A *q*-Queens Problem

Christopher R. H. Hanusa

Queens College, CUNY

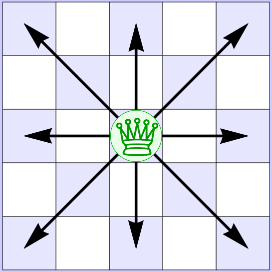
Joint work with

Thomas Zaslavsky, Binghamton University (SUNY)
and Seth Chaiken, University at Albany (SUNY)

`qc.edu/chanusa` > `Research` > `Talks`

When Queens Attack!

A **queen** is a chess piece that can move horizontally, vertically, and diagonally.



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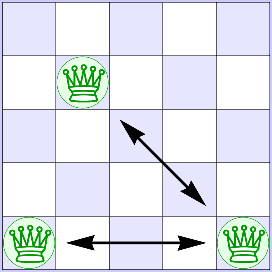

- Two pieces are **attacking** when one piece can move to the other's square.
- A **configuration** is a placement of chess pieces on a chessboard.
- A configuration is **nonattacking** if no two pieces are attacking.

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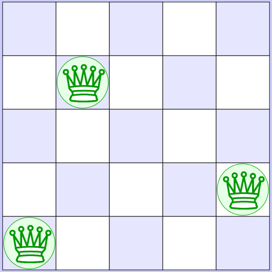

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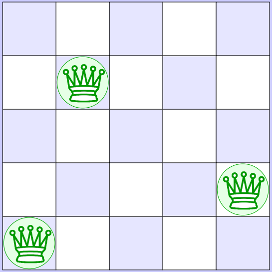

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Question: How many nonattack'g queens MIGHT fit on a chessboard?

The 8-Queens Problem

Q: **Can** you place 8 nonattacking queens on an 8×8 chessboard?

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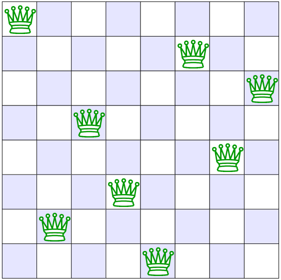
A: Yes!



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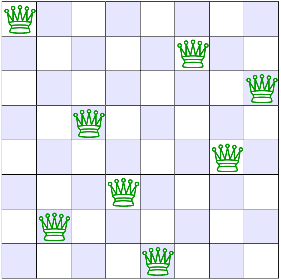

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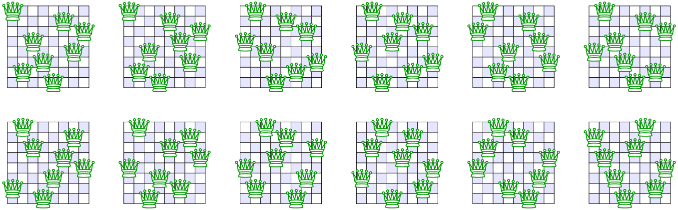
A: 92



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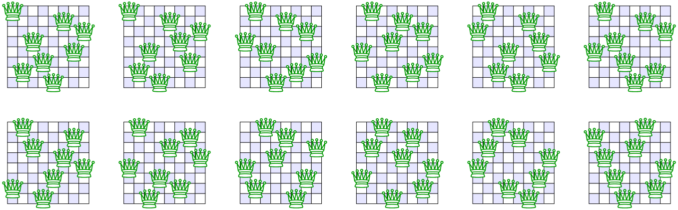

The *n*-Queens Problem: Find a formula for the number of nonattacking configurations of n queens on an $n \times n$ chessboard.

n	1	2	3	4	5	6	7	8	9	10
#	1	0	0	2	10	4	40	92	352	724

From n-Queens to q-Queens

The n-Queens Problem:

nonatt. configs of n queens
on a $n \times n$ square board

From n-Queens to q-Queens

The n-Queens Problem:	**A q-Queens Problem:**
# nonatt. configs of n queens on a $n \times n$ square board	# nonatt. configs of q pieces $\mathbb{P}$ on dilations of a polygonal board $\mathcal{B}$

