^{\Sigma,S}_{\mathcal{O}}$ acting on the span of all Yoshida lifts. The map Φ factors through $\mathbf{T}^{\Sigma,\mathrm{Yosh}}_{\mathcal{O}}$ (cf. Remark 7.17). Obviously, we just need to construct $S \in \mathbf{T}^{\Sigma,\mathrm{Yosh}}_{\mathcal{O}}$ with the desired property.

Write $\mathfrak{m}_Y$ for the maximal ideal of $\mathbf{T}^{\Sigma,\mathrm{Yosh}}_{\mathcal{O}}$ corresponding to $Y(f_1 \otimes f_2)$. Since $\mathbf{T}^{\Sigma,\mathrm{Yosh}}_{\mathcal{O}}$ also decomposes as
$$\mathbf{T}^{\Sigma,\mathrm{Yosh}}_{\mathcal{O}} = \prod_{\mathfrak{m}} \mathbf{T}^{\Sigma,\mathrm{Yosh}}_{\mathfrak{m}},$$
where $\mathfrak{m}$ runs over its maximal ideals and $\mathbf{T}^{\Sigma,\mathrm{Yosh}}_{\mathfrak{m}}$ denotes localization, it is enough to construct $S \in \mathbf{T}^{\Sigma,\mathrm{Yosh}}_{\mathfrak{m}_Y}$. Write $S^{\mathrm{Yosh}} \subset S^S_{k/2+2}(N)$ for the span of $Y(f \otimes g)$ such that the maximal ideal of $\mathbf{T}^{\Sigma,\mathrm{Yosh}}_{\mathcal{O}}$ corresponding to $Y(f \otimes g)$ equals $\mathfrak{m}_Y$. Then we can identify $\mathbf{T}^{\Sigma,\mathrm{Yosh}}_{\mathfrak{m}_Y}$ with the quotient of $\mathbf{T}^{\Sigma,S}$ acting on S^{Yosh}.

For two elements $\alpha, \beta \in \mathcal{O}$, we write $\alpha \equiv \beta$ if $\alpha - \beta$ belongs to the maximal ideal of $\mathcal{O}$.

Lemma 7.19. *Let $f \in S_2(N)$, $g \in S_{k+2}(N)$ be eigenforms for $\mathbf{T}^{\Sigma,(2)}_{\mathcal{O}}$, $\mathbf{T}^{\Sigma,(k+2)}_{\mathcal{O}}$ respectively. If $\lambda_{Y(f\otimes g)}(t) \equiv \lambda_{Y(f_1\otimes f_2)}(t)$ for all $t \in \mathbf{T}^{\Sigma,S}_{\mathcal{O}}$, then either*
$$\lambda_{f_1}(T_l) \equiv \lambda_f(T_l) \quad \text{and} \quad \lambda_{f_2}(T_l) \equiv \lambda_g(T_l)$$

or

$$\lambda_{f_1}(T_l) \equiv \lambda_g(T_l) \quad \text{and} \quad \lambda_{f_2}(T_l) \equiv \lambda_f(T_l)$$

for $l \not\in \Sigma$. Furthermore, if f_1 and f_2 are ordinary, so are f and g.

Proof. As before, the Chebotarev Density Theorem and the Brauer-Nesbitt Theorem imply that $\overline{\rho}_{Y(f_1\otimes f_2)} \cong \overline{\rho}_{Y(f\otimes g)}$, hence $\overline{\rho}_{f_1} \cong \overline{\rho}_f$ and $\overline{\rho}_{f_2} \cong \overline{\rho}_g$ or the other way around. Thus the first part of the assertion follows. Furthermore, since $\ell \nmid 2N$, (5.3) gives

$$\lambda_{f_1}(T_\ell)\lambda_{f_2}(T_\ell) \cong \lambda_f(T_\ell)\lambda_g(T_\ell).$$

Thus f_1 and f_2 being ordinary implies that f and g are ordinary. □

We have a map $\Phi : \mathbf{T}_{\mathcal{O}}^{\Sigma,\mathrm{Yosh}} = \prod_{\mathfrak{m}} \mathbf{T}_{\mathfrak{m}}^{\Sigma,\mathrm{Yosh}} \to \mathbf{T}_{\mathcal{O}}^{\Sigma,(2)} \otimes \mathbf{T}_{\mathcal{O}}^{\Sigma,(k+2)}$.

Let $Y(f\otimes g) \in S^{\mathrm{Yosh}}$. Then by Lemma 7.19, f and g are ordinary, hence (after fixing an appropriate basis)

$$\rho_{Y(f\otimes g)}|_{I_\ell} = \begin{bmatrix} \epsilon^{k/2+1} & * & & \\ & \epsilon^{k/2} & & \\ & & \epsilon^{k+1} & * \\ & & & 1 \end{bmatrix}.$$

Lemma 7.20. *Assume $\ell > k$. Then there exists $\sigma \in I_\ell$ such that $\beta_1 := \epsilon^{k/2+1}(\sigma)$, $\beta_2 := \epsilon^{k/2}(\sigma)$, $\beta_3 = \epsilon^{k+1}(\sigma)$, $\beta_4 := 1$ are all distinct* (mod ℓ)*.*

