CONGRUENCE PRIMES FOR AUTOMORPHIC FORMS ON UNITARY GROUPS AND APPLICATIONS TO THE ARITHMETIC OF IKEDA LIFTS

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ABSTRACT. In this paper we provide a sufficient condition for a prime to be a congruence prime for an automorphic form f on the unitary group $\mathrm{U}(n,n)(\mathbf{A}_F)$ for a large class of totally real fields F via a divisibility of a special value of the standard L-function associated to f. We also study p-adic properties of the Fourier coefficients of an Ikeda lift I_ϕ (of an elliptic modular form ϕ) on $\mathrm{U}(n,n)(\mathbf{A}_{\mathbf{Q}})$ proving that they are p-adic integers which do not all vanish modulo p. Finally we combine these results to show that the condition of p being a congruence prime for I_ϕ is controlled by the p-divisibility of a product of special values of the symmetric square L-function of ϕ.

1. INTRODUCTION

The problem of classifying congruences between automorphic forms has attracted a considerable amount of attention due not only to its inherent interest, but also because of the arithmetic applications of many of these congruences. For instance, a congruence between an elliptic cusp form and an Eisenstein series implies that the residual Galois representation associated to the cusp form is reducible and non-split. This in turn implies one can construct certain field extensions with prescribed ramification properties. One can see Ribet's seminal paper on the converse to Herbrand's theorem for such a construction [25]. The classification of congruence primes for elliptic modular forms was given by Hida in a series of papers [12, 13, 14] where he showed special values of the symmetric square L-function control such congruences.

As in the case of elliptic modular forms, congruences between automorphic forms on other reductive algebraic groups have numerous arithmetic applications. One can see [1, 2, 3, 4, 6, 7, 11, 20, 21, 23, 24, 30, 31] for examples of such applications. As such, there is considerable interest in classifying congruence primes for automorphic forms on these groups. One

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would expect that the congruence primes should be controlled by certain special values of L-functions attached to the automorphic form. While this remains an open problem, partial progress has been made for automorphic forms of full level defined on symplectic groups in the form of a sufficient condition for a prime to be a congruence prime [20]. In this paper we provide a sufficient condition for an automorphic form f (which is an eigenform for the Hecke algebra) defined on the unitary group $\mathrm{U}(n,n)(\mathbf{A}_F)$ to be congruent to an automorphic form on the same group that is orthogonal to f. This condition is expressed in terms of the divisibility of a special value of the standard L-function associated to f. Here $n > 1$ and F is any totally real finite extension of $\mathbf{Q}$ which has the property that every ideal capitulates in the imaginary quadratic extension over which $\mathrm{U}(n,n)_F$ splits (this is true for example if F has class number one). On the other hand the results referenced above [1, 2, 3, 4, 6, 7, 11, 20, 21, 23, 24, 30, 31] on congruence primes for automorphic forms on reductive groups require the automorphic forms to be defined over $\mathbf{Q}$. The results of [20] on congruence primes for automorphic forms defined on $\mathrm{GSp}(2n)(\mathbf{A}_{\mathbf{Q}})$ also require the forms to have full level while our result holds for congruence subgroups of the form $\mathcal{K}_{0,n}(\mathfrak{N})$ for $\mathfrak{N}$ an ideal in $\mathcal{O}_F$.

More precisely, we prove the following result. Let ℓ be a rational prime and fix embeddings $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_\ell \hookrightarrow \mathbf{C}$. Let K/F be an imaginary quadratic extension. Let $\mathfrak{M} \subset \mathcal{O}_F$ be an ideal and f a cuspidal Hecke eigenform of level $\mathfrak{M}$ and parallel weight k defined on $\mathrm{U}(n,n)(\mathbf{A}_F)$ (for precise definitions of level and weight see section 2). Let ξ be a Hecke character of K of infinity type $\xi_{\mathbf{a}}(z) = \left(\frac{z}{|z|}\right)^{-t}$ for $t = (t,\dots,t) \in \mathbf{Z}^{\mathbf{a}}$. We denote the standard L-function associated to f twisted by ξ by $L(s,f,\xi;\mathrm{st})$. We consider the value

$$L^{\mathrm{alg}}(2n+t/2,f,\xi;\mathrm{st}) = \frac{L(2n+t/2,f,\xi;\mathrm{st})}{\pi^{nd(k+2n+t+1)}\langle f,f\rangle}$$

where $d = [F:\mathbf{Q}]$, which was shown to be algebraic by Shimura [29]. The main result of the first part of this paper is that for a prime ℓ up to some technical conditions one has f is congruent modulo ℓ^b to an automorphic form f', orthogonal to f, if $-b = \mathrm{val}_\ell\left(\frac{\pi^{dn^2}}{\mathrm{vol}(\mathcal{F}_{\mathcal{K}_{0,n}(\mathfrak{M})})}\overline{L^{\mathrm{alg}}(2n+t/2,f,\xi;\mathrm{st})}\right) < 0$ where $\mathcal{F}_{\mathcal{K}_{0,n}(\mathfrak{M})}$ is a fundamental domain for the congruence subgroup of level $\mathfrak{M}$. One can see Theorem 4.3 for a self-contained precise statement of the result.

In the second part of the paper we study arithmetic properties of hermitian Ikeda lifts and then apply the results of the first part to construct a congruence between this lift and an orthogonal automorphic form on $\mathrm{U}(n,n)(\mathbf{A})$ of full level. More specifically, let $K = \mathbf{Q}(\sqrt{-D_K})$ be an imaginary quadratic field of discriminant $-D_K$. Let $n > 1$ be an integer and ϕ be either a modular form of level D_K and Nebentypus being the quadratic character associated with K (if n is even) or a modular form of level one (if n is odd).

The Ikeda lift is an automorphic form I_ϕ on $\mathrm{U}(n,n)(\mathbf{A_Q})$ of full level. It was proven in [19] that I_ϕ is non-zero unless $n \equiv 2 \pmod 4$ and ϕ arises from a Hecke character of an imaginary quadratic field. In this paper we are interested in the non-vanishing of I_ϕ modulo a prime ℓ which is important for constructing congruences. We show that I_ϕ has Fourier coefficients which are algebraic integers (Proposition 5.4) and then prove these coefficients do not all vanish modulo ℓ unless $n \equiv 2 \pmod 4$ and the residual $\pmod{\ell}$ Galois representation attached to ϕ restricted to G_K has abelian image (Theorem 5.8). The condition on the image of the Galois representation is not surprising in light of a result of Ribet [26] which says that modular forms whose ℓ-adic (i.e., characteristic zero) Galois representations restricted to the absolute Galois group of an imaginary quadratic field have abelian image arise from Hecke characters. See section 5 for more details.

Given these arithmetic results and an inner product formula due to Katsurada [22] we construct the aforementioned congruence in section 6.1. More precisely, we show that up to some technical hypotheses I_ϕ is congruent to an orthogonal automorphic form f' on $\mathrm{U}(n,n)(\mathbf{A_Q})$ modulo ℓ^b where

$$(1.1)\quad b = \mathrm{val}_\ell(\mathcal{V}) \quad \text{with} \quad \mathcal{V} := \begin{cases} \prod_{i=2}^n \frac{L(i+2k-1, \mathrm{Sym}^2 \phi \otimes \chi_K^{i+1})}{\pi^{2k+2i-1}\langle \phi,\phi\rangle} & n = 2m+1 \\ \prod_{i=2}^n \frac{L(i+2k, \mathrm{Sym}^2 \phi \otimes \chi_K^{i})}{\pi^{2k+2i}\langle \phi,\phi\rangle} & n = 2m. \end{cases}$$

Here we require the choice of an auxiliary character ξ so that ℓ does not divide a product of certain special values of an L-function twisted by ξ. (See Theorem 6.1 for a precise statement.) While this result is in the spirit of the results of Katsurada on congruence primes for Ikeda lifts on $\mathrm{GSp}(2n)(\mathbf{A_Q})$ [21], our result provides more freedom in "missing" the L-values as we can twist by essentially any Hecke character of K while Katsurada's result allows one only to vary the evaluation point of the L-functions. We then prove that replacing $\langle \phi, \phi\rangle$ by the product of the "integral periods" $\Omega_\phi^+\Omega_\phi^-$ we can ensure that f' is not an Ikeda lift itself. This result however requires some expected (but so far to the best of our knowledge absent from the literature) properties of the Hecke algebra and the Galois representations attached to automorphic forms on $\mathrm{U}(n,n)(\mathbf{A}_F)$ to hold. For details see section 7. Our methods should apply directly to $\mathrm{GSp}(2n)(\mathbf{A}_F)$ as well where F is a totally real extension of $\mathbf{Q}$; this will be the subject of future work.

We finish the article with a discussion on how the constructed congruence can be applied to provide evidence for a new case of the Bloch-Kato conjecture, which concerns the motive $\mathrm{ad}^0 \rho_\phi(3)$, where ρ_ϕ is the Galois representation attached to ϕ. Such evidence for the motive $\mathrm{ad}^0 \rho_\phi(2)$ was provided in [23, 24]. We refrain ourselves however from formulating a precise theorem because we can only prove it under an assumption that certain cohomology classes lie in the correct eigenspace of the complex conjugation and such a result appears to us less than satisfying on the one hand and clumsy to state on the other (cf. Section 8).

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2. Notation and terminology

2.1. **Number fields and Hecke characters.** We fix once and for all an algebraic closure $\overline{\mathbf{Q}}$ and an embedding $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. For $z \in \mathbf{C}$ we write $e(z) = e^{2\pi i z}$. Given a number field L, we write $\mathcal{O}_L$ for the ring of integers of L. Given a prime w of L, we write L_w for the completion of L at w, $\mathcal{O}_{L,w}$ for the valuation ring of L_w, and ϖ_w for a uniformizer. If L/E is an extension of number fields and v a prime of E, we write $L_v = E_v \otimes_E L$ and $\mathcal{O}_{L,v} = \mathcal{O}_{E,v} \otimes_{\mathcal{O}_E} \mathcal{O}_L$. We also set $\widehat{\mathcal{O}}_L = \prod_{w \nmid \infty} \mathcal{O}_{L,w}$. Given a finite prime w of L, we let val_w denote the w-adic valuation on L_w. For $\alpha \in L_w$, we set $|\alpha|_{L_w} = q^{-\mathrm{val}_w(\alpha)}$ where q is the cardinality of the residue class field at w. We will write $|\alpha|_w$ for $|\alpha|_{L_w}$ if it is clear from context what is meant.

Let $\mathbf{A}_L$ denote the adeles of L, $\mathbf{f}_L$ the set of finite places of L, and $\mathbf{a}_L$ the set of embeddings of L into $\overline{\mathbf{Q}}$. We regard $\mathbf{a}_L$ as the set of infinite places of L via the fixed embedding $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. We write $\mathbf{A}_{L,\mathbf{a}_L}$ and $\mathbf{A}_{L,\mathbf{f}_L}$ for the infinite and finite part of $\mathbf{A}_L$ respectively. For $\alpha = (\alpha_w) \in \mathbf{A}_L$, set $|\alpha|_L = \prod_w |\alpha_w|_{L_w}$. For $\alpha \in \mathbf{A}_L^\times$ or $\alpha \in \mathbf{C}^{\mathbf{a}_L}$ and $c \in \mathbf{R}^{\mathbf{a}_L}$, we set $\alpha^c = \prod_{w \in \mathbf{a}_L} \alpha_w^{c_w}$.

We say ψ is a Hecke character of L if ψ is a continuous homomorphism from $\mathbf{A}_L^\times$ to $\{z \in \mathbf{C} : |z| = 1\}$ so that $\psi(L^\times) = 1$. We write $\psi_v, \psi_{\mathbf{f}_L}$ and $\psi_{\mathbf{a}_L}$ for its restrictions to $L_v^\times$, $\mathbf{A}_{L,\mathbf{f}_L}^\times$, and $\mathbf{A}_{L,\mathbf{a}_L}^\times$ respectively. Given ψ, there is a unique ideal $\mathfrak{f} \subset \mathcal{O}_L$ satisfying $\psi_v(\alpha) = 1$ if $v \in \mathbf{f}_L$, $\alpha \in \mathcal{O}_{L,v}^\times$ and $\alpha - 1 \in \mathfrak{f}_v$ and if $\mathfrak{f}'$ is another ideal of $\mathcal{O}_L$ with this property then $\mathfrak{f}' \subset \mathfrak{f}$. The ideal $\mathfrak{f}$ is referred to as the conductor of ψ. Given an integral ideal $\mathfrak{N}$ of $\mathcal{O}_L$ and a Hecke character ψ, we write

$$\psi_{\mathfrak{N}} = \prod_{v \mid \mathfrak{N}} \psi_v.$$

Let F be a totally real extension of $\mathbf{Q}$ and K an imaginary quadratic extension of F. Throughout the paper we let $\mathbf{a}$ (resp. $\mathbf{b}$) denote $\mathbf{a}_F$ (resp. $\mathbf{a}_K$) and $\mathbf{f}$ (resp. $\mathbf{k}$) denote $\mathbf{f}_F$ (resp. $\mathbf{f}_K$). Let D_K denote the discriminant of K. Set χ_K to denote the quadratic character associated with the extension K/F. We identify $F_{\mathbf{a}} := \mathbf{R} \otimes_{\mathbf{Q}} F$ with $\mathbf{R}^{\mathbf{a}} := \prod_{v \in \mathbf{a}} \mathbf{R}$ via the map $a \mapsto (\sigma(a))_{\sigma \in \mathbf{a}}$.

2.2. **Unitary group.** Let $\mathbf{G}_a$ denote the additive group scheme and $\mathbf{G}_m$ the multiplicative group scheme. Let Mat_n denote the $\mathbf{Z}$-group scheme of $n \times n$ matrices ($\cong \mathbf{G}_a^{n^2}$). Given a matrix $A \in \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F} \mathrm{Mat}_{n/\mathcal{O}_K}$, we set $A^* = \overline{A}^t$ and $\hat{A} = (A^*)^{-1}$ where bar denotes the action of the nontrivial element of $\mathrm{Gal}(K/F)$. Associated to the imaginary quadratic extension K/F we have the unitary similitude group scheme over $\mathcal{O}_F$:

$$\mathrm{GU}(n,n) = \{A \in \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F} \mathrm{GL}_{2n/\mathcal{O}_K} : AJ_nA^* = \mu_n(A)J_n\}$$

where $J_n = \begin{bmatrix} & -1_n \\ 1_n & \end{bmatrix}$, 1_n is the $n \times n$ identity matrix, and $\mu_n : \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F} \mathrm{GL}_{2n/\mathcal{O}_K} \to \mathbf{G}_{m/\mathcal{O}_F}$ is a morphism of $\mathcal{O}_F$-group schemes. Set $G_n = \ker \mu_n$. For a $2n \times 2n$ matrix g we will write a_g, b_g, c_g, d_g to be the $n \times n$ matrices defined by $g = \begin{bmatrix} a_g & b_g \\ c_g & d_g \end{bmatrix}$.

Write P_n for the standard Siegel parabolic of G_n, i.e.,

$$P_n = \left\{ g = \begin{bmatrix} a_g & b_g \\ 0_n & d_g \end{bmatrix} \in G_n \right\}.$$

We have the Levi decomposition of $P_n = M_{P_n} N_{P_n}$ where

$$M_{P_n} = \left\{ \begin{bmatrix} A & \\ & \hat{A} \end{bmatrix} : A \in \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F} \mathrm{GL}_{n/\mathcal{O}_K} \right\}$$

and

$$N_{P_n} = \left\{ \begin{bmatrix} 1_n & S \\ & 1_n \end{bmatrix} : S \in \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F} \mathrm{Mat}_{n/\mathcal{O}_K}, S^* = S \right\}.$$

Let $\mathfrak{N} \subset \mathcal{O}_F$ be an ideal. For $v \in \mathbf{f}$ we set

$$\mathcal{K}_{0,n,v}(\mathfrak{N}) = \{ g \in G_n(F_v) : a_g, b_g, d_g \in \mathrm{Mat}_n(\mathcal{O}_{K,v}), c_g \in \mathrm{Mat}_n(\mathfrak{N}\mathcal{O}_{K,v}) \}$$

and

$$\mathcal{K}_{1,n,v}(\mathfrak{N}) = \{ g \in \mathcal{K}_{0,n,v}(\mathfrak{N}) : a_g - 1_n \in \mathrm{Mat}_n(\mathfrak{N}\mathcal{O}_{K,v}) \}.$$

We put $\mathcal{K}_{0,n,\mathbf{f}}(\mathfrak{N}) = \prod_{v \in \mathbf{f}} \mathcal{K}_{0,n,v}(\mathfrak{N})$ and similarly for $\mathcal{K}_{1,n,\mathbf{f}}(\mathfrak{N})$. Set

$$\mathcal{K}^+_{0,n,\mathbf{a}} = \left\{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in G_n(\mathbf{R}^{\mathbf{a}}) : A, B \in \mathrm{GL}_n(\mathbf{C}^{\mathbf{a}}), AA^* + BB^* = 1_n, AB^* = BA^* \right\}$$

and $\mathcal{K}_{0,n,\mathbf{a}}$ to be the subgroup of $G_n(\mathbf{R}^{\mathbf{a}})$ generated by $\mathcal{K}^+_{0,n,\mathbf{a}}$ and J_n. Set $\mathcal{K}_{0,n}(\mathfrak{N}) = \mathcal{K}^+_{0,n,\mathbf{a}} \mathcal{K}_{0,n,\mathbf{f}}(\mathfrak{N})$ and $\mathcal{K}_{1,n}(\mathfrak{N}) = \mathcal{K}^+_{0,n,\mathbf{a}} \mathcal{K}_{1,n,\mathbf{f}}(\mathfrak{N})$.