- A number q.
  # of pieces in config.
- A piece $\mathbb{P}$.
  A set of basic moves.
- A board $\mathcal{B}$.
  A convex polygon
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A **piece** $\mathbb{P}$ is defined by its moves $(c, d) \in \mathbf{M}$.

$$(x, y) \longrightarrow (x, y) + \alpha(c, d) \text{ for } \alpha \in \mathbb{Z}$$

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$$\mathbf{M} = \{(1,0),(0,1),(1,1),(1,-1)\}$$

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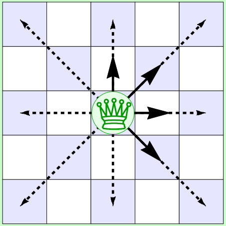

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♗ Bishop:

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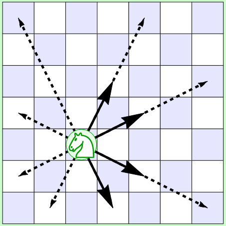
♘ Nightrider:

$$\mathbf{M} = \{(1,2),(1,-2),(2,1),(2,-1)\}$$



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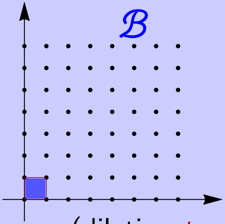

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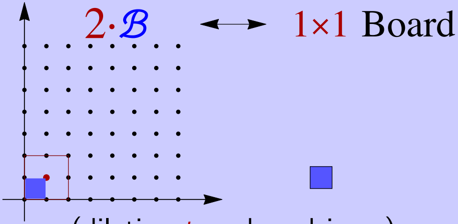

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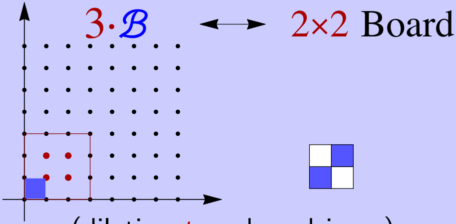

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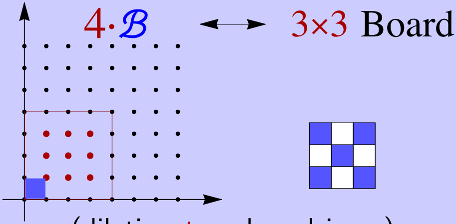

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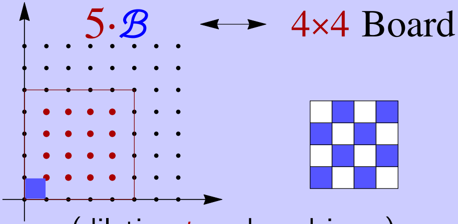

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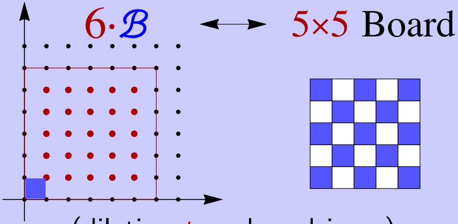

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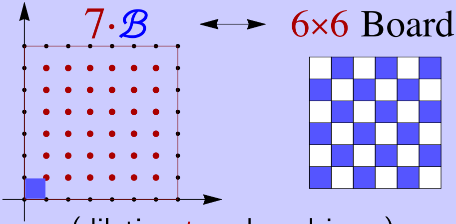

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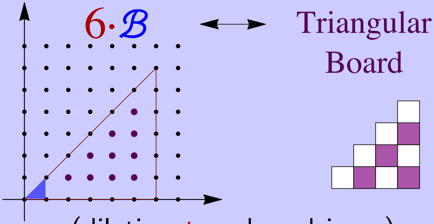

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Our Quest: Find a formula for the number of nonattacking configurations of q pieces $\mathbb{P}$ inside dilations of $\mathcal{B}$.