Proof. The ℓ-adic cyclotomic character gives a surjection $I_\ell \twoheadrightarrow \mathbf{Z}_\ell^\times \cong \mathbf{Z}_\ell \times (\mathbf{Z}/\ell\mathbf{Z})^\times$, hence we just need to show that there exists $\alpha \in (\mathbf{Z}/\ell\mathbf{Z})^\times$ such that $\alpha^{k/2+1}, \alpha^{k/2}, \alpha^{k+1}, 1$ are all distinct. This is equivalent to showing that there is $\alpha \in (\mathbf{Z}/\ell\mathbf{Z})^\times$ such that none of the following $\alpha, \alpha^{k+1}, \alpha^{k/2}, \alpha^{k/2+1}$ is 1. Take α to be any generator of $(\mathbf{Z}/\ell\mathbf{Z})^\times$. We just need to make sure that $\ell - 1$ (which is the order of α) does not divide any of the following $k+1, k/2, k/2+1$.This is clear since $\ell > k > 2$. □

We return to the proof of Proposition 7.16. By Lemma 7.20 there exists $\sigma \in I_\ell$ and a basis of $\rho_{Y(f\otimes g)}$ in which

$$\rho_{Y(f\otimes g)}(\sigma) = \operatorname{diag}(\beta_1, \beta_2, \beta_3, \beta_4).$$

Define

$$e_i := \prod_{j\neq i} \frac{\sigma - \beta_j}{\beta_i - \beta_j} \in \mathcal{O}[G_{\mathbf{Q}}].$$

Set $e_f := e_1 + e_2$.

Let $R' := \prod_{Y(f\otimes g)\in S^{\mathrm{Yosh}}} \mathcal{O}$. Let R be the $\mathcal{O}$-subalgebra of R' generated by $(\lambda_{Y(f\otimes g)}(T))_{Y(f\otimes g)\in S^{\mathrm{Yosh}}}$, where T runs over $\mathbf{T}_{\mathcal{O}}^{\Sigma,\mathrm{Yosh}}$.

Lemma 7.21. *One has*

$$R \cong \mathbf{T}_{\mathfrak{m}_Y}^{\Sigma,\mathrm{Yosh}}.$$

Proof. There exists an $\mathcal{O}$-algebra map $\mathbf{T}_{\mathcal{O}}^{\Sigma,\mathrm{Yosh}} \to R$ defined by sending t to the tuple of eigenvalues of t. This map is clearly surjective by the definition of R. On the other hand since $\mathbf{T}_{\mathcal{O}}^{\Sigma,\mathrm{Yosh}}$ is the quotient of $\mathbf{T}_{\mathcal{O}}^{\Sigma,S}$ acting on the $\mathbf{C}$-vector space S^{Yosh}, the action is faithful. However, since S^{Yosh} is by definition spanned by eigenforms of $\mathbf{T}_{\mathcal{O}}^{\Sigma,S}$ (since by assumption $N \in \Sigma$), if $t \in \mathbf{T}_{\mathcal{O}}^{\Sigma,\mathrm{Yosh}}$ kills all the eigenfoms in S^{Yosh}, it must kill S^{Yosh}, hence injectivity follows by faithfulness of the action of $\mathbf{T}_{\mathcal{O}}^{\Sigma,\mathrm{Yosh}}$ on S^{Yosh}. □

Define

$$\rho = \prod_{Y(f\otimes g)\in S^{\mathrm{Yosh}}} \rho_{Y(f\otimes g)} : G_{\mathbf{Q}} \to \mathrm{GL}_4(R').$$

Extend ρ to an R-algebra map $\rho : R[G_{\mathbf{Q}}] \to M_4(R')$. Let $l \notin \Sigma$ be as in the statement of Proposition 7.16. Set

$$r_f(l) := \mathrm{tr}\,\rho(e_f\,\mathrm{Frob}_l) \in R'.$$

We claim that $r_f(l) \in R$. Note that $\rho(e_f\,\mathrm{Frob}_l)$ is a polynomial in $\rho(\sigma^i\,\mathrm{Frob}_l)$, $i = 0, 1, 2, 3$, with coefficients in $\mathcal{O}$, so it is enough to show that $\mathrm{tr}\,\rho(\sigma^i\,\mathrm{Frob}_l) \in R$. Fix i, set $\tau = \sigma^i\,\mathrm{Frob}_l$. Then by the Chebotarev Density Theorem, $G_{\mathbf{Q}}$ is generated by conjugacy classes of Frobenii away from Σ, so $\mathrm{tr}\,\rho(\tau)$ is the limit of $\mathrm{tr}\,\rho(\mathrm{Frob}_p)$ for some sequence of primes $p \notin \Sigma$. However, as indicated above, for such p, one has

$$\mathrm{tr}\,\rho(\mathrm{Frob}_p) = (\lambda_{Y(f\otimes g)}(T(1,1,p,p)))_{Y(f\otimes g)\in S^{\mathrm{Yosh}}} \in R.$$

By completeness of R we get $\mathrm{tr}\,\rho(\tau) \in R$. So, $r_f(l) \in R$. Define $S \in \mathbf{T}^{\Sigma,\mathrm{Yosh}}_{\mathfrak{m}_Y}$ to be the image of $r_f(l)$ under the isomorphism in Lemma 7.21. It is clear that $\Phi(S) = T_l \otimes 1$. □