2.3. **Automorphic forms.** Define

$$\mathbf{H}_n = \{ Z \in \mathrm{Mat}_n(\mathbf{C}) : -i1_n(Z - Z^*) > 0 \}.$$

For $g_{\mathbf{a}} = (g_v)_{v \in \mathbf{a}} \in G_n(\mathbf{R}^{\mathbf{a}})$ and $Z = (Z_v)_{v \in \mathbf{a}} \in \mathbf{H}_n^{\mathbf{a}}$, set $j(g_{\mathbf{a}}, Z) = (j(g_v, Z_v))_{v \in \mathbf{a}}$ where $j(g_v, Z_v) = \det(c_{g_v} Z_v + d_v)$.

Let $\mathcal{K}$ be an open compact subgroup of $G_n(\mathbf{A}_{F,\mathbf{f}})$. For $k, \nu \in \mathbf{Z}^{\mathbf{a}}$ let $\mathcal{M}_{n,k,\nu}(\mathcal{K})$ denote the $\mathbf{C}$-space of functions $f : G_n(\mathbf{A}_F) \to \mathbf{C}$ satisfying the following:

(i) $f(\gamma g) = f(g)$ for all $\gamma \in G_n(F), g \in G_n(\mathbf{A}_F)$,
(ii) $f(g\kappa) = f(g)$ for all $\kappa \in \mathcal{K}, g \in G_n(\mathbf{A}_F)$,
(iii) $f(gu) = (\det u)^{-\nu} j(u, i1_n)^{-k} f(g)$ for all $g \in G_n(\mathbf{A}_F), u \in \mathcal{K}_{0,n,\mathbf{a}}$,
(iv) $f_c(Z) = (\det g_{\mathbf{a}})^{\nu} j(g_{\mathbf{a}}, i1_n)^k f(g_{\mathbf{a}} c)$ is a holomorphic function of $Z = g_{\mathbf{a}} i1_n \in \mathbf{H}_n^{\mathbf{a}}$ for every $c \in G_n(\mathbf{A}_{F,\mathbf{f}})$ where $g_{\mathbf{a}} \in G_n(\mathbf{R}^{\mathbf{a}})$.

If $n = 1$, one also needs to require that the f_c in (iv) be holomorphic at cusps (cf. [29, p. 31] for what this means). We let $\mathcal{S}_{n,k,\nu}(\mathcal{K})$ denote the space of cusp forms in $\mathcal{M}_{n,k,\nu}(\mathcal{K})$ (cf. [29, p. 33]). If $\nu = 0$ we drop it from the notation.

Let ψ be a Hecke character of K of conductor dividing $\mathfrak{N}$. For $k, \nu \in \mathbf{Z}^{\mathbf{a}}$ we set

$$\mathcal{M}_{n,k,\nu}(\mathfrak{N}, \psi) := \{f \in \mathcal{M}_{n,k,\nu}(\mathcal{K}_{1,n,\mathbf{f}}(\mathfrak{N})) \mid f(g\kappa) = \psi_{\mathfrak{N}}(\det(a_\kappa))^{-1} f(g), \\ g \in G_n(\mathbf{A}_F), \kappa \in \mathcal{K}_{0,n,\mathbf{f}}(\mathfrak{N})\}.$$

If ψ is trivial we drop it from the notation.

Every automorphic form $f \in \mathcal{M}_{n,k,\nu}(\mathcal{K})$ has a Fourier expansion which we now discuss. We first define $e_{\mathbf{A}_F}$ as follows. Let $\alpha = (\alpha_v) \in \mathbf{A}_F$, where v runs over places of F. We set $e_v(\alpha_v) = e(\alpha_v)$ for $v \in \mathbf{a}$ and write $e_{\mathbf{a}}(\alpha) = e\left(\sum_{v \in \mathbf{a}} \alpha_v\right)$. If v is finite, we set $e_v(\alpha_v) = e(-y)$ where $y \in \mathbf{Q}$ is chosen such that $\mathrm{Tr}_{F_v/\mathbf{Q}_p}(\alpha_v) - y \in \mathbf{Z}_p$ if $v \mid p$. We then set $e_{\mathbf{A}_F}(\alpha) = \prod_v e_v(\alpha_v)$. For every $q \in \mathrm{GL}_n(\mathbf{A}_{K,\mathbf{k}})$ and $h \in S_n(F)$ there exist complex numbers $c_f(h, q)$ such that one has (a Fourier expansion of f)

$$f\left(\begin{bmatrix} 1_n & \sigma \\ & 1_n \end{bmatrix} \begin{bmatrix} q & \\ & \hat{q} \end{bmatrix}\right) = \sum_{h \in S_n(F)} c_f(h, q) e_{\mathbf{A}_F}(\mathrm{tr}\, h\sigma)$$

for every $\sigma \in S_n(\mathbf{A}_F)$ where

$$S_n = \{h \in \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F} \mathrm{Mat}_{n/\mathcal{O}_K} : h^* = h\}.$$

One can see [28, Chapter III, Section 18] for further discussion of such Fourier expansions.

Our next step is to define what we mean by a congruence between two automorphic forms. One knows (see e.g., [9, Theorem 3.3.1]) that for any finite subset $\mathcal{B}$ of $\mathrm{GL}_n(\mathbf{A}_{K,\mathbf{k}})$ of cardinality h_K with the property that the canonical projection $c_K : \mathbf{A}_K^\times \to \mathrm{Cl}_K$ restricted to $\det \mathcal{B}$ is a bijection, the following decomposition holds

$$\mathrm{GL}_n(\mathbf{A}_K) = \bigsqcup_{b \in \mathcal{B}} \mathrm{GL}_n(K)\, \mathrm{GL}_n^+(K_{\mathbf{b}}) b\, \mathrm{GL}_n(\widehat{\mathcal{O}}_K).$$

Here $+$ refers to a totally positive determinant. Such a set $\mathcal{B}$ will be called a *base*.

Let Cl_K^- denote the subgroup of Cl_K on which the non-trivial element of $\mathrm{Gal}(K/F)$ acts by inversion.

Lemma 2.1. *If* $(h_K, 2n) = 1$ *and* $\mathrm{Cl}_K^- = \mathrm{Cl}_K$ *then a base* $\mathcal{B}$ *can be chosen so that all* $b \in \mathcal{B}$ *(and hence also* $p_b := \begin{bmatrix} b & \\ & \hat{b} \end{bmatrix}$*) are scalar matrices and* $bb^* = b^*b = 1_n$ *(and hence also* $p_b p_b^* = p_b^* p_b = 1_{2n}$*).*

Proof. This is just a modification of the proof of [24, Corollary 3.9]. Identifying Cl_K with a quotient of $\mathbf{A}_K^\times / K^\times$ and using the fact that raising to the power n acts as automorphism on Cl_K since $(h_K, n) = 1$, it is enough to show that every element of Cl_K can be written in the form $\alpha\overline{\alpha}^{-1}$ for some $\alpha \in \mathbf{A}_K^\times$. Let $\alpha_1, \dots, \alpha_{h_K}$ be representatives of Cl_K. Since $2 \nmid h_K$, $\alpha_1^2, \dots, \alpha_{h_K}^2$ also form a complete set of representatives. By our assumption that $\mathrm{Cl}_K = \mathrm{Cl}_K^-$ we get that for each $1 \leq i \leq h_K$ we have $\overline{\alpha}_i^{-1} = \alpha_i k_i$ for some $k_i \in K^\times$. $\square$

Remark 2.2. The assumption $\mathrm{Cl}_K = \mathrm{Cl}_K^-$ is satisfied if the image of the canonical map $\mathrm{Cl}_K \to \mathrm{Cl}_F$ given by $\mathfrak{a} \mapsto \mathfrak{a}\overline{\mathfrak{a}}$ is trivial, so in particular when $h_F = 1$. Indeed, then $\overline{\mathfrak{a}} = \mathfrak{a}^{-1}k$ for some $k \in F$. The assumption $\mathrm{Cl}_K^- = \mathrm{Cl}_K$ is therefore weaker for allowing $k \in K$ and in fact since h_K is odd it is equivalent to the assumption that every ideal of $\mathcal{O}_F$ capitulates in $\mathcal{O}_K$.

Definition 2.3. Let $\mathcal{B}$ be a base. We say $\mathcal{B}$ is *admissible* if all $b \in \mathcal{B}$ are scalar matrices with $bb^* = 1_n$ (and hence also p_b are scalar matrices with $p_b p_b^* = 1_{2n}$). Furthermore given a prime ℓ we say that $\mathcal{B}$ is *ℓ-admissible*, if it is admissible and for every $b \in \mathcal{B}$ there exists a rational prime $p \nmid 2D_K\ell$ such that $b_\mathfrak{q} = 1_n$ for all $\mathfrak{q} \nmid p$ and $b_\mathbf{b} = 1_n$. The set of primes $\mathfrak{p}$ for which $b_\mathfrak{q} \neq 1_n$ with $\mathfrak{q} \mid \mathfrak{p}$ will be called the support of $\mathcal{B}$.

Note that if $(h_K, 2n) = 1$ and $\mathrm{Cl}_K^- = \mathrm{Cl}_K$, then the Tchebotarev density theorem combined with Lemma 2.1 implies that for every $\ell \nmid D_K$ an ℓ-admissible base always exists.

From now on for the rest of the article for every rational prime ℓ we fix compatible embeddings $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_\ell \hookrightarrow \mathbf{C}$.

Definition 2.4. Let ℓ be a rational prime and $\mathcal{O}$ the ring of integers in some algebraic extension E of $\mathbf{Q}_\ell$ with maximal ideal λ.

(1) Let $f \in \mathcal{M}_{n,k,\nu}(\mathfrak{N}, \psi)$. We say f has *$\mathcal{O}$-integral* Fourier coefficients (with respect to $\mathcal{B}$) if there exists a base $\mathcal{B}$ so that for all $b \in \mathcal{B}$ and all $h \in S_n(F)$ we have $e_\mathbf{a}(-i\mathrm{tr}\, h_\mathbf{a})c_f(h, b) \in \mathcal{O}$.
(2) Let $f, g \in \mathcal{M}_{n,k,\nu}(\mathfrak{N}, \psi)$ and suppose that f and g both have $\mathcal{O}$-integral Fourier coefficients with respect to a base $\mathcal{B}$. Let E' be a finite extension of $\mathbf{Q}_\ell$ with $E' \subset E$ and write $\mathcal{O}'$ for its ring of integers and ϖ for a uniformizer. We say f is congruent to g modulo ϖ^n if for all $b \in \mathcal{B}$ and all $h \in S_n(F)$ we have $e_\mathbf{a}(-i\mathrm{tr}\, h_\mathbf{a})c_f(h, b) - e_\mathbf{a}(-i\mathrm{tr}\, h_\mathbf{a})c_g(h, b) \in \varpi^n\mathcal{O}'$. We denote this by $f \equiv g \pmod{\varpi^n}$.

Remark 2.5. Our definition of a congruence between automorphic forms on $G_n(\mathbf{A}_F)$ allows for the situation when one knows that each individual Fourier coefficient of f is an algebraic integer, but one does not know if collectively they generate a number field. This is certainly known for modular forms on GL_2 due to a straightforward connection between Hecke eigenvalues and Fourier coefficients. The connection in the case of higher rank groups is much more delicate.

Remark 2.6. If f and g are congruent $\pmod{\varpi^n}$ with respect to one admissible base, say $\mathcal{B}$, and $\mathcal{B}'$ is another admissible base such that f and

g both have $\mathcal{O}$-integral Fourier coefficients with respect to $\mathcal{B}'$, then we have $e_{\mathbf{a}}(-i\mathrm{tr}\, h_{\mathbf{a}})c_f(h,b) \equiv e_{\mathbf{a}}(-i\mathrm{tr}\, h_{\mathbf{a}})c_g(h,b) \pmod{\varpi^n}$ for all $b \in \mathcal{B}'$ and all $h \in S_n(F)$. This is easily proved using admissibility and a generalization (to the setting of totally real F and rank n) of formula (5.4) in [24] using the references indicated in the proof of Lemma 5.5 in [24]. Formula (5.4) in [loc.cit.] can also be used to show that if f and g have $\mathcal{O}$-integral Fourier coefficients with respect to one admissible base, then they have $\mathcal{O}$-integral Fourier coefficients with respect to all admissible bases. Hence we see that the congruence relation between automorphic forms is transitive, and thus an equivalence relation.

2.4. ***L*-functions.** We collect here the definitions of the L-functions needed for this paper. Let F be a number field and, as before, let $\mathbf{f}_F$ denote the set of all finite places of F.

Given an integral ideal $\mathfrak{N}$ of $\mathcal{O}_F$ and an Euler product

$$L(s) = \prod_{v \in \mathbf{f}_F} L_v(s),$$

we write

$$L^{\mathfrak{N}}(s) = \prod_{\substack{v \in \mathbf{f}_F \\ v \nmid \mathfrak{N}}} L_v(s)$$

and

$$L_{\mathfrak{N}}(s) = \prod_{v \mid \mathfrak{N}} L_v(s).$$

Let χ be a Hecke character of F. We set

$$L(s,\chi) = \prod_{\substack{v \in \mathbf{f}_F \\ v \nmid \mathrm{cond}(\chi)}} (1 - \chi(\varpi_v)|\varpi_v|_v^s)^{-1},$$

where we identify a uniformizer ϖ_v with its image in $\mathbf{A}_F^\times$.

Now, let F be a totally real field, K an imaginary quadratic extension of F and ψ and χ Hecke characters of K. Let $f \in \mathcal{S}_{n,k,\nu}(\mathfrak{N},\psi)$ be an eigenform with parameters $\lambda_{v,1},\dots,\lambda_{v,n}$ determined by the Satake isomorphism [29, Section 20.6]. The standard L-function of f is defined as in [29, p. 171] by

$$L(s,f,\chi;\mathrm{st}) = \prod_{\substack{v \in \mathbf{f} \\ v \nmid \mathcal{O}_F \cap \mathrm{cond}(\chi)}} L_v(s,f,\chi;\mathrm{st})$$

where for $v \in \mathbf{f}$ and $v \nmid \mathfrak{N}$ we set

$$L_v(s,f,\chi;\mathrm{st}) = \begin{cases} \prod_{i=1}^n \left[(1-\lambda_{v,i}\chi(\varpi_w)|\varpi_w|_w^{s-n+1})(1-\lambda_{v,i}^{-1}\chi(\varpi_w)|\varpi_w|_w^{s-n})\right]^{-1} & K\otimes_F F_v \text{ is a field} \\ \prod_{i=1}^{2n} \left[(1-\lambda_{v,i}^{-1}\chi(\varpi_{v_1})|\varpi_v|_v^{s-2n})(1-\lambda_{v,i}\chi(\varpi_{v_2})|\varpi_v|_v^{s+1})\right]^{-1} & K\otimes_F F_v \cong F_v\times F_v \end{cases}$$

and when $v \mid \mathfrak{N}$ we set

$$L_v(s, f, \chi; \mathrm{st}) = \begin{cases} \prod_{i=1}^n \left[1 - \lambda_{v,i}\chi(\varpi_w)|\varpi_w|_w^{s-n+1}\right]^{-1} & K \otimes_F F_v \text{ is a field} \\ \prod_{i=1}^n \left[(1 - \lambda_{v,i}\chi(\varpi_{w_1})|\varpi_{w_1}|_{w_1}^{s-n+1})(1 - \lambda_{v,n+i}\chi(\varpi_{w_2})|\varpi_{w_2}|_{w_2}^{s-n+1})\right]^{-1} & K \otimes_F F_v \cong F_v \times F_v. \end{cases}$$

Here ϖ_v denotes a uniformizer of F_v, ϖ_w denotes a uniformizer of $K \otimes F_v$ if the latter is a field and $\varpi_{v_1}, \varpi_{v_2} \in K \otimes_F F_v$ correspond to $(\varpi_v, 1)$ and $(1, \varpi_v)$ respectively under the isomorphism $K \otimes_F F_v \cong F_v \times F_v$ if such an isomorphism holds. Here again we identify $K \otimes_F F_v$ with its image in $\mathbf{A}_K^\times$.