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Theorem: (CZ'05, CHZ'14)
Given q, $\mathbb{P}$, and $\mathcal{B}$, the number of nonattacking configurations of q pieces $\mathbb{P}$ inside $t\mathcal{B}$ is a quasipolynomial function of t.

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Definition: A **quasipolynomial** is a function $f(t)$ on $t \in \mathbb{Z}_+$ s.t. $f(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_0$, where each c_i is periodic in t.

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Example. The number of ways to place two nightriders on an $n \times n$ chessboard is:

$$u_{♘}(2; n) = \begin{cases} \frac{n^4}{2} - \frac{5n^3}{6} + \frac{3n^2}{2} - \frac{2n}{3} & \text{for even } n \\ \frac{n^4}{2} - \frac{5n^3}{6} + \frac{3n^2}{2} - \frac{7n}{6} & \text{for odd } n \end{cases}$$

Proof uses Inside-out polytopes

Two pieces $\mathbb{P}$ in positions (x_i, y_i) and (x_j, y_j) inside $t\mathcal{B}$ are attacking if:

$$(x_i, y_i) - (x_j, y_j) = \alpha(c, d) \quad \overset{\text{move eqn.}}{\longleftrightarrow} \quad d(x_i - x_j) = c(y_i - y_j)$$

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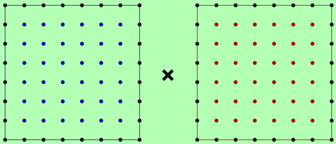
With two pieces, a move equation defines a *forbidden hyperplane* in $\mathcal{B}^2 \subset \mathbb{R}^4$.



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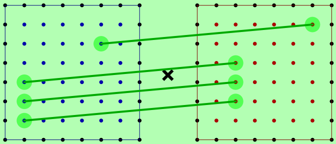

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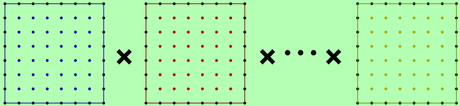
With q pieces, a move equation defines $\binom{q}{2}$ *forbidden hyperplanes* in $\mathcal{B}^q \subset \mathbb{R}^{2q}$.



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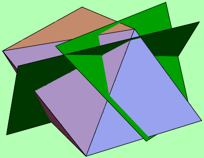

Our quest becomes: Count lattice points inside $\mathcal{B}^q$ that avoid forbidden hyperplanes.

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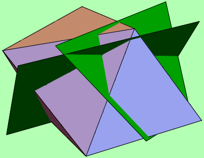

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Inside-out polytope!
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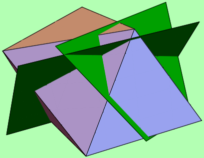

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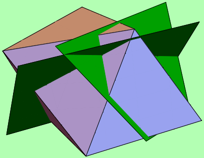

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- Inclusion-Exclusion for exact formula (later!)

Computing formulas experimentally

Restatement: The number of ways to place q $\mathbb{P}$-pieces inside a t dilation of $\mathcal{B}$ is a quasipolynomial:

$$u_{\mathbb{P}}(q;t) = \left\{ \begin{array}{lll} c_{2q,0}\, t^{2q} + \cdots + c_{1,0}\, t + c_{0,0} & t \equiv 0 & \bmod p \\ c_{2q,1}\, t^{2q} + \cdots + c_{1,1}\, t + c_{0,1} & t \equiv 1 & \bmod p \\ \vdots & & \\ c_{2q,p-1}\, t^{2q} + \cdots + c_{1,p-1}\, t + c_{0,p-1} & t \equiv p-1 & \bmod p \end{array} \right\}$$

Computing formulas experimentally

Restatement: The number of ways to place q $\mathbb{P}$-pieces inside a t dilation of $\mathcal{B}$ is a quasipolynomial:

$$u_{\mathbb{P}}(q;t) = \left\{ \begin{array}{ll} c_{2q,0}\, t^{2q} + \cdots + c_{1,0}\, t + c_{0,0} & t \equiv 0 \mod p \\ c_{2q,1}\, t^{2q} + \cdots + c_{1,1}\, t + c_{0,1} & t \equiv 1 \mod p \\ \vdots & \\ c_{2q,p-1}\, t^{2q} + \cdots + c_{1,p-1}\, t + c_{0,p-1} & t \equiv p-1 \mod p \end{array} \right\}$$

Consequence: If we can prove what the period is (or a bound), then with enough data we can solve for the coefficients!