This completes the proof of Theorem 7.15. □

Corollary 7.22. *Let Σ be a finite set of rational primes. Let N be a prime such that $\ell \nmid N(N+1)$. Assume $N, \ell \in \Sigma$. Let $k \in \{2, 4, 6, 8, 12\}$. Suppose that $\ell > k$. Let $f_1 \in S_2(N)$, $f_2 \in S_{k+2}(N)$ be newforms, ordinary at ℓ, whose ℓ-adic Galois representations are residually absolutely irreducible when restricted to $G_{\mathbf{Q}(\sqrt{(-1)^{(\ell-1)/2}\ell})}$. Write Y_{f_1,f_2} for the subspace of $S^S_{k/2+2}(N)$ consisting of common eigenforms for all $t \in \mathbf{T}^{\Sigma,S}_{\mathcal{O}}$ whose Hecke eigenvalues coincide with those of $Y(f_1\otimes f_2)$ for all $t \in \mathbf{T}^{\Sigma,S}_{\mathcal{O}}$. Then there exists $T^S \in \mathbf{T}^{\Sigma,S}_{\mathcal{O}}$ such that both of the following hold:*

- *$T^S F = \eta_1\eta_2 F$ for every $F \in Y_{f_1,f_2}$;*
- *$T^S F = 0$ for all $F \in S^S_{k/2+2}(N)$ such that F is orthogonal to Y_{f_1,f_2} and F is a linear combination of common eigenforms F' for all $t \in \mathbf{T}^{\Sigma,S}_{\mathcal{O}}$ such that the ℓ-adic Galois representation attached to F' is reducible.*

Proof. This follows immediately from Theorem 7.15, Corollary 7.14 and Proposition 7.9. □

Proposition 7.23. *The space Y_{f_1,f_2} is one-dimensional.*

Proof. The following argument is essentially due to Neil Dummigan. Let $G \in Y_{f_1,f_2}$ be a $\mathbf{T}^{\Sigma,S}_{\mathcal{O}}$-eigenform. We want to show that G is a scalar multiple of $Y(f_1 \otimes f_2)$. Let Π be the automorphic representation in which G lies. Then using hypothesis A (4) and (6) of [39], we see that Π must be associated to a Yoshida lift in the sense that there exists an automorphic representation Π' containing a Yoshida lift such that $\Pi \cong \Pi'$ (classicially this means that there is a Yoshida lift whose associated Hecke eigenform at all places has the same eigenvalues as Π). However, the Hecke eigenvalues of a Yoshida lift are completely determined by the Hecke eigenvalues of the elliptic normalized modular eigenforms, say g_1, g_2 from which it is lifted. Since Hecke eigenvalues of $Y(f_1\otimes f_2)$ can differ from those of Π (and hence Π') only at the primes Σ, we conclude that the Hecke eigenvalues of f_1, f_2 differ from those of g_1, g_2 only at the primes in Σ. Using strong multiplicity one on GL_2, we get that $f_1 = g_1$

and $f_2 = g_2$, hence $\Pi \cong \Pi_Y$, where Π_Y is the automorphic representation containing $Y(f_1 \otimes f_2)$. Now, use Hypothesis A (6) in [39] to conclude that the multiplicity of Π_Y in the discrete spectrum is one, so we must have $\Pi = \Pi' = \Pi_Y$. Note that both G and $Y(f_1 \otimes f_2)$ are vectors lying in the subspace of Π_Y fixed by the group $K_0(N)$ and having the correct behavior at infinity (holomorphic, correct weight). The behavior at infinity implies that the infinite components of the automorphic forms attached to G and $Y(f_1 \otimes f_2)$ agree. Moreover, clearly away from N the finite local components (at l, say) lie in the subspaces fixed by $\mathrm{GSp}_4(\mathbf{Z}_l)$, hence are one-dimensional, because at those places Π_Y is spherical. At N they lie in the subspace fixed by $K_0(N) \cap \mathrm{GSp}_4(\mathbf{Z}_N)$. So, it remains to show that this subspace is one-dimensional. Since $Y(f_1 \otimes f_2)$ comes from forms which are new at N, i.e., whose automorphic representations π_1 and π_2 have the Steinberg representation as a local component at N, we know that $\Pi_Y = \theta(\mathit{Steinberg} \otimes \mathit{Steinberg})$ in the notation of [37], and that therefore Π_Y is a twist of $\tau(S, \nu^{-1/2})$ see [37], Lemme 1.2.10(ii). Hence using [32], table 3(VIa), we see that the space of vectors fixed by $K_0(N) \cap \mathrm{GSp}_4(\mathbf{Z}_N)$ is one-dimensional (note that his P_1 is the same as our $K_0(N) \cap \mathrm{GSp}_4(\mathbf{Z}_N)$ - see [32], page 267 for notation). Hence we are done. □

8. Galois representations and Selmer groups

Let the notation and assumptions be as in Assumption 6.1. In this section we will give a lower bound on the order of (the Pontryagin dual of) the Selmer group of

$$\mathrm{Hom}(\rho_{f_2}, \rho_{f_1}(k/2)) \cong \rho_{f_1}(k/2) \otimes \rho_{f_2}^{\vee} \cong \rho_{f_1} \otimes \rho_{f_2}(-k/2-1)$$

in terms of the Yoshida ideal (Definition 6.9) as well as in terms of the special L-value $L^{N,\mathrm{alg}}(2 + k/2, f_1 \times f_2)$. Most of the arguments are now standard (see e.g., [38], [4] or [25], section 9). We will often refer the reader to [25], section 9 for details.