3. An inner product relation

Fix a totally real extension $F/\mathbf{Q}$ and an imaginary quadratic extension K/F. Let $d = [F : \mathbf{Q}]$. In this section we generalize a certain Rankin-Selberg type formula which for the case of $F = \mathbf{Q}$ was derived in [24, Section 7]. Since all the calculations carried out in [loc.cit.] carry over without difficulty to the general case, we include here only the necessary setup as well as indicate where changes need to be made and refer the reader to [loc.cit.] for a less condensed version (see also [5]).

3.1. **Eisenstein series.** In this section we define the hermitian Siegel Eisenstein series and recall its properties needed to prove our congruence result.

Let $\mathfrak{N} \subset \mathcal{O}_F$ be an ideal. Let $X_{m,\mathfrak{N}}$ be the set of Hecke characters ψ' of K of conductor dividing $\mathfrak{N}$ satisfying

$$\psi'_{\mathbf{a}}(x) = (x/|x|)^m \tag{3.1}$$

for $m \in \mathbf{Z}^{\mathbf{a}}$. Here and later we regard Hecke characters of K as characters of $(\mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\mathbf{G}_{m/\mathcal{O}_K})(\mathbf{A}_F)$. Let $\psi \in X_{m,\mathfrak{N}}$. Recall we set $\psi_{\mathfrak{N}} = \prod_{v|\mathfrak{N}} \psi_v$.

For a place v of F and an element $p = \begin{bmatrix} a_p & b_p \\ 0 & d_p \end{bmatrix} \in P_n(F_v)$, set

$$\delta_{P_n,v}(p) = |\det(d_p d_p^*)|_v$$

and $\delta_{P_n} = \prod_v \delta_{P_n,v}$. Set $\mu_{P_n} = \prod_v \mu_{P_n,v} : G_n(\mathbf{A}_F) \to \mathbf{C}$ to be a function vanishing outside of $P_n(\mathbf{A}_F)\mathcal{K}_{0,n}(\mathfrak{N})$ where for $p_v \in P_n(F_v)$ and $k_v \in \mathcal{K}_{0,n,v}(\mathfrak{N})$ we set

$$\mu_{P_n,v}(p_v k_v) = \begin{cases} \psi_v(\det d_{q_v})^{-1} & v \in \mathbf{f}, v \nmid \mathfrak{N} \\ \psi_v(\det d_{q_v})^{-1}\psi_v(\det d_{k_v}) & v \in \mathbf{f}, v \mid \mathfrak{N} \\ \psi_{\mathbf{a}}(\det d_{q_{\mathbf{a}}})^{-1} j(k_{\mathbf{a}}, i)^{-m} & v \in \mathbf{a}. \end{cases}$$

The hermitian Siegel Eisenstein series of weight $m \in \mathbf{Z}^{\mathbf{a}}$, level $\mathfrak{N}$, and character ψ is defined by

$$E(g, s, m, \psi, \mathfrak{N}) = \sum_{\gamma \in P_n(F)\backslash G_n(F)} \mu_{P_n}(\gamma g)\delta_{P_n}(\gamma g)^{-s}$$

for $\mathrm{Re}(s) >> 0$. The meromorphic continuation of $E(g,s,m,\psi,\mathfrak{N})$ to all $s \in \mathbf{C}$ is given by Shimura [28, Proposition 19.1]. We will make use of the following normalized Eisenstein series

$$D(g,s,m,\psi,\mathfrak{N}) = \prod_{j=1}^{n} L^{\mathfrak{N}}(2s-j+1, \psi_F \chi_K^{j-1}) E(g,s,m,\psi,\mathfrak{N})$$

where ψ_F denotes the restriction of ψ to $\mathbf{A}_F^{\times}$.

Let $\eta_{\mathbf{f}} \in G_n(\mathbf{A}_F)$ be the matrix with the infinity components equal to 1_{2n} and finite components equal to $J_n = \begin{bmatrix} & -1_n \\ 1_n & \end{bmatrix}$. Given a function $f : G_n(\mathbf{A}_F) \to \mathbf{C}$, set $f^*(g) = f(g\eta_{\mathbf{f}}^{-1})$. If $f \in \mathcal{M}_{n,k}(\mathcal{K})$, we have $f^* \in \mathcal{M}_{n,k}(\eta_{\mathbf{f}}^{-1}\mathcal{K}\eta_{\mathbf{f}})$.

Write

$$D^*\left(\begin{bmatrix} q & \sigma q \\ & \hat{q} \end{bmatrix}, n-m/2, m, \psi, \mathfrak{N}\right) = \sum_{h \in S_n(F)} c_{D^*}(h,q) e_{\mathbf{A}_F}(h\sigma).$$

We have the following result giving the integrality of the Fourier coefficients of D^*.

Proposition 3.1. *Let ℓ be a prime and assume $\ell \nmid D_K \, \mathrm{N}_{F/\mathbf{Q}}(\mathfrak{N})(n-2)!$, $(h_K, 2n) = 1$, and that $\mathrm{Cl}_K^- = \mathrm{Cl}_K$. Let $\mathcal{B}$ be an ℓ-admissible base whose support is relatively prime to $\mathrm{cond}(\psi)$. There exists a finite extension E of $\mathbf{Q}_\ell$ so that*

$$\pi^{-n(n+1)d/2} e_{\mathbf{a}}(-i \mathrm{tr}\, h_{\mathbf{a}}) c_{D^*}(h,b) \in \mathcal{O}_E$$

for all $h \in S_n(F)$ and all $b \in \mathcal{B}$.

Proof. This is proved as in [24, Theorem 7.8] and [24, Corollary 7.11]. There are only two significant changes to the proof given in [24]. The first is the power of π has an additional $d = [F : \mathbf{Q}]$, which follows immediately from [28, Equation 18.11.5]. The second is the reference that $L(n, \psi')$ lies in the ring of integers of a finite extension of $\mathbf{Q}_\ell$ for all $n \in \mathbf{Z}_{<0}$ must be changed from [33, Corollary 5.13] to [16, Theorem 1, p. 104] as our characters are now defined over F instead of $\mathbf{Q}$. □

3.2. **Theta series.** Let $k \in \mathbf{Z}^{\mathbf{a}}$ be such that $k_v > 0$ for all $v \in \mathbf{a}$ and let ξ be a Hecke character of K with conductor $\mathfrak{f}_\xi$ and infinity type $|x|^t x^{-t}$ for $t \in \mathbf{Z}^{\mathbf{a}}$ with $t_v \geq -k_v$ for each $v \in \mathbf{a}$. In [29, Section A5] and [28, Section A7] Shimura defines a theta series on $G_n(\mathbf{A})$ associated to a quadruple consisting of k (as above), the Hecke character ξ, a matrix $r \in \mathrm{GL}_n(\mathbf{A}_{K,\mathbf{k}})$ and a matrix $\tau \in S_n^+(F) = \{h \in S_n(F) \subset S_n(\mathbf{A}_F) : h_v > 0 \text{ for all } v \in \mathbf{a}\}$ which satisfies the following condition

$$\{g^* \tau g \mid g \in r\mathcal{O}_K^n\} = \mathfrak{d}_{F/\mathbf{Q}}^{-1}, \tag{3.2}$$

where $\mathfrak{d}_{F/\mathbf{Q}}$ is the relative different of F over $\mathbf{Q}$ (see Remark 3.2). Let φ be a Hecke character of K with infinity type $\prod_{v \in \mathbf{a}} \frac{|x_v|}{x_v}$ such that the restriction of φ to $\mathbf{A}_F^\times$ equals χ_K. Such a character always exists [29, Lemma A5.1]. Set $\psi' = \xi^{-1}\varphi^{-n}$. Then the theta series, which we (following Shimura) will denote by θ_ξ (so in particular we will suppress k and τ from notation) belongs to the space $\mathcal{S}_{n,t+k+n}(\mathfrak{N}_{\mathfrak{t}}, \psi')$ (cf. [28, Proposition A7.16]; cuspidality follows from our assumption that $k_v > 0$ [28, p. 277]). Here

$$\mathfrak{N}_{\mathfrak{t}} = (\mathfrak{t}D_{K/F}\,\mathrm{N}_{K/F}(\mathfrak{f}_\xi) \cap \mathfrak{d}_{F/\mathbf{Q}}^{-1} D_{K/F} \cap \mathfrak{d}_{F/\mathbf{Q}}^{-1}\mathfrak{f}_\xi)\mathfrak{d}_{F/\mathbf{Q}} \tag{3.3}$$

is an ideal of $\mathcal{O}_F$, where $D_{K/F}$ is the relative discriminant of K over F and

$$\mathfrak{t}^{-1} := \{g^*\tau^{-1}g \mid g \in \mathcal{O}_K\}$$

is a fractional ideal of F (cf. [29, Section A5.5] and [28, Prop. A7.16]).

Remark 3.2. The theta series considered in [28, 29] are of a more general type and are not required to satisfy (3.2). However, in this article we treat only the case of congruence subgroups of the form $\mathcal{K}_{1,n}(\mathfrak{M})$ as opposed to the more general kind of $\mathcal{K}_{1,n}(\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{M})$ considered in [loc.cit.]. The assumption (3.2) implies that for the level of the theta series we have $\mathfrak{b} = 1$. Furthermore, we will later choose r and τ (and hence a theta series associated to them) in such a way the (τ, r)-Fourier coefficient $c_f(\tau, r)$ of a form $f \in \mathcal{M}_{n,k}(\mathfrak{M})$ (to which we will construct congruences - cf. Theorem 4.3) will be an ℓ-adic unit. However, it follows from [28, Proposition 18.3(2)] that $c_f(\tau, r) = 0$ if $(r^*\tau r)_v \notin (\mathfrak{d}_{F/\mathbf{Q}}^{-1})_v S_n(\mathcal{O}_{F,v})$ for all $v \in \mathbf{f}$. From this it is easy to show that for a pair (τ, r) where $c_f(\tau, r) \neq 0$ (and these are the only pairs we are interested in) one has $\{g^*\tau g \mid g \in r\mathcal{O}_K^n\} \subset \mathfrak{d}_{F/\mathbf{Q}}^{-1}$, hence the assumption (3.2) is often satisfied. Let us also remark that the purpose of this restriction is to merely simplify the exposition which would require a great deal of additional notation should we consider the general case. However, the method is entirely general and the main result (Theorem 4.3) remains true in the case of $\mathfrak{b} \neq 1$ provided one makes the necessary modifications of level of the forms involved.

Proposition 3.3 ([24], Corollary 7.13)**.** *Let k, ξ, r, τ be a quadruple as above determining a theta series θ_ξ. Assume that $r_w = 1_n$ for all $w \mid \mathfrak{f}_\xi D_K \ell$. For an ℓ-admissible base $\mathcal{B}$ there exists a finite extension E of $\mathbf{Q}_\ell$ so that $e_{\mathbf{a}}(-i\mathrm{tr}\, h_{\mathbf{a}})c_{\theta_\xi}(h, b) \in \mathcal{O}_E$ for all $b \in \mathcal{B}$ and all $h \in S_n(F)$.*

Remark 3.4. One should note that [24, Corollary 7.13] is given for the case $F = \mathbf{Q}$, but the results of Shimura it depends on are valid in our case. The proof given in [24] then goes through.

3.3. **Inner product.** Let k, ξ, r, τ and hence θ_ξ be fixed as above. We fix $\mathfrak{N}$ an ideal of $\mathcal{O}_F$ with $\mathfrak{N}_{\mathfrak{t}} \mid \mathfrak{N}$ so that $\theta_\xi \in \mathcal{S}_{n,t+k+n}(\mathfrak{N}, \psi')$. We now give the inner product relation that forms the key to constructing the congruences in this paper.

Set

$$\Gamma((s)) = (4\pi)^{-nd\frac{2s+k+l}{2}}\Gamma_n\left(s + (k+l)/2\right)^d$$

where

$$\Gamma_n(s) = \pi^{n(n-1)/2} \prod_{j=0}^{n-1} \Gamma(s-j).$$

Let $Y_{\rm re} = \{h \in \mathrm{Mat}_n(\mathbf{C_a}) : h = h^*\}/\{h \in \mathrm{Mat}_n(\mathcal{O}_K) : h = h^*\}$ and $Y_{\rm im} = \{h \in \mathrm{Mat}_n(\mathbf{C_a}) : h = h^*, h > 0\}/\sim$ where $h \sim h'$ if there exists $g \in \mathrm{GL}_n(\mathcal{O}_K)$ such that $h' = ghg^*$. Then one has $Y_{\rm re} \times Y_{\rm im}$ is commensurable with $\mathcal{K}_{0,n}(\mathfrak{N}) \cap P_n(F) \backslash \mathbf{H}_n^{\mathbf{a}}$, i.e., the ratio of their volumes is a rational number [29, p.179]. Set $A_{\mathfrak{N}}$ to be this rational number times $\mathrm{vol}(Y_{\rm re})^{-1}$. Set

$$C(s) = \frac{\# X_{m,\mathfrak{N}} h_K \Gamma((\overline{s}-n))(\det \tau)^{-\overline{s}+n-(k+l)/2} |\det r|_K^{\overline{s}-n/2} e_{\mathbf{a}}(-i \mathrm{tr}\, \tau_{\mathbf{a}}) c_f(\tau, r)}{A_{\mathfrak{N}} [\mathcal{K}_{0,n}(\mathfrak{N}) : \mathcal{K}^1_{1,n}(\mathfrak{N})] \prod_{v \in \mathbf{c}} g_v(\xi(\varpi_v)|\varpi_v|_v^{2\overline{s}+n})}$$

where $\mathbf{c}$ is a finite subset of $\mathbf{f}$ as given in [28, Proposition 19.2] and $\mathcal{K}^1_{1,n}(\mathfrak{N}) = \{\kappa \in \mathcal{K}_{1,n}(\mathfrak{N}) : \det \kappa = 1\}$. Note this differs from [24, (7.29)] as the term h_K is omitted there.

For $f_1, f_2 \in \mathcal{M}_{n,k,\nu}(\mathcal{K})$ (with at least one of them a cusp form) we define the inner product $\langle f_1, f_2 \rangle$ as in [28, (10.9.6)] and set

$$\langle f_1, f_2 \rangle_{\mathcal{K}} := \mathrm{vol}(\mathcal{F}_{\mathcal{K}}) \langle f_1, f_2 \rangle, \tag{3.4}$$

where $\mathcal{F}_{\mathcal{K}} := (G_n(F) \cap p_b \mathcal{K} p_b^{-1}) \setminus \mathbf{H}_n^{\mathbf{a}}$ for $b \in \mathcal{B}$ (we refer the reader to [28, p. 81] for the definition of the volume and the explanation that the volume is independent of the choice of b). We note here that $\langle f_1, f_2 \rangle$ is independent of the choice of level $\mathcal{K}$ and satisfies $\langle \rho(g) f_1, \rho(g) f_2 \rangle = \langle f_1, f_2 \rangle$ for all $g \in G_n(\mathbf{A}_F)$ where ρ is the right regular representation [28, (10.9.3)]. The key input into our congruence result is the following inner product relation.

Theorem 3.5 ([24], Theorem 7.7)**.** *Assume $(h_K, 2n) = 1$ and $\mathrm{Cl}_K^- = \mathrm{Cl}_K$. Let $f \in \mathcal{S}_{n,k}(\mathfrak{N})$ be a Hecke eigenform and let ξ and ψ' be as above. Then*

$$\langle D(\cdot, s, m, (\psi')^c, \mathfrak{N}) \theta_\xi, f \rangle_{\mathcal{K}_{0,n}(\mathfrak{N})} = \overline{C(s)} \cdot \overline{L(\overline{s} + n/2, f, \xi; \mathrm{st})}$$

where $C(s)$ is as given above.