8.1. **Galois representations.**

To an elliptic newform as well as to a Siegel eigenform one can attach an ℓ-adic Galois representation. As the elliptic case is well presented in the literature we will only record the relevant theorem in the Siegel modular case.

Theorem 8.1 (Weissauer, Laumon, Urban)**.** *Let $F \in S^S_{k/2+2}(N)$ be a Siegel eigenform. Let ℓ be a prime not dividing N. There exists a finite extension E_F of $\mathbf{Q}_\ell$ and a 4-dimensional continuous semisimple representation $\rho_F : G_{\mathbf{Q}} \to \mathrm{GL}_4(E_F)$ unramified away from the primes dividing $N\ell$ and such that*

- (i) *For any prime l such that $l \nmid N\ell$, the characteristic polynomial of $\rho_F(\mathrm{Frob}_l)$ coincides with the polynomial on the right-hand side of (5.5) if one substitutes X for l^{-s}.*
- (ii) *the representation $\rho_F|_{D_\ell}$ is crystalline (cf. section 8.2).*
- (iii) *If $\ell > k/2 + 2$, then the representation $\rho_F|_{D_\ell}$ is short. (For a definition of short we refer the reader to* [16]*, section 1.1.2.)*

Proof. For everything except part (iii), see e.g., [37], Theoreme 3.1.3 and 3.1.4. For (iii) see e.g., [11], Theorem 8.2 and references cited there. □

As before, let E be a sufficiently large finite extension of $\mathbf{Q}_\ell$ with valuation ring $\mathcal{O}$, uniformizer ϖ and residue field $\mathbf{F} = \mathcal{O}/\varpi$. In this section we will make the following additional assumption:

Assumption 8.2. We will assume that the Galois representation $\mathrm{Hom}(\rho_{f_2}, \rho_{f_1}(k/2))$ is absolutely irreducible modulo ϖ.

Let ϵ denote the ℓ-adic cyclotomic character. The following isomorphism is a consequence of (5.7)

$$\rho_{Y(f_1\otimes f_2)} \cong (\rho_{f_1} \otimes \epsilon^{k/2}) \oplus \rho_{f_2}.$$

8.2. **Selmer group.** Let $\mathcal{N}^Y$ be those vectors of the $F_1, F_2, \dots, F_r$ (cf. section 6) whose associated ℓ-adic Galois representation is irreducible. Let $\Sigma = \{N, \ell\}$. Let $\mathcal{M}$ denote the set of maximal ideals of the Hecke algebra $\mathbf{T}^{\Sigma,S}_{\mathcal{O}}$ and $\mathcal{M}^Y$ the set of maximal ideals of the quotient $\mathbf{T}^Y_{\mathcal{O}} := \mathbf{T}^{\Sigma,S,Y}_{\mathcal{O}}$ of $\mathbf{T}^{\Sigma,S}_{\mathcal{O}}$ acting on the space generated by $\mathcal{N}^Y$. We have $\mathbf{T}^Y_{\mathcal{O}} = \prod_{\mathfrak{m}\in\mathcal{M}^Y} \mathbf{T}^Y_{\mathfrak{m}}$, where $\mathbf{T}^Y_{\mathfrak{m}}$ denotes the localization of $\mathbf{T}^Y_{\mathcal{O}}$ at $\mathfrak{m}$. Let $\phi : \mathbf{T}^{\Sigma,S}_{\mathcal{O}} \to \mathbf{T}^Y_{\mathcal{O}}$ be the natural projection. We have $\mathcal{M} = \mathcal{M}^c \sqcup \mathcal{M}^{nc}$, where $\mathcal{M}^c$ consists of those $\mathfrak{m} \in \mathcal{M}$ which are preimages (under ϕ) of elements of $\mathcal{M}^Y$ and $\mathcal{M}^{nc} := \mathcal{M}\setminus\mathcal{M}^c$. Note that ϕ factors into a product $\phi = \prod_{\mathfrak{m}\in\mathcal{M}^c} \phi_{\mathfrak{m}} \times \prod_{\mathfrak{m}\in\mathcal{M}^{nc}} 0_{\mathfrak{m}}$, where $\phi_{\mathfrak{m}} : \mathbf{T}^{\Sigma,S}_{\mathfrak{m}} \to \mathbf{T}^{\Sigma,S}_{\mathfrak{m}'}$ is the projection, with $\mathfrak{m}' \in \mathcal{M}^Y$ being the unique maximal ideal such that $\phi^{-1}(\mathfrak{m}') = \mathfrak{m}$ and $0_{\mathfrak{m}}$ is the zero map. For F_i as above we denote by $\mathfrak{m}_{F_i}$ (respectively $\mathfrak{m}^Y_{F_i}$) the element of $\mathcal{M}$ (resp. of $\mathcal{M}^Y$) corresponding to F_i. In particular, $\mathfrak{m}^Y_0 := \mathfrak{m}^Y_{F_0} := \mathfrak{m}^Y_{Y(f_1\otimes f_2)} \in \mathcal{M}^Y$ is such that $\phi^{-1}(\mathfrak{m}^Y_0) = \mathfrak{m}_{F_0}$. However, to ease notation we will write $\mathbf{T}^Y_{\mathfrak{m}_0}$ instead of $\mathbf{T}^Y_{\mathfrak{m}^Y_0}$.

We now define the Selmer group relevant for our purposes. For a profinite group $\mathcal{G}$ and a $\mathcal{G}$-module M (where we assume the action of $\mathcal{G}$ on M to be continuous) we will consider the group $H^1_{\mathrm{cont}}(\mathcal{G}, M)$ of cohomology classes of continuous cocycles $\mathcal{G} \to M$. To shorten notation we will suppress the subscript 'cont' and simply write $H^1(\mathcal{G}, M)$. For a field L, and a $\mathrm{Gal}(\overline{L}/L)$-module M (with a continuous action of $\mathrm{Gal}(\overline{L}/L)$) we sometimes write $H^1(L, M)$ instead of $H^1_{\mathrm{cont}}(\mathrm{Gal}(\overline{L}/L), M)$. We also write $H^0(L, M)$ for the submodule $M^{\mathrm{Gal}(\overline{L}/L)}$ consisting of the elements of M fixed by $\mathrm{Gal}(\overline{L}/L)$.

Let $\Sigma \supset \{\ell\}$ be a finite set of primes of $\mathbf{Q}$ and denote by G_Σ the Galois group of the maximal Galois extension $\mathbf{Q}_\Sigma$ of $\mathbf{Q}$ unramified outside of Σ. Let V be a finite dimensional E-vector space with a continuous G_Σ-action. Let $T \subset V$ be a G_Σ-stable $\mathcal{O}$-lattice. Set $W := V/T$.

We begin by defining local Selmer groups. For every $p \in \Sigma$ and a G_Σ-module M set

$$H^1_{\mathrm{un}}(\mathbf{Q}_p, M) := \ker\{H^1(\mathbf{Q}_p, M) \xrightarrow{\mathrm{res}} H^1(I_p, M)\}.$$

Define the local p-Selmer group (for V) by

$$H^1_{\mathrm{f}}(\mathbf{Q}_p, V) := \begin{cases} H^1_{\mathrm{un}}(\mathbf{Q}_p, V) & p \in \Sigma \setminus \{\ell\} \\ \ker\{H^1(\mathbf{Q}_p, V) \to H^1(\mathbf{Q}_p, V \otimes B_{\mathrm{crys}})\} & p = \ell. \end{cases}$$

Here B_{crys} denotes Fontaine's ring of ℓ-adic periods (cf. [18]).

For every p, define $H^1_{\mathrm{f}}(\mathbf{Q}_p, W)$ to be the image of $H^1_{\mathrm{f}}(\mathbf{Q}_p, V)$ under the natural map $H^1(\mathbf{Q}_p, V) \to H^1(\mathbf{Q}_p, W)$. Using the fact that $\mathrm{Gal}(\overline{\mathbf{F}}_p : \mathbf{F}_p) = \hat{\mathbf{Z}}$ has cohomological dimension 1, one easily sees that if W is unramified at p and $p \neq \ell$, then $H^1_{\mathrm{f}}(\mathbf{Q}_p, W) = H^1_{\mathrm{un}}(\mathbf{Q}_p, W)$.

For a $\mathbf{Z}_\ell$-module M, we write $M^\vee$ for its Pontryagin dual defined as

$$M^\vee = \mathrm{Hom}_{\mathrm{cont}}(M, \mathbf{Q}_\ell/\mathbf{Z}_\ell).$$

Moreover, if M is a Galois module, we denote by $M(n) := M \otimes \epsilon^n$ its n-th Tate twist.

Definition 8.3. For each finite set $\Sigma' \subset \Sigma \setminus \{\ell\}$, the group

$$\mathrm{Sel}_\Sigma(\Sigma', W) := \ker \left\{ H^1(G_\Sigma, W) \xrightarrow{\mathrm{res}} \bigoplus_{p \in \Sigma' \cup \{\ell\}} \frac{H^1(\mathbf{Q}_p, W)}{H^1_{\mathrm{f}}(\mathbf{Q}_p, W)} \right\}$$

is called the (global) *Selmer group of the triple* (Σ, Σ', W). We also set $S_\Sigma(\Sigma', W) := \mathrm{Sel}_\Sigma(\Sigma', W)^\vee$. Define $\mathrm{Sel}_\Sigma(\Sigma', V)$ in the same way with V instead of W.

The group $\mathrm{Sel}_\Sigma(\Sigma \setminus \{\ell\}, W)$ is the standard Selmer group $H^1_{\mathrm{f}}(\mathbf{Q}, W)$ defined by Bloch and Kato [6], section 5.

Let Σ, Σ' be as above. Let $\rho : G_\Sigma \to \mathrm{GL}_E(V)$ denote the representation giving the action of G_Σ on V. The following two lemmas are easy (cf. [31], Lemma 1.5.7 and [36]).