4. Congruence

For the moment fix a quadruple k, ξ, r, τ satisfying the conditions of section 3.2 and an ideal $\mathfrak{N}$ of $\mathcal{O}_F$ such that $\mathfrak{N}_{\mathfrak{t}} \mid \mathfrak{N}$ and let $\theta_\xi \in \mathcal{S}_{n,t+k+n}(\mathfrak{N}, \psi')$ denote the associated theta series as in that section. We ease the notation in this section by writing $D(g) = D(g, n - m/2, m, (\psi')^c, \mathfrak{N})$ and $D^*(g) = D(g\eta_{\mathbf{f}}^{-1}, n - m/2, m, (\psi')^c, \mathfrak{N})$. Set $l = t + k + n$ as above and put $m = k - l$. Since $\theta_\xi \in \mathcal{S}_{n,l}(\mathfrak{N}, \psi')$ and $D \in \mathcal{M}_{n,m}(\mathfrak{N}, (\psi')^{-1})$, we have $D\theta_\xi \in \mathcal{S}_{n,k}(\mathfrak{N})$ and $(D\theta_\xi)^* \in \mathcal{S}_{n,k}(\eta_{\mathbf{f}}^{-1} \mathcal{K}_{0,n,\mathbf{f}}(\mathfrak{N}) \eta_{\mathbf{f}})$.

Given $\mathcal{K}_1 \subset \mathcal{K}_2$, we define a trace operator

$$\begin{aligned}\operatorname{tr}{}^{\mathcal{K}_2}_{\mathcal{K}_1} : \mathcal{M}_{n,k}(\mathcal{K}_1) &\to \mathcal{M}_{n,k}(\mathcal{K}_2)\\ f &\mapsto \sum_{\kappa \in \mathcal{K}_1 \backslash \mathcal{K}_2} \rho(\kappa) f\end{aligned}$$

where we recall that ρ is the right regular representation. If f has $\mathcal{O}$-integral Fourier coefficients the q-expansion principle implies that $\operatorname{tr} f$ also has $\mathcal{O}$-integral Fourier coefficients [18, Section 8.4].

Now we fix one more ideal $\mathfrak{M}$ of $\mathcal{O}_F$ with $\mathfrak{M} \mid \mathfrak{N}$. For $g \in G_n(\mathbf{A}_K)$, write $\mathcal{K}^g := g^{-1}\mathcal{K}g$. Set

$$\Xi := \operatorname{tr}{}^{\mathcal{K}_{0,n}(\mathfrak{M})^{\eta_{\mathbf{f}}}}_{\mathcal{K}_{0,n}(\mathfrak{N})^{\eta_{\mathbf{f}}}} (\pi^{-nd(n+1)/2}(D\theta_\xi)^*). \tag{4.1}$$

Lemma 4.1. *Let ℓ be a prime. Assume $(h_K, 2n) = 1$, $\mathrm{Cl}_K^- = \mathrm{Cl}_K$. Further assume that $\ell \nmid D_K \mathrm{N}_{F/\mathbf{Q}}(\mathfrak{N})(n-2)!$ and that r satisfies $r_w = 1_n$ for all $w \mid D_K \ell \mathrm{cond}(\xi)$. If $\mathcal{B}$ is an ℓ-admissible base with support relatively prime to $\mathrm{cond}(\psi)$ there exists a finite extension E of $\mathbf{Q}_\ell$ so that $e_{\mathbf{a}}(-i\mathrm{tr}\, h_{\mathbf{a}})c_\Xi(h, b) \in \mathcal{O}_E$ for all $h \in S_n(F)$ and $b \in \mathcal{B}$.*

Proof. The result follows immediately upon combining Proposition 3.1, Proposition 3.3, and the q-expansion principle as explained above. □

Lemma 4.2. *For any eigenform $f \in \mathcal{S}_{n,k}(\mathfrak{M})$ with $\mathfrak{M} \mid \mathfrak{N}$ we have*

$$\langle \Xi, f^* \rangle = \pi^{-nd(n+1)/2}[\mathcal{K}_{0,n}(\mathfrak{M}) : \mathcal{K}_{0,n}(\mathfrak{N})]\langle D\theta_\xi, f\rangle.$$

Proof. We have

$$\begin{aligned}\pi^{nd(n+1)/2}\langle \Xi, f^*\rangle &= \sum_{\kappa \in \mathcal{K}_{0,n}(\mathfrak{N})^{\eta_{\mathbf{f}}} \backslash \mathcal{K}_{0,n}(\mathfrak{M})^{\eta_{\mathbf{f}}}} \langle \rho(\kappa)(D\theta_\xi)^*, f^*\rangle\\ &= \sum_{\kappa \in \mathcal{K}_{0,n}(\mathfrak{N})^{\eta_{\mathbf{f}}} \backslash \mathcal{K}_{0,n}(\mathfrak{M})^{\eta_{\mathbf{f}}}} \langle (D\theta_\xi)^*, \rho(\kappa^{-1})f^*\rangle\\ &= [\mathcal{K}_{0,n}(\mathfrak{M}) : \mathcal{K}_{0,n}(\mathfrak{N})]\langle (D\theta_\xi)^*, f^*\rangle\\ &= [\mathcal{K}_{0,n}(\mathfrak{M}) : \mathcal{K}_{0,n}(\mathfrak{N})]\langle D\theta_\xi, f\rangle.\end{aligned}$$

□

We now apply Lemma 4.2 together with Theorem 3.5 to obtain the following result. For the convenience of the reader we make the statement of Theorem 4.3 self-contained, repeating all the assumptions and making some of the choices in a slightly different order than in the above narrative.

Theorem 4.3. *Let $F/\mathbf{Q}$ be a totally real extension of degree d, K/F an imaginary quadratic extension, and assume $\mathrm{Cl}_K^- = \mathrm{Cl}_K$. Let n be a positive integer with $(h_K, 2n) = 1$. Let $k = (k, \dots, k) \in \mathbf{Z}^{\mathbf{a}}$ be a parallel weight so that $k > 0$ and let ℓ be a rational prime so that $\ell > k$ and $\ell \nmid D_K h_K$. Let $f \in \mathcal{S}_{n,k}(\mathfrak{M})$ be a Hecke eigenform with $\mathcal{O}$-integral Fourier coefficients for $\mathcal{O}$ the ring of integers in some algebraic extension $E/\mathbf{Q}_\ell$. Let ξ be a*

Hecke character of K so that $\xi_{\mathbf{a}}(z) = \left(\frac{z}{|z|}\right)^{-t}$ for some $t = (t, \dots, t) \in \mathbf{Z}^{\mathbf{a}}$ with $-k \leq t < \min\{-6, -4n\}$. Assume there exists $\tau \in S_n^+(\mathcal{O}_F)$ and $r \in \mathrm{GL}_n(\mathbf{A}_{K,\mathbf{k}})$ such that $r_w = 1_n$ for all $w \mid D_K \ell \mathrm{cond}(\xi)$, the condition (3.2) holds and $\mathrm{val}_{\varpi}(e_{\mathbf{a}}(-i\mathrm{tr}\, h_{\mathbf{a}})c_f(\tau, r)) = \mathrm{val}_{\varpi}(\det r) = 0$. Let $\mathfrak{N}$ be an ideal of $\mathcal{O}_F$ such that $\mathfrak{N}_{\mathfrak{t}} \mid \mathfrak{N}$ and $\mathfrak{M} \mid \mathfrak{N}$ with $\mathfrak{N}_{\mathfrak{t}}$ defined as in (3.3). Then there exists a field subextension $\mathbf{Q}_\ell \subset E' \subset E$, finite over $\mathbf{Q}_\ell$ with uniformizer ϖ, containing the algebraic value

$$L^{\mathrm{alg}}(2n + t/2, f, \xi; \mathrm{st}) = \frac{L(2n + t/2, f, \xi; \mathrm{st})}{\pi^{n(k+2n+t+1)d}\langle f, f\rangle}$$

such that if

$$\mathrm{val}_{\varpi}\left(\#(\mathcal{O}_K/\mathfrak{N}\mathcal{O}_K)\#(\mathcal{O}_K/\mathfrak{N}\mathcal{O}_K)^\times[\mathcal{K}_{0,n}(\mathfrak{M}) : \mathcal{K}_{0,n}(\mathfrak{N})]\right) = 0 \quad \text{and} \quad \mathrm{val}_{\varpi}(A_{\mathfrak{N}}) \geq 0$$

and

$$-b = \mathrm{val}_{\varpi}\left(\frac{\pi^{dn^2}}{\mathrm{vol}(\mathcal{F}_{\mathcal{K}_{0,n}(\mathfrak{M})})}\overline{L^{\mathrm{alg}}(2n + t/2, f, \xi; \mathrm{st})}\right) < 0,$$

then there exists $f' \in \mathcal{S}_{n,k}(\mathfrak{M})$, orthogonal to f, with $\mathcal{O}$-integral Fourier coefficients so that $f \equiv f' \pmod{\varpi^b}$.

Proof. We begin by noting that [29, Theorem 28.8] gives that $L^{\mathrm{alg}}(2n + t/2, f, \xi; \mathrm{st}) \in \overline{\mathbf{Q}}$ and [28, Theorem 24.7] gives that $\pi^{dn^2}/\mathrm{vol}(\mathcal{F}_{\mathcal{K}_{0,n}(\mathfrak{M})}) \in \overline{\mathbf{Q}}$. Note here that [28, Theorem 24.7] is stated for anisotropic forms, but one can easily show $G_n(F_{\mathbf{a}})$ is isomorphic as a Lie group to the unitary group of an anisotropic form so it applies in our case as well.

For the quadruple k, ξ, r, τ we define the theta series θ_ξ as in section 3.2. The assumption that $\mathfrak{N}_{\mathfrak{t}} \mid \mathfrak{N}$ guarantees that $\theta_\xi \in \mathcal{S}_{n,t+k+n}(\mathfrak{N}, \psi')$. We also attach the Eisenstein series $D = D(g, n - (k - l)/2, (\psi')^c, \mathfrak{N})$ to the ideal $\mathfrak{N}$ and the character ψ'. We then define Ξ as in (4.1). By Lemma 4.1 our assumptions guarantee that Ξ has $\mathcal{O}'$-integral Fourier coefficients for $\mathcal{O}'$ the ring of integers in some finite extension E' of $\mathbf{Q}_\ell$. We will write ϖ for a uniformizer of E'. By extending E' (and thus $\mathcal{O}'$) if necessary we may also assume that $L^{\mathrm{alg}}(2n + t/2, f, \xi; \mathrm{st})$ and $\pi^{dn^2}/\mathrm{vol}(\mathcal{F}_{\mathcal{K}_{0,n}(\mathfrak{M})})$ both lie in E'. Indeed, note that here we are forcing $\ell > n$, hence the assumption that $\mathrm{val}_{\varpi}((n-2)!) = 0$ in Lemma 4.1 is unnecessary. Write

$$\Xi = C_f f^* + g^* \tag{4.2}$$

for some $g \in \mathcal{S}_{n,k}(\mathfrak{M})$ where $\langle f, g\rangle = 0$ and $C_f = \frac{\langle \Xi, f^*\rangle}{\langle f^*, f^*\rangle}$.

We will be interested in the ℓ-adic valuation of C_f so we do not require the exact value. Lemma 4.2 and Theorem 3.5 give

$$\begin{aligned}\langle \Xi, f^* \rangle &= \pi^{-nd(n+1)/2}[\mathcal{K}_{0,n}(\mathfrak{M}) : \mathcal{K}_{0,n}(\mathfrak{N})]\langle D\theta_\xi, f\rangle \\ &= \frac{[\mathcal{K}_{0,n}(\mathfrak{M}) : \mathcal{K}_{0,n}(\mathfrak{N})]}{\pi^{nd(n+1)/2}\operatorname{vol}(\mathcal{F}_{\mathcal{K}_{0,n}(\mathfrak{M})})}\langle D\theta_\xi, f\rangle_{\mathcal{K}_{0,n}(\mathfrak{M})} \\ &= \frac{[\mathcal{K}_{0,n}(\mathfrak{M}) : \mathcal{K}_{0,n}(\mathfrak{N})]}{\pi^{nd(n+1)/2}\operatorname{vol}(\mathcal{F}_{\mathcal{K}_{0,n}(\mathfrak{M})})}\overline{C((3n+t)/2)}\cdot\overline{L(2n+t/2, f, \xi; \mathrm{st})}.\end{aligned}$$

It now remains to simplify $C((3n+t)/2)$. We have

$$C((3n+t)/2) = \frac{\# X_{-t-n,\mathfrak{N}} h_K \Gamma((\frac{n+t}{2}))(\det\tau)^{-(n+t+k)}|\det r|_K^{n+t/2} e_{\mathbf{a}}(-i\operatorname{tr}\tau_{\mathbf{a}})c_f(\tau, r)}{A_{\mathfrak{N}}[\mathcal{K}_{0,n}(\mathfrak{N}) : \mathcal{K}^1_{1,n}(\mathfrak{N})]\prod_{v\in\mathbf{c}} g_v(\xi(\varpi_v)|\varpi_v|_v^{4n+t})}.$$

Note here the fact that $l = t+k+n$ has been used in simplifying this expression. We have

$$\Gamma\left(\left(-\frac{t+n}{2}\right)\right) = (4\pi)^{-nd(t+k+n)}\pi^{nd(n-1)/2}\prod_{j=0}^{n-1}\Gamma(t+k+n-j)^d.$$

Note the largest value of the argument of Γ in the above product is $t+k+1$. By our assumption on the allowable range of the values of t we see that this value is never greater than k, which is less than ℓ. Hence the product of the Γ-factors is a ϖ-adic unit. Observe also that $\prod_{v\in\mathbf{c}} g_v(\xi(\varpi_v)|\varpi_v|_v^{4n+t})$ is a finite product and the g_v are polynomials with coefficients in $\mathbf{Z}$ and a constant term of 1 [29, Lemma 20.5]. Thus as long as $-t > 4n$ we have this lies in a finite extension of $\mathbf{Z}_\ell$. Thus, (extending E' if necessary) we get $\operatorname{val}_\varpi(1/\prod_{v\in\mathbf{c}} g_v(\xi(\varpi_v)|\varpi_v|_v^{4n+t})) \leq 0$. Moreover, $[\mathcal{K}_{0,n}(\mathfrak{N}) : \mathcal{K}^1_{1,n}(\mathfrak{N})] \in \mathbf{Z}$, so $\operatorname{val}_\varpi(1/[\mathcal{K}_{0,n}(\mathfrak{N}) : \mathcal{K}^1_{1,n}(\mathfrak{N})]) \leq 0$. We also have $\det\tau \in \mathfrak{d}^{-1}_{K/F}$. However, since $\ell \nmid D_K$ and $-(n+t+k) < 0$ we have (assuming $K \subset E'$) that $\operatorname{val}_\varpi((\det\tau)^{-(n+t+k)}) \leq 0$. Since $\mathfrak{N} \neq (1)$ the proof of [28, Lemmas 11.14, 11.15] (and the remarks that follow these lemmas) give that $\# X_{-t-n,\mathfrak{N}}$ equals the index of the group $\{x \in \mathbf{A}_K^\times : x_w \in \mathcal{O}_{K,w}^\times \text{ and } x_w - 1 \in \mathfrak{N}\mathcal{O}_{K,w} \text{ for every } w \in \mathbf{k}\}$ in $\mathbf{A}_K^\times$. Thus, $\operatorname{val}_\varpi(\# X_{-t-n,\mathfrak{N}}) = 0$. Finally, note that $|\det r|_K$ as well as h_K are also ϖ-adic units by our assumptions. We can now conclude that

$$\frac{\langle \Xi, f^*\rangle}{\langle f^*, f^*\rangle} = (*)\left(\frac{\pi^{dn^2}}{\operatorname{vol}(\mathcal{F}_{\mathcal{K}_{0,n}(\mathfrak{M})})}\overline{L^{\mathrm{alg}}(2n+t/2, f, \xi; \mathrm{st})}\right)$$

where $(*) \in E'$ with $\operatorname{val}_\varpi(*) \leq 0$ and we have used that $\langle f, f\rangle = \langle f^*, f^*\rangle$. As the ϖ-valuation of $\frac{\pi^{dn^2}}{\operatorname{vol}(\mathcal{F}_{\mathcal{K}_{0,n}(\mathfrak{M})})}\overline{L^{\mathrm{alg}}(2n+t/2, f, \xi; \mathrm{st})}$ is assumed to be $-b < 0$ we see there exists some positive integer $a \geq b$ so that $C((3n+t)/2) = u\varpi^{-a}$ with $u \in (\mathcal{O}')^\times$. Thus, we have

$$uf^* = \varpi^a \Xi - \varpi^a g^*. \tag{4.3}$$

Since the Fourier coefficients of f are $\mathcal{O}$-integral by assumption, so are the Fourier coefficients of f^* by the q-expansion principle. As the Fourier coefficients of Ξ are $\mathcal{O}'$-integral and thus also $\mathcal{O}$-integral by Lemma 4.1, we get that the Fourier coefficients of $\varpi^a g^*$ are $\mathcal{O}$-integral. Rearranging (4.3) and using the $\mathcal{O}'$-integrality of the Fourier coefficients of Ξ we get that the Fourier coefficients of $uf^* + \varpi^a g^*$ are $\mathcal{O}'$-integral and (since $u \in (\mathcal{O}')^\times$) we get

$$f^* \equiv -u^{-1}\varpi^a g^* \pmod{\varpi^a}.$$

Thus, we obtain

$$f \equiv -u^{-1}\varpi^a g \pmod{\varpi^a} \tag{4.4}$$

and $-u^{-1}\varpi^a g$ can be taken as the form f' in the statement of the theorem. □

Remark 4.4. The form f' in the statement Theorem 4.3 can a priori be zero. However, it is non-zero whenever $f \not\equiv 0 \pmod{\varpi^b}$. This follows immediately from (4.4).

5. Arithmetic properties of Ikeda lifts

For the rest of the paper we restrict our attention to the case when $F = \mathbf{Q}$ and $K = \mathbf{Q}(\sqrt{-D_K})$ is an imaginary quadratic extension of $\mathbf{Q}$ with discriminant $-D_K$. We write $\mathbf{A}$ for $\mathbf{A}_\mathbf{Q}$. In this section we will show that the Fourier coefficients of Ikeda lifts (with respect to some base) are integral, generate a number field and we formulate a condition on a certain (mod ℓ) Galois representation which will ensure they are also non-vanishing mod ℓ. This will provide a complementary result to Ikeda's non-vanishing result (cf. Theorem 5.7).

5.1. **Generalities on Ikeda lifts.** In this context we will need the base change and symmetric square L-functions as well; we recall the definitions here. For positive integers k, N and a Dirichlet character $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ we will denote by $S_k(N,\chi)$ the space of (classical) elliptic cusp forms of weight k, level N and nebentypus χ. If χ is trivial we will omit it. For $p \nmid N$ and $\phi \in S_k(N,\chi)$ a primitive eigenform, let $\alpha_{\phi,p}, \beta_{\phi,p}$ be the p-Satake parameters of ϕ normalized arithmetically, i.e., so that $\alpha_{\phi,p}\beta_{\phi,p} = p^{k-1}\chi(p)$. For a Dirichlet character $\psi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ and $s \in \mathbf{C}$ with $\mathrm{Re}(s)$ sufficiently large the (partial) symmetric square L-function is defined by

$$L^N(s, \mathrm{Sym}^2\phi \otimes \psi) = \prod_{p \nmid N} \left[(1-\alpha_{\phi,p}^2\psi(p)p^{-s})(1-\alpha_{\phi,p}\beta_{\phi,p}\psi(p)p^{-s})(1-\beta_{\phi,p}^2\psi(p)p^{-s})\right]^{-1}.$$

If N or ψ are 1 we drop them from notation.

We define the (twisted) base change L-function from $\mathbf{Q}$ to K as follows. Let ψ be a Hecke character of K of conductor dividing N. For a place w of K of residue characteristic $p \nmid N$, set $\alpha_{\phi,w} = \alpha_{\phi,p}^d$ and $\beta_{\phi,w} = \beta_{\phi,p}^d$ where $d = [\mathcal{O}_{K,w}/\varpi_w\mathcal{O}_{K,w} : \mathbf{F}_p]$ and ϖ_w denotes a uniformizer of K_w. For $s \in \mathbf{C}$

with $\mathrm{Re}(s)$ sufficiently large the (partial) base change L-function is defined by

$$L^N(s, \mathrm{BC}(\phi) \otimes \psi) = \prod_{w \nmid N} [(1 - \alpha_{\phi,w}\psi(\varpi_w)|\varpi_w|_w^s)(1 - \beta_{\phi,w}\psi(\varpi_w)|\varpi_w|_w^s)]^{-1}.$$

Here again we identify ϖ_w with its image in $\mathbf{A}_K^\times$.

Let $n = 2m$ (resp. $n = 2m + 1$). Let ϕ be a newform in $S_{2k+1}(D_K, \chi_K)$ (resp. in $S_{2k}(1)$). Ikeda has shown ([19, Section 5], see also [22, Theorems 2.1, 2.2] and note that the product of the L-functions in [loc.cit.] agrees with the base change L-function below) that there exists a Hecke eigenform $I_\phi \in \mathcal{S}_{n,2k+2m,-k-m}(G_n(\hat{\mathbf{Z}}))$ such that for any Hecke character ψ of K we have

$$L^{D_K}(s, I_\phi, \psi; \mathrm{st}) = \prod_{i=1}^{n} L^{D_K}(s + k + m - n - i + 1, \mathrm{BC}(\phi) \otimes \psi). \tag{5.1}$$

Remark 5.1. The reader will note that we use the classical and adelic language somewhat inconsistenly reserving the first one for the elliptic modular form ϕ while using the second one for its Ikeda lift I_ϕ. This is however the convention used by both Ikeda [19] and Katsurada [22] and we chose not to alter it here in order to make the references which we use more readily applicable to our situation.

Remark 5.2. In [22] an automorphic normalization of the standard L-function is used, which we temporarily denote by $L_0(s)$. However, here and in the rest of the paper we use Shimura's normalization. The translation is given by $L(s) = L_0(s-n+1/2)$. In particular in the normalization given in [22] in (5.1) one has factors of the form $L_0^{D_K}(s+k+m-i+1/2, \mathrm{BC}(f)\otimes\psi)$ which in our normalization equals $L^{D_K}(s + k + m - n - i + 1, \mathrm{BC}(f) \otimes \psi)$. We also note that our use of letters n and m is reversed from that in [22].

It has been proven by Katsurada [22, Theorem 2.2] that one has

$$\langle I_\phi, I_\phi \rangle_{G_n(\hat{\mathbf{Z}})} = (*) \langle \phi, \phi \rangle \times \tag{5.2}$$
$$\begin{cases} \prod_{i=2}^n L(i + 2k - 1, \mathrm{Sym}^2 \phi \otimes \chi_K^{i+1}) L(i, \chi_K^i) \Gamma_{\mathbf{C}}(i + 2k - 1) \Gamma_{\mathbf{C}}(i)^2, & n = 2m + 1 \\ \prod_{i=2}^n L(i + 2k, \mathrm{Sym}^2 \phi \otimes \chi_K^i) L(i, \chi_K^i) \Gamma_{\mathbf{C}}(i + 2k) \Gamma_{\mathbf{C}}(i)^2, & n = 2m \end{cases}$$

where $(*)$ is an integer divisible only by powers of 2 and D_K, and $\Gamma_{\mathbf{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$. Here for two forms ϕ, ϕ' of weight m and level Γ with at least one of them a cusp form we set

$$\langle \phi, \phi' \rangle := \frac{1}{[\overline{\mathrm{SL}_2(\mathbf{Z})} : \overline{\Gamma}]} \int_{\Gamma \backslash \mathbf{H}} \phi(z) \overline{\phi'(z)} y^{m-2} dx\, dy,$$

where $\mathbf{H}$ is the complex upper half-plane, $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$, $\overline{\mathrm{SL}_2(\mathbf{Z})} := \mathrm{SL}_2(\mathbf{Z})/\langle -1_2 \rangle$ and $\overline{\Gamma}$ is the image of Γ in $\overline{\mathrm{SL}_2(\mathbf{Z})}$.

Remark 5.3. Shimura [27] and Sturm [32] (whose results we will later use) define the inner product differently by dividing the integral by the volume of the fundamental domain. If we denote the inner product used in [27, 32] by $\langle\cdot,\cdot\rangle_S$, then one has $\langle\cdot,\cdot\rangle = \frac{\pi}{3}\langle\cdot,\cdot\rangle_S$.

5.2. **Integrality of the Fourier coefficients of Ikeda lift.** Let ϕ be as above and write I_ϕ for its Ikeda lift to $G_n(\mathbf{A})$.

Proposition 5.4. *Let $\mathcal{B}$ be an admissible base. Then for every $b \in \mathcal{B}$ and $h \in S_n(\mathbf{Q})$ the normalized Fourier coefficient $e(-i\mathrm{tr}\, h)c_{I_\phi}(h, p_b)$ is an algebraic integer.*

Proof. Let $\mathcal{B}$ be an admissible base. For $b \in \mathcal{B}$ the formulas of [22, p. 7] give that

$$e(-i\mathrm{tr}\, h)c_{I_\phi}(h, p_b) = |\gamma(h)|^x \prod_{p|\gamma(h)} \tilde{F}_p(h, \alpha'_{\phi,p}). \tag{5.3}$$

Here $x = k$ if $n = 2m$ (resp. $x = k - 1/2$ if $n = 2m + 1$), $\gamma(h) = (-D_K)^{\lfloor n/2\rfloor} \det h \in \mathbf{Z}$, $\tilde{F}_p(h, X)$ is a Laurent polynomial with coefficients in $\mathbf{Z}$ whose top degree term is $X^{\mathrm{val}_p(\gamma(h))}$ and the bottom degree term is a root of unity times $X^{-\mathrm{val}_p(\gamma(h))}$ (cf. [19, p.1112]). Finally, we set

$$\alpha'_{\phi,p} := \begin{cases} p^{-k}\alpha_{\phi,p} & n = 2m \\ p^{-k+1/2}\alpha_{\phi,p} & n = 2m+1, \end{cases} \tag{5.4}$$

where $\alpha_{\phi,p}$ is any Satake parameter of ϕ at p if $n = 2m + 1$ or if $n = 2m$ and $p \nmid D_K$ and equals the pth Fourier coefficient of ϕ if $n = 2m$ and $p \mid D_K$. As noted by Ikeda [19, p. 1118] the independence of $\tilde{F}_p(h, \alpha'_{\phi,p})$ from the choice of a Satake parameter at p follows from the functional equation satisfied by $\tilde{F}_p(h, X)$ [19, Lemma 2.2]. The above formula shows that $e(-i\mathrm{tr}\, h)c_{I_\phi}(h, p_b)$ lies in some number field K', hence it remains to show that its $\mathfrak{p}$-adic valuation is non-negative for each prime $\mathfrak{p}$ of K'. Let us only show the claim in the case when $\mathfrak{p}$ lies over $p \nmid D_K$ and $n = 2m$ (the other cases are handled analogously). Suppose $p \nmid D_K$, $n = 2m$ and set $y := \mathrm{val}_p(\gamma(h)) \geq 0$. Then $|\gamma(h)|^k = p^{yk}u$ for some $u \in \mathbf{Z}$ with $p \nmid u$. We will prove that $\mathrm{val}_{\mathfrak{p}}(\tilde{F}_l(h, \alpha'_{\phi,l})) \geq 0$ for all $l \mid \gamma(h)$ with $l \neq p$ and that $\mathrm{val}_{\mathfrak{p}}(|\gamma(h)|^k \tilde{F}_p(h, \alpha'_{\phi,p})) \geq 0$. First consider the case of $l \neq p$ with $\ell \mid \gamma(h)$. Since $\alpha_{\phi,l}\beta_{\phi,l} = \chi_K(l)l^{2k}$, where $\beta_{\phi,l}$ stands for the other l-Satake parameter of ϕ, we must have that $\mathrm{val}_{\mathfrak{p}}(\alpha'_{\phi,l}) = 0$. The claim now follows from the fact that $\tilde{F}_l(h, X)$ has coefficients in $\mathbf{Z}$. Hence it remains to consider that case of the prime p. For simplicity write α for $\alpha_{\phi,p}$ and β for the other p-Satake parameter. Write e for the ramification index of $\mathfrak{p}$ over p. It suffices to show that both $V_1 := \mathrm{val}_{\mathfrak{p}}(|\gamma(h)|^k (p^{-k}\alpha)^{\mathrm{val}_p(\gamma(h))}) \geq 0$ and $V_2 := \mathrm{val}_{\mathfrak{p}}(|\gamma(h)|^k (p^{-k}\alpha)^{-\mathrm{val}_p(\gamma(h))}) \geq 0$. A direct calculation yields $V_1 = eyk + y(-ke + \mathrm{val}_{\mathfrak{p}}(\alpha)) = y\,\mathrm{val}_{\mathfrak{p}}(\alpha)$ and $V_2 = eyk - y(-ke + \mathrm{val}_{\mathfrak{p}}(\alpha)) = y(2ek - \mathrm{val}_{\mathfrak{p}}(\alpha))$. Note that since α and β are algebraic integers, we have

$\mathrm{val}_{\mathfrak{p}}(\alpha) \geq 0$ and $\mathrm{val}_{\mathfrak{p}}(\beta) \geq 0$. This alone shows that $V_1 \geq 0$. We also have $\alpha\beta = \chi_K(p)p^{2k}$, which yields $\mathrm{val}_{\mathfrak{p}}(\alpha) + \mathrm{val}_{\mathfrak{p}}(\beta) = 2ke$ and hence $\mathrm{val}_{\mathfrak{p}}(\alpha) \leq 2ke$, which implies that $V_2 \geq 0$. □

Corollary 5.5. *Let $\mathcal{B}$ be an admissible base. Let L be a number field containing all the Fourier coefficients of the newform ϕ. Write $\mathcal{O}_L$ for the ring of integers of L. Then for all $b \in \mathcal{B}$ and $h \in S_n(\mathbf{Q})$ one has $e(-i\mathrm{tr}\, h)c_{I_\phi}(h, p_b) \in \mathcal{O}_L$.*

Proof. First note that it is a well-known fact that the Fourier coefficients of any newform are all contained in finite extension of $\mathbf{Q}$. By Lemma 5.4 it is enough to show that $e(-i\mathrm{tr}\, h)c_{I_\phi}(h, p_b) \in L$ for all $b \in \mathcal{B}$ and $h \in S_n(\mathbf{Q})$. Fix such a pair h, b. Then it suffices to show that $\tilde{F}_p(h, \alpha'_{\phi,p}) \in L$ for all $p \mid \gamma(h)$. Since $\alpha'_{\phi,p} \in L'$, where $L' = L$ or L'/L is a quadratic extension, we get that $\tilde{F}_p(h, \alpha'_{\phi,p}) \in L'$. If $L' = L$ we are done. Otherwise let σ be the non-trivial element of $\mathrm{Gal}(L'/L)$ and write $\beta'_{\phi,p}$ for the other p-Satake parameter of ϕ (normalized as $\alpha'_{\phi,p}$). Then one clearly has $\sigma(\alpha'_{\phi,p}) = \beta'_{\phi,p}$ and thus $\sigma(\tilde{F}_p(h, \alpha'_{\phi,p})) = \tilde{F}_p(h, \beta'_{\phi,p}) = \tilde{F}_p(h, \alpha'_{\phi,p})$, where the last equality follows from the independence of $\tilde{F}_p(h, \alpha'_{\phi,p})$ of the choice of a particular p-Satake parameter (see proof of Lemma 5.4). Hence the claim follows. □

Corollary 5.6. *Let ϕ, ϕ' be two newforms in $S_{2k+1}(D_K, \chi_K)$ (if $n = 2m$) or in $S_{2k}(1)$ (if $n = 2m + 1$). Let $\mathfrak{p}$ be a prime of a number field L containing the Fourier coefficients of both ϕ and ϕ'. Suppose that $\phi \equiv \phi'$ (mod $\mathfrak{p}^r$). Then $I_\phi \equiv I_{\phi'}$ (mod $\mathfrak{p}^r$).*

Proof. Since ϕ, ϕ' are newforms, their Hecke eigenvalues are congruent mod $\mathfrak{p}^r$. From this it is easy to see that for every prime p not dividing the level, the corresponding p-Satake parameters (which are integral over L) are also congruent mod $\mathfrak{p}^r$. Then it follows from the proof of Lemma 5.4 that $\tilde{F}_p(h, \alpha'_{\phi,p})$ (which live in $\mathcal{O}_L$ by Lemma 5.5) are congruent if $\mathfrak{p} \nmid p$ and $|\gamma(h)|^x \tilde{F}_p(h, \alpha'_{\phi,p})$ are congruent if $\mathfrak{p} \mid p$. The case of p dividing the level is immediate. □

5.3. **Non-vanishing of the Fourier coefficients of Ikeda lift mod ℓ.** The following result is due to Ikeda.