Lemma 8.4. *$S_\Sigma(\Sigma', W)$ is a finitely generated $\mathcal{O}$-module.*

Lemma 8.5. *If the mod ϖ reduction $\overline{\rho}$ of ρ is absolutely irreducible, then the length of $S_\Sigma(\Sigma', W)$ as an $\mathcal{O}$-module is independent of the choice of the lattice T.*

Remark 8.6. For an $\mathcal{O}$-module M, $\mathrm{val}_\ell(\# M) = [\mathcal{O}/\varpi : \mathbf{F}_\ell] \, \mathrm{length}_{\mathcal{O}}(M)$.

Example 8.7. Let $\Sigma = \{N, \ell\}$ and let V denote the representation space of $\rho = \mathrm{Hom}(\rho_{f_2}, \rho_{f_1}(k/2))$ of $G_{\mathbf{Q}}$. Let $T \subset V$ be some choice of a $G_{\mathbf{Q}}$-stable lattice. Set $W = V/T$. Note that the action of $G_{\mathbf{Q}}$ on V factors through G_Σ. Since the mod ϖ reduction of ρ is absolutely irreducible by assumption, $\mathrm{val}_\ell(S_\Sigma(\{N\}, W))$ is independent of the choice of T.

Denote the image of the ideal I_{f_1,f_2} inside $\mathbf{T}^Y_{\mathfrak{m}_0}$ in the same way. Our goal is to prove the following theorem.

Theorem 8.8. *Let Σ and W be as in Example 8.7 and let the notation and assumptions be as in Assumptions 6.1 and 8.2. Then*

$$\mathrm{val}_\ell(\# S_\Sigma(\{N\}, W)) \geq \mathrm{val}_\ell(\# \mathbf{T}^Y_{\mathfrak{m}_0} / I_{f_1,f_2}).$$

Corollary 8.9. *Let Σ and W be as in Example 8.7 and let the notation and assumptions be as in Assumptions 6.1 and 8.2. Let M be as in Theorem 6.5. With the same assumptions as before we have*

$$\mathrm{val}_\ell(\# S_\Sigma(\{N\}, W)) \geq M.$$

If in addition the conditions in Remark 6.6 are satisfied then

$$\mathrm{val}_\ell(\# S_\Sigma(\{N\}, W)) \geq \mathrm{val}_\ell(\# \mathcal{O}/L^{\mathrm{alg}}(k/2+2, f_1 \times f_2)).$$

Proof. The corollary follows immediately from Theorem 8.8. □

Remark 8.10. Note that $S_\Sigma(\{N\}, W)$ is just the Pontryagin dual of the Bloch-Kato Selmer group $H^1_{\mathrm{f}}(\mathbf{Q}, W)$. The Bloch-Kato conjecture for the convolution L-function $L(s, f_1 \times f_2)$ predicts that

$$\mathrm{val}_\ell(\# S_\Sigma(\{N\}, W) \cdot c) = \mathrm{val}_\ell(\# \mathcal{O}/L^{\mathrm{alg}}(k/2+2, f_1 \times f_2)), \tag{8.1}$$

where c is the product of the so called *Tamagawa factor* and the orders of the spaces of $G_{\mathbf{Q}}$-invariants of the modules $A(k/2+2)$ and $A(-k/2-1)$, where $A =$

$\mathrm{Hom}(\rho_{f_2}, \rho_{f_1}(k/2))$ - see also [7], Conjecture 3.1. In [7] the authors also prove that these spaces of invariants are trivial in our case ([loc. cit.], Lemma 3.4) and that the Tamagawa factor is an ℓ-adic unit under some mild conditions ([loc. cit.], Lemma 3.2). Thus Corollary 8.9 yields one inequality in (8.1) hence providing evidence for the Bloch-Kato conjecture.

8.3. **Proof of Theorem 8.8.** In this section we will mainly follow [25], sections 9.4 and 9.5 as the arguments presented there can be easily adapted to the current case. As in [loc. cit.] the key ingredient in the proof of Theorem 8.8 is a result due to Urban [38], which we state as Lemma 8.11 below. However, we first need some notation. Let $\Sigma \supset \{\ell\}$ be a finite set of primes of $\mathbf{Q}$. Let $n', n'' \in \mathbf{Z}_{\geq 0}$ and $n := n' + n''$. Let V' (respectively V'') be an E-vector space of dimension n' (resp. n''), affording a continuous absolutely irreducible representation $\rho' : G_\Sigma \to \mathrm{Aut}_E(V')$ (resp. $\rho'' : G_\Sigma \to \mathrm{Aut}_E(V'')$). Assume that the residual representations $\overline{\rho}'$ and $\overline{\rho}''$ are also absolutely irreducible (hence well-defined) and non-isomorphic. Let $V_1, \dots, V_m$ be n-dimensional E-vector spaces each of them affording an absolutely irreducible continuous representation $\rho_i : G_\Sigma \to \mathrm{Aut}_E(V_i)$, $i = 1, \dots, m$. Moreover assume that the mod ϖ reductions $\overline{\rho}_i$ (with respect to some G_Σ-stable lattice in V_i and hence with respect to all such lattices) satisfy

$$\overline{\rho}_i^{\mathrm{ss}} \cong \overline{\rho}' \oplus \overline{\rho}''.$$

For $\sigma \in G_\Sigma$, let $\sum_{j=0}^n a_j(\sigma) X^j \in \mathcal{O}[X]$ be the characteristic polynomial of $(\rho' \oplus \rho'')(\sigma)$ and let $\sum_{j=0}^n c_j(i,\sigma) X^j \in \mathcal{O}[X]$ be the characteristic polynomial of $\rho_i(\sigma)$. Put $c_j(\sigma) := \begin{bmatrix} c_j(1,\sigma) \\ \dots \\ c_j(m,\sigma) \end{bmatrix} \in \mathcal{O}^m$ for $j = 0, 1, \dots, n-1$. Let $\mathbf{T} \subset \mathcal{O}^m$ be the $\mathcal{O}$-subalgebra generated by the set $\{c_j(\sigma) \mid 0 \leq j \leq n-1, \sigma \in G_\Sigma\}$. By continuity of the ρ_i this is the same as the $\mathcal{O}$-subalgebra of $\mathcal{O}^m$ generated by $\{c_j(\mathrm{Frob}_\mathfrak{p}) \mid 0 \leq j \leq n-1, \mathfrak{p} \notin \Sigma\}$. Note that $\mathbf{T}$ is a finite $\mathcal{O}$-algebra. Let $I \subset \mathbf{T}$ be the ideal generated by the set $\{c_j(\mathrm{Frob}_\mathfrak{p}) - a_j(\mathrm{Frob}_\mathfrak{p}) \mid 0 \leq j \leq n-1, \mathfrak{p} \notin \Sigma\}$. From the definition of I it follows that the $\mathcal{O}$-algebra structure map $\mathcal{O} \to \mathbf{T}/I$ is surjective. Let J be the kernel of this map, so we have $\mathcal{O}/J = \mathbf{T}/I$. The following lemma is due to Urban ([38], Theorem 1.1; see also [25], Lemma 9.21 for the statement concerning the Fitting ideal).

Lemma 8.11. *Suppose $\mathbf{F}^\times$ contains n distinct elements. Then there exists a G_Σ-stable $\mathbf{T}$-submodule $\mathcal{L} \subset \bigoplus_{i=1}^m V_i$, $\mathbf{T}$-submodules $\mathcal{L}', \mathcal{L}'' \subset \mathcal{L}$ (not necessarily G_Σ-stable) and a finitely generated $\mathbf{T}$-module $\mathcal{T}$ such that*

1. *as $\mathbf{T}$-modules we have $\mathcal{L} = \mathcal{L}' \oplus \mathcal{L}''$ and $\mathcal{L}'' \cong \mathbf{T}^{n''}$;*
2. *$\mathcal{L}$ has no $\mathbf{T}[G_\Sigma]$-quotient isomorphic to $\overline{\rho}'$;*
3. *$\mathcal{L}'/I\mathcal{L}'$ is G_Σ-stable and there exists a $\mathbf{T}[G_\Sigma]$-isomorphism*

$$\mathcal{L}/(I\mathcal{L} + \mathcal{L}') \cong M'' \otimes_\mathcal{O} \mathbf{T}/I$$

   for any G_Σ-stable $\mathcal{O}$-lattice $M'' \subset V''$.

4. $\mathrm{Fitt}_\mathbf{T}(\mathcal{T}) = 0$ *and there exists a $\mathbf{T}[G_\Sigma]$-isomorphism*

$$\mathcal{L}'/I\mathcal{L}' \cong M' \otimes_\mathcal{O} \mathcal{T}/I\mathcal{T}$$

   for any G_Σ-stable $\mathcal{O}$-lattice $M' \subset V'$.

We will now show how Lemma 8.11 implies Theorem 8.8. For this we set

- $n' = n'' = 2$;
- $\Sigma = \{\ell, N\}$, $\Sigma' := \{N\}$;
- $\rho' = \rho_{f_1} \otimes \epsilon^{k/2}$, $\rho'' = \rho_{f_2}$, V', V'' = representation spaces of ρ', ρ'' respectively;
- $\mathbf{T} = \mathbf{T}^Y_{\mathfrak{m}_0}$;
- $\mathcal{N}^Y_0 = \{F_i(i = 1, 2, \dots, r) \mid \phi^{-1}(\mathfrak{m}^Y_{F_i}) = \mathfrak{m}_0\}$ (by reordering the F_i's we may assume that $\mathcal{N}^Y_0 = \{F_1, F_2, \dots, F_m\}$ for some $m \geq 0$);
- $I = I_{f_1,f_2}$
- (V_i, ρ_i) = the representation ρ_{F_i}, $i = 1, \dots, m$.

Lemma 8.11 guarantees the existence of $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{L}''$ and $\mathcal{T}$ with properties (1)-(4) as in the statement of the lemma. Let M' (resp. M'') be a G_Σ-stable $\mathcal{O}$-lattice inside V' (resp. V''). The split short exact sequence of $\mathbf{T}$-modules (cf. Lemma 8.11, (1))

$$0 \to \mathcal{L}' \to \mathcal{L} \to \mathcal{L}/\mathcal{L}' \to 0 \tag{8.2}$$

gives rise to a short exact sequence of $(\mathbf{T}/I)[G_\Sigma]$-modules, which splits as a sequence of $\mathbf{T}/I$-modules (cf. Lemma 8.11, (3) and (4))

$$0 \to M' \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T} \to \mathcal{L}/I\mathcal{L} \to M'' \otimes_{\mathcal{O}} \mathbf{T}/I \to 0. \tag{8.3}$$