Theorem 5.7. *The Ikeda lift I_ϕ is non-zero unless $n \equiv 2$ (mod 4) and ϕ arises from a Hecke character of some imaginary quadratic field.*

Proof. This follows from Corollary 14.2 and Corollary 15.21 of [19]. □

In this section we prove a mod ℓ-version of this result. Write $\overline{\rho}_\phi : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$ for the semi-simple residual Galois representation attached to ϕ.

Theorem 5.8. *Let $\ell \nmid 2D_K$ be a prime. Let $\mathcal{B}$ be an admissible base. If $n \not\equiv 2$ (mod 4), then with respect to $\mathcal{B}$ at least one of the Fourier coefficients of I_ϕ has ℓ-adic valuation equal to zero. If $n \equiv 2$ (mod 4) and $\overline{\rho}_\phi(G_K)$ is a* *non-abelian subgroup of* $\mathrm{GL}_2(\overline{\mathbf{F}}_p)$*, then with respect to* $\mathcal{B}$ *at least one of the Fourier coefficients of* I_ϕ *has* ℓ*-adic valuation equal to zero.*

Remark 5.9. A connection between a modular eigenform ϕ arising from a Hecke character of an imaginary quadratic field K' and the condition that its ℓ-adic (so, in particular characteristic zero) Galois representation ρ_ϕ has abelian image when restricted to $G_{K'}$ was proved by Ribet [26]. This combined with Theorem 5.7 above yields a connection between non-vanishing of the Ikeda lift I_ϕ and the condition that $\rho_\phi(G_K)$ be non-abelian. Our result in essence provides an analogous connection on a mod ℓ level.

Proof of Theorem 5.8. We pick an admissible base $\mathcal{B}$. If $n \not\equiv 2 \pmod 4$, then the assertion follows from [19, Lemmas 11.1 and 11.2] because they ensure that there exists h for which $\gamma(h) = 1$ and thus we get $e(-i\mathrm{tr}\, h)c_{I_\phi}(h, p_b) = 1$ for all $b \in \mathcal{B}$. Hence for the rest of the proof we assume $n \equiv 2 \pmod 4$. We will prove Theorem 5.8 by a sequence of lemmas.

Lemma 5.10. *Suppose* p *is an odd prime such that* $p \equiv -1 \pmod{D_K}$*. Then* p *is inert in* K*.*

Proof. Recall that p being inert in K is equivalent to $\left(\frac{-D_K}{p}\right) = -1$. The result now follows easily from properties of the Legendre symbol. □

Lemma 5.11. *For every odd prime* p *with* $p \equiv -1 \pmod{D_K}$ *there exists a matrix* h_p *such that* $e(-i\mathrm{tr}\, h_p)c_{I_\phi}(h_p, p_b) = a_\phi(p)$ *for every* $b \in \mathcal{B}$*. Here* $a_\phi(p) = \alpha_{\phi,p} + \beta_{\phi,p}$ *denotes the pth Fourier coefficient of* ϕ*.*

Proof. By [19, Lemma 11.4] and its proof for every such p there exists a matrix h_p such that $\gamma(h_p) = -p$, where γ is as in (5.3). Hence by that same formula, we get that

$$e(-i\mathrm{tr}\, h_p)c_{I_\phi}(h_p, p_b) = |\gamma(h_p)|^k \tilde{F}_p(h_p, \alpha'_{\phi,p}).$$

By [19, p. 1112] we know that $\tilde{F}_p(h_p, X)$ is a Laurent polynomial whose highest degree term is X and whose lowest degree term is $\underline{\chi}_p(\gamma(h_p))^{n-1}X^{-1}$, i.e., $\tilde{F}_p(h_p, X) = X + a + \underline{\chi}_p(\gamma(h_p))1/X$ for some integer a. Here $\underline{\chi}_p(a) := \left(\frac{-D_K, a}{\mathbf{Q}_p}\right)$ (cf. [19, p. 1110]), where the latter denotes the Hilbert symbol. Using the assumptions on p, we get $\underline{\chi}_p(\gamma(h_p)) = -1$. By [19, Lemma 2.2] we have that the functional equation for $\tilde{F}_p$ reads

$$\tilde{F}_p(h_p; 1/X) = \underline{\chi}_p(\gamma(h_p))\tilde{F}_p(h_p; X).$$

So, we have

$$1/X + a - X = \tilde{F}_p(h_p; 1/X) = -\tilde{F}_p(h_p, X) = (-1)(X + a - 1/X).$$

From this it follows that $\tilde{F}_p(h_p, X) = X - 1/X$.

Thus

$$e(-i\mathrm{tr}\, h_p)c_{I_\phi}(h_p, p_b) = |\gamma(h_p)|^k \tilde{F}_p(h_p, p^{-k}\alpha_{\phi,q}) = p^k \left(\alpha_{\phi,p}p^{-k} - \frac{1}{\alpha_{\phi,p}p^{-k}}\right).$$

Since $\alpha_{\phi,p}\beta_{\phi,p} = p^{2k}\chi_K(p) = -p^{2k}$, we get

$$e(-i\mathrm{tr}\, h_p)c_{I_\phi}(h_p, p_b) = \alpha_{\phi,p} + \beta_{\phi,p} = a_p(\phi).$$

□

Note that one has $K \subset \mathbf{Q}(\zeta_{D_K})$. Write L for the splitting field of $\overline{\rho}_\phi$.

Lemma 5.12. *One has* $L \cap \mathbf{Q}(\zeta_{D_K}) = K$.

Proof. For simplicity set $D = D_K$. For any $p \mid D$ and any prime $\mathfrak{p}$ of L lying over p, write $I_\mathfrak{p}$ for the inertia subgroup of $\mathfrak{p}$. One has that

$$\overline{\rho}_\phi|_{I_\mathfrak{p}} \cong 1 \oplus \chi'_K, \tag{5.5}$$

where $\chi'_K : \mathrm{Gal}(\mathbf{Q}(\zeta_D)/\mathbf{Q}) \to \overline{\mathbf{Z}}_\ell^\times$ is the Galois character obtained from $\chi_K : (\mathbf{Z}/D\mathbf{Z})^\times \to \mathbf{C}^\times$ via the canonical identification $\mathrm{Gal}(\mathbf{Q}(\zeta_D)/\mathbf{Q}) \cong (\mathbf{Z}/D\mathbf{Z})^\times$ ([10, Theorem 3.1(e)] or [17, Theorem 3.26(3)]).

Set $K' = L \cap \mathbf{Q}(\zeta_D)$. One has the following tower of fields $\mathbf{Q} \subset K \subset K' \subset L$. Note that K' is a Galois extension of $\mathbf{Q}$, so $\mathrm{Gal}(L/K')$ is a normal subgroup of $\mathrm{Gal}(L/K)$. Since $K' \subset \mathbf{Q}(\zeta_D)$, we have the following maps:

$$(\mathbf{Z}/D\mathbf{Z})^\times \cong \mathrm{Gal}(\mathbf{Q}(\zeta_D)/\mathbf{Q}) \twoheadrightarrow \mathrm{Gal}(K'/\mathbf{Q}).$$

From this we see that $\mathrm{Gal}(K'/K)$ is generated by the images of the inertia subgroups $I_\mathfrak{p}$ for $\mathfrak{p}$ lying over p with $p \mid D$. We also have that $\overline{\rho}_\phi$ induces an isomorphism $\mathrm{Gal}(K'/\mathbf{Q}) \xrightarrow{\rho'} \rho(G_\mathbf{Q})/\rho(G_{K'})$. However, since K is the splitting field of χ'_K, we see that (5.5) implies that $\rho'|_{\mathrm{Gal}(K'/K)}$ is trivial and hence we must have that $K' = K$. □

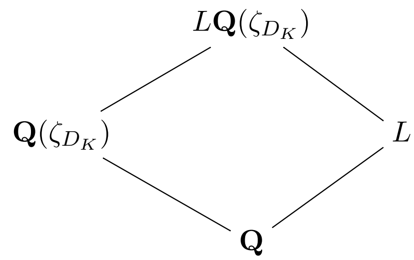
Let ϖ be a uniformizer of a finite extension E of $\mathbf{Q}_\ell$ in which all the Fourier coefficients of ϕ lie. Suppose that $e(-i\mathrm{tr}\, h)c_{I_\phi}(h, p_b) \equiv 0 \pmod{\varpi}$ for all matrices h (this makes sense by Corollary 5.5). By Lemma 5.11 this implies that $a_\phi(p) \equiv 0 \pmod{\varpi}$ for all odd primes $p \equiv -1 \pmod{D_K}$. Since for every prime $p \nmid \ell D_K$ we have $\mathrm{tr}\, \rho_\phi(\mathrm{Frob}_p) = a_p(\phi)$, we see that we must have $\mathrm{tr}\, \overline{\rho}_\phi(\mathrm{Frob}_p) \equiv 0 \pmod{\varpi}$ for all odd primes $p \nmid \ell D_K$ with $p \equiv -1 \pmod{D_K}$.

Lemma 5.13. *One has* $\mathrm{Gal}(L/\mathbf{Q}) = G \sqcup cG$*, where*

$$G := \{\text{conjugates of } \mathrm{Frob}_p \mid p \equiv 1 \pmod{D_K}\} = \mathrm{Gal}(L/K)$$

and $cG = \{\text{conjugates of } \mathrm{Frob}_p \mid p \neq 2, p \equiv -1 \pmod{D_K}\}$. *Here c is complex conjugation.*

Proof. Consider the following diagram of fields



By the Tchebotarev Density Theorem we know that the conjugates of Frob_p (as p runs over all primes not dividing ℓD_K) generate $\mathrm{Gal}(L\mathbf{Q}(\zeta_{D_K})/\mathbf{Q})$. Furthermore for any $\tau \in \mathrm{Gal}(L\mathbf{Q}(\zeta_{D_K})/\mathbf{Q})$ the image of $\tau\,\mathrm{Frob}_p\,\tau^{-1}$ in $\mathrm{Gal}(\mathbf{Q}(\zeta_{D_K})/\mathbf{Q}) \cong (\mathbf{Z}/D_K\mathbf{Z})^\times$ equals n if and only if $p \equiv n \pmod{D_K}$. Write $\varphi : \mathrm{Gal}(L\mathbf{Q}(\zeta_{D_K})/\mathbf{Q}) \twoheadrightarrow \mathrm{Gal}(L/\mathbf{Q})$ for the canonical quotient map. Since $L \cap \mathbf{Q}(\zeta_{D_K}) = K$ by Lemma 5.12, the image $\varphi(\mathrm{Gal}(L\mathbf{Q}(\zeta_{D_K})/\mathbf{Q}(\zeta_{D_K})))$ is an index two subgroup G of $\mathrm{Gal}(L/\mathbf{Q})$ and in fact this subgroup must be $\mathrm{Gal}(L/K)$. Since $\mathrm{Gal}(L\mathbf{Q}(\zeta_{D_K})/\mathbf{Q}(\zeta_{D_K}))$ is generated by the set

$$\{\text{conjugates of } \mathrm{Frob}_p \mid p \equiv 1 \pmod{D_K}\},$$

we get the same for its image in $\mathrm{Gal}(L/\mathbf{Q})$.

Now note that c itself is a conjugate of Frob_p with $p \equiv -1 \pmod{D_K}$. Consider $c\,\mathrm{Gal}(L\mathbf{Q}(\zeta_{D_K})/\mathbf{Q}(\zeta_{D_K})) \subset \mathrm{Gal}(L\mathbf{Q}(\zeta_{D_K})/\mathbf{Q})$. On the one hand we have that $\varphi(c\,\mathrm{Gal}(L\mathbf{Q}(\zeta_{D_K})/\mathbf{Q}(\zeta_{D_K}))) = cG$ and on the other hand we must have that the image of $c\,\mathrm{Gal}(L\mathbf{Q}(\zeta_{D_K})/\mathbf{Q}(\zeta_{D_K}))$ in $\mathrm{Gal}(\mathbf{Q}(\zeta_{D_K})/\mathbf{Q})$ is $\{-1\}$. Thus every element of $c\,\mathrm{Gal}(L\mathbf{Q}(\zeta_{D_K})/\mathbf{Q}(\zeta_{D_K}))$ (and hence also of cG) is a conjugate of Frob_p with $p \equiv -1 \pmod{D_K}$. Finally, by the Tchebotarev Density Theorem we can omit Frob_2 from this set. □

We will now finish the proof of Theorem 5.8. Let $\sigma \in G = \mathrm{Gal}(L/K)$. Then $c\sigma = \tau\,\mathrm{Frob}_p\,\tau^{-1}$ for some $\tau \in \mathrm{Gal}(L/\mathbf{Q})$ and some $p \equiv -1 \bmod D_K$. Hence

$$\mathrm{tr}\,\overline{\rho}_\phi(c\sigma) = \mathrm{tr}\,\overline{\rho}_\phi(\tau\,\mathrm{Frob}_p\,\tau^{-1}) = \mathrm{tr}\,\overline{\rho}_\phi(\mathrm{Frob}_p) = 0.$$

In some basis we have $\overline{\rho}_\phi(c) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$ and thus if we write $\overline{\rho}_\phi(\sigma) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then we have $\overline{\rho}_\phi(c\sigma) = \begin{bmatrix} a & b \\ -c & -d \end{bmatrix}$. Since $\mathrm{tr}\,\overline{\rho}_\phi(c\sigma) = 0$ we get $a = d$. So, we have now proved that with respect to some fixed basis all elements $\sigma \in G$ have the property that $\overline{\rho}_\phi(\sigma)$ has its upper-left and its lower-right entries equal. This ensures that $\overline{\rho}_\phi(G)$ is abelian (see the last few lines of the proof of [23, Proposition 8.13] for details). This finishes the proof of the theorem. □

Corollary 5.14. *Fix an admissible base $\mathcal{B}$. If $n \not\equiv 2 \pmod 4$, then for every $b \in \mathcal{B}$ there exists $h \in S_n^+(\mathbf{Q})$ with $\gamma(h) = 1$ such that $\mathrm{val}_\varpi(\det h) = \mathrm{val}_\varpi(e(-i\mathrm{tr}\,h)c_{I_\phi}(h, p_b)) = 0$. If $n \equiv 2 \pmod 4$ then for every $b \in \mathcal{B}$ there* *exists a prime* $p \nmid 2\ell D_K$*, inert in* K *and* $h \in S_n(\mathbf{Q})$ *with* $\gamma(h) = -p$ *such that* $\mathrm{val}_\varpi(\det h) = \mathrm{val}_\varpi(e(-i\mathrm{tr}\, h)c_{I_\phi}(h, p_b)) = 0$.

Proof. If $n \not\equiv 2 \pmod 4$, then [19, Lemmas 11.1 and 11.2] allow us to find h with $\gamma(h) = 1$ hence the claim follows from the definition of γ, formula (5.3) and the assumption that $\ell \nmid D_K$. If $n \equiv 2 \pmod 4$, then the existence of h follows from the proof of Theorem 5.8. □

6. Congruence to Ikeda lift

6.1. **Congruence with respect to the periods** $\langle \phi, \phi \rangle$**.** We keep the notation and assumptions from section 5. Set $\mathcal{J}(K) = \frac{1}{2}\#\mathcal{O}_K^\times$. Note that $\mathcal{J}(K) = 1$ when $D_K > 12$. It was shown in [24, Proposition 3.13] that $\mathcal{M}_{n,k}(\mathcal{K}) \cong \mathcal{M}_{n,k,\nu}(\mathcal{K})$ provided that $\mathcal{J}(K) \mid \nu$ and $(2n, h_K) = 1$. The isomorphism between the two spaces is Hecke-equivariant and is given by a function $\Psi_\beta : f \mapsto \beta \otimes f$, where β is an everywhere unramified Hecke character of K of infinity type $\left(\frac{z}{|z|}\right)^{-2\nu}$. For details we refer the reader to [24, p. 811-812].

From now on we assume that $(h_K, 2n) = 1$ and $\mathcal{J}(K) \mid \nu$ so that $\Psi_\beta^{-1}(I_\phi) \in \mathcal{S}_{n,2k+2m}(G_n(\hat{\mathbf{Z}}))$, where from now on we fix β as above with $-2\nu = 2k + 2m$.