(Note that $\mathcal{L}/I\mathcal{L} \cong \mathcal{L} \otimes_{\mathbf{T}} \mathbf{T}/I \cong \mathcal{L} \otimes_{\mathcal{O}} \mathbf{T}/I$, hence (8.3) recovers the sequence from Theorem 1.1 of [38].) Let $s : M'' \otimes_{\mathcal{O}} \mathbf{T}/I \to \mathcal{L}/I\mathcal{L}$ be a section of $\mathbf{T}/I$-modules. Define a class $c \in H^1(G_\Sigma, \mathrm{Hom}_{\mathbf{T}/I}(M'' \otimes_{\mathcal{O}} \mathbf{T}/I, M' \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T}))$ by

$$g \mapsto (m'' \otimes t \mapsto s(m'' \otimes t) - g \cdot s(g^{-1} \cdot m'' \otimes t)).$$

The following lemma will be used in the proof of Lemma 8.13 and is proved in [7], Proposition 5.1(3).

Lemma 8.12. *Let I_N denote the inertia group at N. We have $c|_{I_N} = 0$.*

Note that $\mathrm{Hom}_{\mathbf{T}/I}(M'' \otimes_{\mathcal{O}} \mathbf{T}/I, M' \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T}) \cong \mathrm{Hom}_{\mathcal{O}}(M'', M') \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T}$, so c can be regarded as an element of

$$H^1(G_\Sigma, \mathrm{Hom}_{\mathcal{O}}(M'', M') \otimes_{\mathcal{O}} \mathcal{T}/I\mathcal{T}).$$

Define a map

$$\begin{aligned} \iota : \mathrm{Hom}_{\mathcal{O}}(\mathcal{T}/I\mathcal{T}, E/\mathcal{O}) &\to H^1(G_\Sigma, \mathrm{Hom}_{\mathcal{O}}(M'', M') \otimes_{\mathcal{O}} E/\mathcal{O}) \\ f &\mapsto (1 \otimes f)(c). \end{aligned} \tag{8.4}$$

Note that $\tilde{T} := \mathrm{Hom}_{\mathcal{O}}(M'', M')$ is a G_Σ-stable $\mathcal{O}$-lattice inside $\tilde{V} = \mathrm{Hom}(\rho_{f_2}, \rho_{f_1}(k/2)) = \mathrm{Hom}_E(V'', V')$. Then $\tilde{W} = \mathrm{Hom}_{\mathcal{O}}(M'', M') \otimes_{\mathcal{O}} E/\mathcal{O} = W$, where W is as in Theorem 8.8.

Since the mod ϖ reduction of the representation $\tilde{V}$ is absolutely irreducible, Lemma 8.5 implies that our conclusion is independent of the choice of T. Hence we can work with $\tilde{T}$ chosen as above.

The following two lemmas are proved exactly as Lemma 9.25 and 9.26 in [25], so we will omit their proofs here.

Lemma 8.13. *The image of ι is contained inside* $\mathrm{Sel}_\Sigma(\{N\}, \tilde{W})$.

Lemma 8.14. $\ker(\iota)^\vee = 0$.

Let us now show how Lemma 8.13 and Lemma 8.14 imply Theorem 8.8.

Proof of Theorem 8.8. It follows from Lemma 8.13 that

$$\mathrm{val}_\ell(\# S_\Sigma(\{N\}, \tilde{W})) \geq \mathrm{val}_\ell(\# \mathrm{Im}(\iota)^\vee),$$

and from Lemma 8.14 that

$$\mathrm{val}_\ell(\# \mathrm{Im}(\iota)^\vee) = \mathrm{val}_\ell(\# \mathrm{Hom}_\mathcal{O}(\mathcal{T}/I\mathcal{T}, E/\mathcal{O})^\vee). \tag{8.5}$$

Since $\mathrm{Hom}_\mathcal{O}(\mathcal{T}/I\mathcal{T}, E/\mathcal{O})^\vee \cong (\mathcal{T}/I\mathcal{T})^{\vee\vee} = \mathcal{T}/I\mathcal{T}$ (cf. [22], page 98), we have

$$\mathrm{val}_\ell(\# \mathrm{Im}(\iota)^\vee) = \mathrm{val}_\ell(\# \mathcal{T}/I\mathcal{T}).$$

So, it remains to show that $\mathrm{val}_\ell(\#\mathcal{T}/I\mathcal{T}) \geq \mathrm{val}_\ell(\#\mathbf{T}/I)$. Since $\mathrm{Fitt}_\mathbf{T}(\mathcal{T}) = 0$ (Lemma 8.11 (4)), we have $\mathrm{Fitt}_\mathbf{T}(\mathcal{T} \otimes_\mathbf{T} \mathbf{T}/I) \subset I$ and thus $\mathrm{val}_\ell(\#(\mathcal{T} \otimes_\mathbf{T} \mathbf{T}/I)) \geq \mathrm{val}_\ell(\#\mathbf{T}/I)$. As $\mathrm{val}_\ell(\#\mathcal{T}/I\mathcal{T}) = \mathrm{val}_\ell(\#(\mathcal{T} \otimes_\mathbf{T} \mathbf{T}/I))$, the claim follows. □

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