The aim of this section is to prove the following result. Set

$$\mathcal{V} = \begin{cases} \prod_{i=2}^n \frac{L(i+2k-1, \mathrm{Sym}^2\,\phi \otimes \chi_K^{i+1})}{\pi^{2k+2i-1}\langle \phi, \phi\rangle} & n = 2m+1 \\ \prod_{i=2}^n \frac{L(i+2k, \mathrm{Sym}^2\,\phi \otimes \chi_K^{i})}{\pi^{2k+2i}\langle \phi, \phi\rangle} & n = 2m. \end{cases}$$

By a result of Sturm [32, p. 220-221] for $n = 2m+1$ (i.e., the weight of ϕ equals $2k$) we have

$$\frac{L(i+2k-1, \mathrm{Sym}^2\,\phi \otimes \chi_K^{i+1})}{\pi^{2k+2i-1}\langle \phi, \phi\rangle} \in \overline{\mathbf{Q}} \tag{6.1}$$

and for $n = 2m$ (i.e., the weight of ϕ equals $2k+1$) we have

$$\frac{L(i+2k, \mathrm{Sym}^2\,\phi \otimes \chi_K^{i})}{\pi^{2k+2i}\langle \phi, \phi\rangle} \in \overline{\mathbf{Q}} \tag{6.2}$$

for $2 \leq i \leq 2k-1$ (cf. Remark 5.3 for the discrepancy in the exponent of π between here and in [32]). Indeed, let us note that regardless of the parity of n our points of evaluation are always in the second subset of what is called S_1 on page 220 of [32]. The inequalities there translate to exactly the above range for the values of i. Hence to apply this result to the L-factors appearing in $\mathcal{V}$ we need to (and will from now on) make the assumption that $n \leq 2k-1$. Thus in particular $\mathcal{V} \in \overline{\mathbf{Q}}$.

Theorem 6.1. *Assume* $n \leq 2k-1$*. Let* $\ell \nmid 2h_K D_K$ *be a rational prime with* $\ell > 2k + 2m$ *and write* ϖ *for a choice of a uniformizer in some sufficiently large finite extension of* $\mathbf{Q}_\ell$*. Let* ξ *be a Hecke character of* K *such that*

$\mathrm{val}_\varpi(\mathrm{cond}\,\xi) = 0$, $\xi_\infty(z) = \left(\frac{z}{|z|}\right)^{-t}$ *for some* $t \in \mathbf{Z}$ *with* $-2k - 2m \leq t < \min\{-6, -4n\}$. *Then*

(6.3)

$$\mathcal{U} := \frac{\prod_{i=1}^{n} \pi^{-2n-2k-2m-t+2i-2} L^{D_K}(n + t/2 + k + m - i + 1, \mathrm{BC}(\phi) \otimes \xi^{-1}\beta)}{\langle \phi, \phi \rangle^n} \times \prod_{i=2}^{n} \frac{L(i, \chi_K^i)}{\pi^i} \times L_{D_K}(2n + t/2, \Psi_\beta^{-1}(I_\phi), \xi^{-1}; \mathrm{st})$$

belongs to $\overline{\mathbf{Q}}$.

Let $\tau \in S_n(\mathbf{Q})$ *be as in Corollary 5.14 and set* $N = TD_K h_K \,\mathrm{N}_{K/\mathbf{Q}}(\mathrm{cond}\,\xi)$, *where* $T \in \mathbf{Z}$ *is a generator of the inverse of the fractional ideal* $\{g^*\tau^{-1}g \mid g \in \mathcal{O}_K^n\}$ *of* $\mathbf{Q}$. *Assume that* $\mathrm{val}_\varpi\,(T\#(\mathcal{O}_K/N\mathcal{O}_K)^\times) = 0$ *and* $\mathrm{val}_\varpi(A_N) \geq 0$ *(for definition of* A_N *see section 3.3). If* $\mathrm{val}_\varpi(\mathcal{U}) = 0$ *and* $b := \mathrm{val}_\varpi(\mathcal{V}) > 0$, *then there exists a non-zero* $f' \in \mathcal{S}_{n,2k+2m}(G_n(\hat{\mathbf{Z}}))$, *orthogonal to* $\Psi_\beta^{-1}(I_\phi)$, *such that* $f' \equiv \Psi_\beta^{-1}(I_\phi) \pmod{\varpi^b}$.

Remark 6.2. The assumptions in Theorem 6.1 ensure that the assumptions of Theorem 4.3 are satisfied. Note that in the current setup $F = \mathbf{Q}$ and since we will apply Theorem 4.3 for the eigenform $\Psi_\beta^{-1}(I_\phi)$, the weight k in Theorem 4.3 is replaced by $2k + 2m$ here. Also note that Lemma 5.4 guarantees that the Fourier coefficients of I_ϕ are $\mathcal{O}$-integral. Furthermore, we pick r in Theorem 4.3 to be I_n and we choose τ to be as in Corollary 5.14. The conditions that $\gamma(\tau) = 1$ or p easily imply that condition (3.2) is satisfied, i.e., that $\{g^*\tau g \mid g \in \mathcal{O}_K^n\} = \mathbf{Z}$. Hence Corollary 5.14 guarantees that for this choice of r and τ we get $\mathrm{val}_\varpi(e(-i\mathrm{tr}\,\tau)c_{I_\phi}(\tau, r)) = 0$. Also note that τ and T enter the statement of Theorem 6.1 only via the assumptions on the valuations of $\#(\mathcal{O}_K/N\mathcal{O}_K)^\times$ and A_N.

Remark 6.3. Since Theorem 6.1 holds for any character ξ with the specified properties, it is likely that the assumption $\mathrm{val}_\varpi(\mathcal{U}) = 0$ always holds for some choice of ξ. For a more detailed explanation cf. e.g. [8, Section 5].

Proof. In this proof set $\mathcal{K} = G_n(\hat{\mathbf{Z}})$. As noted in Theorem 4.3 (and using (3.4)) we have

$$L^{\mathrm{alg}}(2n+t/2, \Psi_\beta^{-1}(I_\phi), \xi; \mathrm{st}) = \frac{L(2n + t/2, \Psi_\beta^{-1}(I_\phi), \xi; \mathrm{st})\,\mathrm{vol}(\mathcal{F}_\mathcal{K})}{\pi^{n(2n+(2k+2m)+t+1)}\langle \Psi_\beta^{-1}(I_\phi), \Psi_\beta^{-1}(I_\phi)\rangle_\mathcal{K}} \in \overline{\mathbf{Q}}.$$

One easily checks (cf. e.g. [24, Lemma 8.6]) that $\langle \Psi_\beta^{-1}(I_\phi), \Psi_\beta^{-1}(I_\phi)\rangle_\mathcal{K} = \langle I_\phi, I_\phi\rangle_\mathcal{K}$. Using (5.1) and (5.2) we see that

$$\overline{L_{D_K}(2n + t/2, \Psi_\beta^{-1}(I_\phi), \xi; \mathrm{st}) L^{\mathrm{alg}}(2n + t/2, \Psi_\beta^{-1}(I_\phi), \xi; \mathrm{st})}$$

equals (cf. also [24, p. 849])

(6.4)
$$\frac{\pi^{-n(2n+(2k+2m)+t+1)}\prod_{i=1}^{n}L^{D_K}(n+t/2+k+m-i+1,\mathrm{BC}(\phi)\otimes\xi^{-1}\beta)}{\langle\phi,\phi\rangle\prod_{i=2}^{n}L(i+2k-1,\mathrm{Sym}^2\phi\otimes\chi_K^{i+1})L(i,\chi_K^i)\Gamma_{\mathbf{C}}(i+2k-1)\Gamma_{\mathbf{C}}(i)^2}\mathrm{vol}(\mathcal{F}_{\mathcal{K}})$$

for $n=2m+1$ and

(6.5)
$$\frac{\pi^{-n(2n+(2k+2m)+t+1)}\prod_{i=1}^{n}L^{D_K}(n+t/2+k+m-i+1,\mathrm{BC}(\phi)\otimes\xi^{-1}\beta)}{\langle\phi,\phi\rangle\prod_{i=2}^{n}L(i+2k,\mathrm{Sym}^2\phi\otimes\chi_K^{i})L(i,\chi_K^i)\Gamma_{\mathbf{C}}(i+2k)\Gamma_{\mathbf{C}}(i)^2}\mathrm{vol}(\mathcal{F}_{\mathcal{K}})$$

for $n=2m$.

Using the definition of $\Gamma_{\mathbf{C}}$ expressions (6.4) and (6.5) become (here u is some ϖ-adic unit)

(6.6)
$$\frac{\pi^{-n(n+(2k+2m)+t+1)}\prod_{i=1}^{n}L^{D_K}(n+t/2+k+m-i+1,\mathrm{BC}(\phi)\otimes\xi^{-1}\beta)u}{\langle\phi,\phi\rangle\prod_{i=2}^{n}L(i,\chi_K^i)\pi^{-i}L(i+2k-1,\mathrm{Sym}^2\phi\otimes\chi_K^{i+1})\pi^{-2k-2i+1}}\frac{\mathrm{vol}(\mathcal{F}_{\mathcal{K}})}{\pi^{n^2}}$$

for $n=2m+1$ and

(6.7)
$$\frac{\pi^{-n(n+(2k+2m)+t+1)}\prod_{i=1}^{n}L^{D_K}(n+t/2+k+m-i+1,\mathrm{BC}(\phi)\otimes\xi^{-1}\beta)u}{\langle\phi,\phi\rangle\prod_{i=2}^{n}L(i,\chi_K^i)\pi^{-i}L(i+2k,\mathrm{Sym}^2\phi\otimes\chi_K^{i})\pi^{-2k-2i}}\frac{\mathrm{vol}(\mathcal{F}_{\mathcal{K}})}{\pi^{n^2}}$$

for $n=2m$.

For a Hecke character ψ of K of infinity type $(z/|z|)^u$ $(u\leq 0)$ we will write

$$g_\psi=\sum_{j=1}^{\infty}a_{g_\psi}(j)q^j \quad \text{with} \quad a_{g_\psi}(j)=\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_K \text{ ideal}\\ N(\mathfrak{a})=j}}j^{u/2}\psi(\mathfrak{a})$$

for the associated modular form of weight $-u+1$ (which is a cusp form if $u<0$). Observe that we have $L(s,g_\psi)=L(s-u/2,\psi)$. It is easy to check one has

$$L^{D_K}(s,\mathrm{BC}(\phi)\otimes\psi)=L^{D_K}(s-u/2,\phi\otimes g_\psi),$$

where $L^{D_K}(s,\phi\otimes g_\psi)$ is the convolution L-function which for $s\in\mathbf{C}$ with sufficiently large real part is defined by

$$\prod_{p\nmid D_K}\{(1-\alpha_\phi\alpha_{g_\psi}p^{-s})(1-\alpha_\phi\beta_{g_\psi}p^{-s})(1-\beta_\phi\alpha_{g_\psi}p^{-s})(1-\beta_\phi\beta_{g_\psi}p^{-s})\}^{-1}$$

with α_ϕ,β_ϕ and $\alpha_{g_\psi},\beta_{g_\psi}$ the Satake parameters of ϕ and g_ψ respectively where, as above, the Satake parameters are normalized arithmetically.

We note that the character $\xi^{-1}\beta$ has infinity type $(z/|z|)^{2k+2m+t}$, hence the character $\xi\beta^{-1}$ has infinity type $(z/|z|)^{-2k-2m-t}$ and the number $-2k-2m-t$ is by our assumption on t a negative number. Thus the cusp form corresponding to $\xi\beta^{-1}$ is $g=g_{\xi\beta^{-1}}$ which is of weight $2k+2m+t+1>0$. Hence the L-function in the numerator of (6.4) equals

$$\begin{aligned}
&\overline{L^{D_K}(n+t/2+k+m-i+1, \mathrm{BC}(\phi)\otimes\xi\beta^{-1})}\\
(6.8)\qquad &=\overline{L^{D_K}(n+t+2k+2m-i+1, \phi\otimes g)}\\
&=L^{D_K}(n+t+2k+2m-i+1, \phi^c\otimes g^c),
\end{aligned}$$

where c denotes conjugating the Fourier coefficients.

Since $t<\min\{-6,-4n\}$ we see that for every $i\in\{1,2,\dots,n\}$ the point of evaluation $n+2k+2m+t-i+1$ satisfies

$$2k+2m+t<2k+2m+t+n-i+1<2k,$$

i.e., the points of evaluation lie strictly between the weights of ϕ and g, hence they are critical points in the sense of Deligne. We have the following result due to Shimura.

Theorem 6.4. *If ϕ and ϕ' are two cuspidal eigenforms of weights l,l' respectively, and $l>l'$ then for all integers M such that $l'\leq M<l$ one has*

$$\frac{\pi^{l'-1-2M}L(M,\phi\otimes\phi')}{\langle\phi,\phi\rangle}\in\overline{\mathbf{Q}}.$$

The values of M in the above ranges are critical.

Proof. This is stated on page 218 of [27]. For the discrepancy in the exponent of π between our statement and the formula in [loc.cit.], see Remark 5.3. $\square$

Applied to our case ($l=2k$ if $n=2m+1$, $l=2k+1$ if $n=2m$, and $l'=2k+2m+t+1$) this implies that

$$\frac{\pi^{-2n-2k-2m-t+2i-2}L(n+2k+2m+t-i+1,\phi\otimes g)}{\langle\phi,\phi\rangle}\in\overline{\mathbf{Q}}\quad\text{for } i\in\{1,2,\dots,n\}.$$

Note that the power of π in $\mathcal{U}$ (not involved in normalizing the Dirichlet L-function) equals

$$\begin{aligned}
\sum_{i=1}^{n}(-2n-2k-2m-t+2i-2)&=-n(2n+2k+2m+t+2)+2\sum_{i=1}^{n}i\\
&=-n(n+2k+2m+t+1)
\end{aligned}$$

and that $L(i,\chi_K^i)\pi^{-i}\in\overline{\mathbf{Q}}$. Thus we have proved that $\mathcal{U}\in\overline{\mathbf{Q}}$. Hence both (6.6) and (6.7) equal

$$u\frac{\mathcal{U}}{\mathcal{V}}\frac{\mathrm{vol}(\mathcal{F}_{\mathcal{K}})}{\pi^{n^2}},$$

where u is a ϖ-adic unit. To summarize we have shown

$$\begin{aligned}
\mathrm{val}_{\varpi}\left(L^{\mathrm{alg}}(2n+t/2,\Psi_\beta^{-1}(I_\phi),\xi^{-1};\mathrm{st})\frac{\pi^{n^2}}{\mathrm{vol}(\mathcal{F}_{\mathcal{K}})}\right)&=\mathrm{val}_{\varpi}(\mathcal{U}/\mathcal{V})\\
&=-\mathrm{val}_{\varpi}(\mathcal{V})\\
&=-b.
\end{aligned}$$

The result now follows directly from Theorem 4.3 and the fact that $f' \neq 0$ follows from Remark 4.4 and Theorem 5.8. □

Remark 6.5. Theorem 5.8 ensures that there is a genuine congruence to depth b between I_ϕ and some orthogonal form f', i.e., that no part of that congruence comes from the fact that I_ϕ itself is congruent to zero to some depth. We also note that Theorem 5.8 is indeed necessary for the construction of such a congruence, because our method does not allow for any rescaling of the lift as such a rescaling would change the inner products used in the derivation of the congruence.

7. Congruence with respect to integral periods

It is possible that the congruence constructed in Theorem 6.1 is "inherited" directly from a congruence between ϕ and another newform $\phi' \in S_{2k}(1)$ (or $S_{2k+1}(D_K, \chi_K)$). It is well-known [15] that for ℓ as in the last section (i.e., $\ell > 2k+1$) the mod ℓ^r congruences between ϕ and another ϕ' are controlled by the ratio

$$\frac{\langle \phi, \phi \rangle}{\Omega_\phi^+ \Omega_\phi^-} \in \overline{\mathbf{Q}}_\ell$$

at least when ϕ is ordinary (see e.g., [6, Section 5] for definitions of $\Omega_\phi^\pm$ as well as the discussion of the ratio). Thus by replacing the Petersson norm with the product of the so called integral periods $\Omega_\phi^\pm$ we can formulate conditions which ensure that the resulting congruence is between I_ϕ and an automorphic form $f' \in \mathcal{M}_{n,2k+2m}(G_n(\hat{\mathbf{Z}}))$, which does not arise as an Ikeda lift. However, in doing so, we will make use of the following conjecture.

Conjecture 7.1. *Fix a rational prime ℓ. Let $f \in \mathcal{S}_{n,k,\nu}(\mathcal{K})$ be a Hecke eigenform of central character ω for $\mathcal{K} \subset G_n(\mathbf{A}_{\mathbf{Q},\mathbf{f}})$ an open compact subgroup. Then there exists a continuous semi-simple representation $\rho_f : G_K \to \mathrm{GL}_{2n}(\overline{\mathbf{Q}}_\ell)$ such that*

(i) *ρ_f is unramified at all finite places not dividing D_K;*
(ii) *One has $L^{D_K}(s, \rho_f) = L^{D_K}(s, \mathrm{BC}(f) \otimes \omega^c; \mathrm{st})$,*
(iii) *If $\ell \nmid D_K$, then for any $\mathfrak{p} \mid \ell$, the representation $\rho_f|_{D_\mathfrak{p}}$ is crystalline.*

Remark 7.2. The above conjecture is widely regarded as a known result, however the only reference we know of in which the existence of Galois representations attached to automorphic forms on $G_n(\mathbf{A}_F)$, for F a totally real field, is proven is the article of Skinner [31, Theorem 7.1] which excludes the parallel weight case. Let us also mention that a similar remark to Remark 5.2 regarding different normalization of L-functions concerns [31].

From now on as in the last section, let $\ell \nmid 2D_K$, $\ell > n + 2k - 1$ be a prime and ϕ be a newform in $S_{2k}(1)$ (if $n = 2m+1$) or in $S_{2k+1}(D_K, \chi_K)$ (if $n = 2m$). Let $\rho_\phi : G_\mathbf{Q} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$ be the ℓ-adic Galois representation associated to ϕ by Deligne et al. and write $\rho_\phi(j)$ for its jth Tate twist. We will also assume that the residual Galois representation $\overline{\rho}_\phi : G_\mathbf{Q} \to$

$\mathrm{GL}_2(\overline{\mathbf{F}}_\ell)$ is irreducible when restricted to G_K. This combined with (5.1) and Conjecture 7.1 implies that ρ_{I_ϕ} is isomorphic to

$$\epsilon^{-k-m+n} \otimes \bigoplus_{j=0}^{n-1} \rho_\phi(j)|_{G_K}. \tag{7.1}$$

Let $S_0' = \{\phi_1 = \phi, \phi_2, \dots, \phi_{r'}\}$ be the subset of a basis of newforms of $S_{2k}(1)$ (if $n = 2m+1$) or of $S_{2k+1}(D_K, \chi_K)$ (if $n = 2m$) consisting of forms congruent to ϕ (mod ϖ). Write $S_1' := \{f_1 = I_\phi, \dots, f_{r'} = I_{\phi_{r'}}\}$ for the corresponding set of Ikeda lifts. Since $\overline{\rho}_\phi|_{G_K}$ is irreducible, it must have a non-abelian image, hence it follows from Theorem 5.8 that all $f_i \in S_1'$ are non-zero. We choose a subset S_0 of S_0' so that the elements of $S_1 := \{f_i = I_{\phi_i} \in S_1' \mid \phi_i \in S_0\}$ are linearly independent and span the same subspace of $\mathcal{S}_{2k+2m}(G_n(\hat{\mathbf{Z}}))$ as S_1'. Renumbering the elements of S_0' and hence of S_1' if necessary we may assume that $S_0 = \{\phi_1 = \phi, \phi_2, \dots, \phi_r\}$ and $S_1 = \{f_1 = I_\phi, f_2, \dots f_r\}$ for some $r \leq r'$. By Corollary 5.6 we know that $f_i \equiv f_1$ (mod ϖ) for all $1 \leq i \leq r$. We complete this set to a basis $f_1, \dots, f_r, f_{r+1}, \dots, f_s$ of the subspace $\mathcal{W}$ of $\mathcal{S}_{2k+2m}(G_n(\hat{\mathbf{Z}}))$ containing all eigenforms which are congruent to f_1 (mod ϖ) (in the sense of Definition 2.4) and note that by Theorem 5.8 we must have $f_i \not\equiv 0$ (mod ϖ) for all $1 \leq i \leq s$.

For $\sigma \in G_K$ let $\sum_{j=0}^{2n} c_j(i,\sigma) X^j \in \mathcal{O}[X]$ be the characteristic polynomial of $\rho_{f_i}(\sigma)$, where ρ_{f_i} is the Galois representation attached to f_i (Conjecture 7.1). Here $\mathcal{O}$ is the valuation ring of some sufficiently large finite extension of $\mathbf{Q}_\ell$. Put $c_j(\sigma) := \begin{bmatrix} c_j(1,\sigma) \\ \vdots \\ c_j(s,\sigma) \end{bmatrix} \in \mathcal{O}^s$ for $j = 0, 1, \dots, 2n$. Let $\mathbf{T}$ be the $\mathcal{O}$-subalgebra of $\mathcal{O}^s$ generated by the set $\{c_j(\sigma) \mid 0 \leq j \leq 2n, \sigma \in G_K\}$. The algebra $\mathbf{T}$ acts on $\mathcal{W}$ by

$$Tf = \begin{bmatrix} t_1 \\ \vdots \\ t_s \end{bmatrix} \sum_{i=1}^{s} \alpha_i f_i := \sum_{i=1}^{s} \alpha_i t_i f_i.$$

We will say that $\mathbf{T}$ *preserves the integrality of Fourier coefficients* if whenever $f \in \mathcal{W}$ has $\mathcal{O}$-integral Fourier coefficients, so does Tf for all $T \in \mathbf{T}$ (cf. Definition 2.4).

Remark 7.3. It follows from the Tchebotarev Density Theorem that $\mathbf{T}$ is generated by the set $\{c_j(\mathrm{Frob}_\mathfrak{p}) \mid 0 \leq j \leq 2n, \mathfrak{p} \nmid D_K\}$. By Conjecture 7.1 (ii), each $c_j(\mathrm{Frob}_\mathfrak{p})$ is a polynomial in Hecke operators. The action of each such Hecke operator on Fourier coefficients (albeit tedious) should be straightforward to compute (we refer the reader for example to [23, (5.6)] where this is done for $n = 2$) and from this one should be able to see that these polynomials preserve the integrality of Fourier coefficients. Hence in fact we expect that $\mathbf{T}$ always preserves the integrality of Fourier coefficients, but we do not pursue the full proof here since the theorem below (for which the integrality is used) is already conditional on Conjecture 7.1 and a full proof may be computationally involved.

Theorem 7.4. *Assume that Conjecture 7.1 holds. Keep the assumptions of Theorem 6.1, but replace all occurrences of* $\langle\phi,\phi\rangle$ *in the definitions of* $\mathcal{U}$ *and* $\mathcal{V}$ *by* $\Omega_\phi^+\Omega_\phi^-$*. Suppose furthermore that* ϕ *is ordinary at* ℓ *with* $\overline{\rho}_\phi|_{G_K}$ *absolutely irreducible and that the algebra* $\mathbf{T}$ *preserves the integrality of Fourier coefficients. Then the claims of Theorem 6.1 hold true, but the form* $f' \in \mathcal{M}_{n,2k+2m}(G_n(\hat{\mathbf{Z}}))$ *congruent to* $\Psi_\beta^{-1}(I_\phi) \pmod{\varpi^b}$ *is not an Ikeda lift of any newform* $\phi' \in S_{2k}(1)$ *if* $n = 2m+1$ *(resp.* $\phi' \in S_{2k+1}(D_K, \chi_K)$ *if* $n = 2m$*).*

Proof. As in [6] to achieve our goal we will show the existence of a certain 'idempotent-like' Hecke operator in the Hecke algebra on $G_n(\mathbf{A})$. However, instead of deducing its existence from explicit relations between Hecke operators on G_n and GL_2 (as in [loc.cit.]) we will construct it from the knowledge of the Galois representation of I_ϕ (a variant of this method was used in [23, Section 5.5]). As the arguments for $n = 2m$ are entirely analogous, we limit ourselves to the case $n = 2m+1$.

Denote by $\mathbf{T}_f$ the quotient of $\mathbf{T}$ defined in the same way as $\mathbf{T}$ but with r in place of s. After choosing some $G_{\mathbf{Q}}$-invariant lattice we may assume that ρ_{ϕ_i} is valued in $\mathrm{GL}_2(\mathcal{O})$. Write $\rho : \mathcal{O}[G_K] \to \mathrm{Mat}_2(\mathcal{O}^r)$ for the $\mathcal{O}$-algebra map sending $g = \sum_l a_l g_l \in \mathcal{O}[G_K]$ to $\sum_l a_l \begin{bmatrix} \rho_{\phi_1}(g_l) \\ \vdots \\ \rho_{\phi_r}(g_l) \end{bmatrix}$. By a standard argument (using the irreducibility of $\overline{\rho}_{\phi_i}|_{G_K}$) one can show that the image of ρ is contained in $\mathrm{Mat}_2(\mathbf{T}_0)$, where $\mathbf{T}_0$ is the $\mathcal{O}$-Hecke algebra acting on $S_{2k}(1)$ (cf. e.g. [10, Lemma 3.27]). In particular $\mathrm{tr}\,\rho(g)$ can be interpreted as a Hecke operator. It thus follows from [23, Proposition 8.14] that there exists $g_0 \in \mathcal{O}[G_K]$ such that $\mathrm{tr}\,\rho(g_0)\phi_1 = \frac{\langle\phi,\phi\rangle}{\Omega_\phi^+\Omega_\phi^-}\phi_1$ and $\mathrm{tr}\,\rho(g_0)\phi_i = 0$ for all $i > 1$. The proposition in [loc.cit.] as stated applies only to $K = \mathbf{Q}(i)$, but the deformation-theoretic arguments in the proof are not sensitive to these restrictions and the result of Hida they reduce the proof to is also valid in full generality.

Now let $\tilde{\rho} : \mathcal{O}[G_K] \to \mathrm{Mat}_{2n}(\mathcal{O}^r)$ be the $\mathcal{O}$-algebra map sending $g = \sum_l a_l g_l \in \mathcal{O}[G_K]$ to $\sum_l a_l \begin{bmatrix} \rho_{I_1}(g_l) \\ \vdots \\ \rho_{I_r}(g_l) \end{bmatrix}$. By (7.1) we see that $\tilde{\rho}$ decomposes as a direct product of Tate twists of ρ, hence one concludes that the image of $\tilde{\rho}$ is contained in $\mathrm{Mat}_{2n}(\mathbf{T}_f)$. Using (7.1) again and the fact that ϕ is ordinary at ℓ and working as in the proof of [23, Proposition 5.14] we see that there exists a projector $e \in \mathcal{O}[G_K]$ such that

$$\tilde{\rho}(e) = \begin{bmatrix} \rho(e) & & & \\ & \rho(1)(e) & & \\ & & \ddots & \\ & & & \rho(n-1)(e) \end{bmatrix} = \begin{bmatrix} \rho(e) & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}.$$

Set $T_\phi := \operatorname{tr} \tilde{\rho}(eg_0)$ with g_0 as above and let T be any lift of T_ϕ to $\mathbf{T}$. It is clear that this T has the desired properties. Now applying T to both sides of (4.2) and amending the periods in the L-functions appearing in C_{f_1} we get the desired result. We refer the reader to [6] or [23] for details. □

8. Speculations about consequences for the Bloch-Kato conjecture

Let us now specialize to the case $n = 3$. The relevant L-values in this case are

$$L_1 := \frac{L(2k+1, \operatorname{Sym}^2 \phi \otimes \chi_K)}{\pi^{2k+3}\Omega_\phi^+\Omega_\phi^-} \in \overline{\mathbf{Q}}, \quad L_2 := \frac{L(2k+2, \operatorname{Sym}^2 \phi)}{\pi^{2k+5}\Omega_\phi^+\Omega_\phi^-} \in \overline{\mathbf{Q}}.$$

For $i = 1, 2$ set $v_i := \operatorname{val}_\ell(\#\mathcal{O}/L_i)$. The ($\varpi$-part of the) Bloch-Kato conjecture for a G_F-module (with F a number field) predicts that $\operatorname{val}_\ell(\#H^1_f(F, V^\vee(1)))$ and $\operatorname{val}_\ell(\#\mathcal{O}/L^{\mathrm{alg}}(V, 0))$ (where $L^{\mathrm{alg}}(V, 0)$ is the appropriately normalized L-value of V at zero) should be related (in fact equal up to certain canonically defined factors - see e.g. [24, Section 9] for a precise statement). We have

$$\begin{aligned} V_1 &:= \operatorname{ad}^0 \rho_\phi(2) \otimes \chi_K, \quad V_1^\vee = \operatorname{ad}^0 \rho_\phi(-2) \otimes \chi_K, \\ V_2 &:= \operatorname{ad}^0 \rho_\phi(3), \quad V_2^\vee = \operatorname{ad}^0 \rho_\phi(-3). \end{aligned} \tag{8.1}$$

So, the Bloch-Kato conjecture for V_1 relates $\operatorname{val}_\ell(\#H^1_f(\mathbf{Q}, \operatorname{ad}^0 \rho_\phi(-1) \otimes \chi_K))$ to $\operatorname{val}_\ell(\#\mathcal{O}/L^{\mathrm{alg}}(\operatorname{ad}^0 \rho_\phi(2) \otimes \chi_K, 0)) = v_1$, and for V_2 it relates $\operatorname{val}_\ell(\#H^1_f(\mathbf{Q}, \operatorname{ad}^0 \rho_\phi(-2)))$ to $\operatorname{val}_\ell(\#\mathcal{O}/L^{\mathrm{alg}}(\operatorname{ad}^0 \rho_\phi(3), 0)) = v_2$.

By Theorem 7.4 there exists $f' \in \mathcal{S}_{2k+2}(G_3(\hat{\mathbf{Z}}))$ orthogonal to the space spanned by all the Ikeda lifts such that $f' \equiv I_\phi \pmod{\varpi^{v_1+v_2}}$. Assume for simplicity that $v_1 = 0$ and $v_2 = 1$. By a standard Ribet-style argument [25] it is possible to show that there exists a non-semi-simple representation $\rho : G_K \to \mathrm{GL}_6(\overline{\mathbf{F}}_\ell)$ of the form

$$\rho(g) = \epsilon(g)^{2-k} \begin{bmatrix} \rho_\phi(g) & a(g) & b(g) \\ & \rho_\phi(g)\epsilon(g) & c(g) \\ & & \rho_\phi(g)\epsilon^2(g) \end{bmatrix}.$$

This representation is unramified away from primes dividing ℓ and crystalline at ℓ. First note that both a and c give rise to elements in $H^1_f(K, \operatorname{ad}^0 \rho_\phi(-1)|_{G_K})$. One has a natural splitting

$$H^1_f(K, \operatorname{ad}^0 \rho_\phi(-1)) = H^1_f(K, \operatorname{ad}^0 \rho_\phi(-1))^+ \oplus H^1_f(K, \operatorname{ad}^0 \rho_\phi(-1))^-,$$

where the superscripts indicate the sign of the action of the complex conjugation.

Suppose now one can show that $a, c \in H^1_f(K, \operatorname{ad}^0 \rho_f(-1))^-$. Let us explain how one can use our result to conclude that the validity of the Bloch-Kato conjecture for V_1 gives evidence for its validity for V_2. The inflation-restriction sequence gives an identification

$$H^1(K, \operatorname{ad}^0 \rho_f(-1))^- = H^1(\mathbf{Q}, \operatorname{ad}^0 \rho_f(-1) \otimes \chi_K)$$

and using this one can show that the classes a, c lie in the Selmer group of $\operatorname{ad}^0 \rho_\phi(-1) \otimes \chi_K$ which by the Bloch-Kato conjecture for V_1 and our assumption that $v_1 = 0$ is trivial. So, a, c must be trivial classes. Hence for ρ to be non-semi-simple, b has to give rise to a non-trivial class in $H^1(K, \operatorname{ad}^0 \rho_f(-2)|_{G_K})$. Splitting a and c and rearranging the diagonal elements in ρ, we see that $\begin{bmatrix} \rho_f|_{G_K} & b \\ & \rho_f(2)|_{G_K} \end{bmatrix}$ is a subrepresentation of ρ and hence also crystalline. Thus, b in fact gives rise to a non-zero element in the Selmer group of $\operatorname{ad}^0 \rho_f(-2)|_{G_K}$. This provides evidence for the Bloch-Kato conjecture for V_2. In fact this argument can be applied with V_1 and V_2 switched. Furthermore, with some more work it could be extended to arbitrary values of v_1 and v_2 to obtain the full equivalence of the validity of the Bloch-Kato conjectures for V_1 and V_2. However, we do not proceed with the general argument here, because we are unable to show that the classes a and c indeed lie in the minus part of the Selmer group, which at this stage reduces our argument to a mere speculation.

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