Gives a proof of correctness for $u_{\mathbb{P}}(q;t)$!

Enough data?

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n	2 impalas	3 impalas	4 impalas	5 impalas
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4	96	276	412	340
5	244	1220	3472	5958
6	526	4128	19465	58740
7	1008	11596	83982	409740
8	1768	28136	290676	2084888
9	2896	61032	854496	8495772
10	4494	121180	2208797	29071456
11	6676	224172	5158998	86879914
12	9568	391416	11098648	232827336
13	13308	651396	22307354	570364522
14	18046	1041052	42347901	1296078688
15	23944	1607280	76581532	2763279296
16	31176	2408552	132822748	5577897568
17	39928	3516656	222156028	10738451380
18	50398	5018556	359938909	19834780832
19	62796	7018372	567017466	35323608130
20	77344	9639480	871181912	60902798384
21	94276	13026732	1308891718	102010521470
22	113838	17348796	1927301333	166480366144
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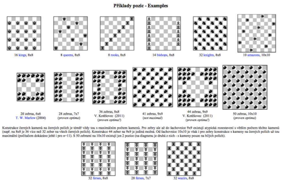

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A108792 - **5 Queens**, board *n* x *n* (*V.Kotěšovec, 4.4.2010*):

$$\frac{1}{120}n^{10}-\frac{5}{18}n^9+\frac{301}{72}n^8-\frac{1679}{45}n^7+\frac{78383}{360}n^6-\frac{77519}{90}n^5+\frac{1867681}{810}n^4-\frac{6499681}{1620}n^3+\frac{5324093}{1296}n^2-\frac{12758453}{6480}n+\frac{13038851}{64800}$$
$$+\left(\frac{1}{8}n^5-\frac{143}{48}n^4+\frac{82}{3}n^3-\frac{5647}{48}n^2+\frac{10475}{48}n-\frac{3547}{32}\right)*(-1)^n+\left(\frac{29}{2}n-\frac{35}{2}\right)*\cos\left(\frac{\pi n}{2}\right)+(2n+15)*\sin\left(\frac{\pi n}{2}\right)$$
$$+\left(\frac{32}{27}n^3-\frac{1328}{81}n^2+\frac{6328}{81}n-\frac{5488}{81}\right)*\cos\left(\frac{2\pi n}{3}\right)+\left(\frac{40}{81}n^2-\frac{1496}{243}n+\frac{7024}{243}\right)*\sqrt{3}*\sin\left(\frac{2\pi n}{3}\right)$$
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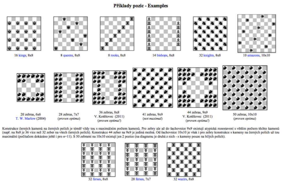
Dedicated to memory of Gauss, Lucas and Landau.



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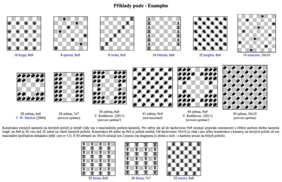

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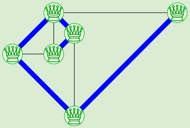
Discrete Fibonacci spiral!



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Deriving formulas theoretically

Our Quest: Count lattice points inside $\mathcal{P}$ avoiding hyperplanes.

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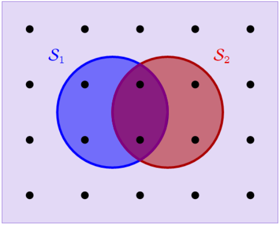

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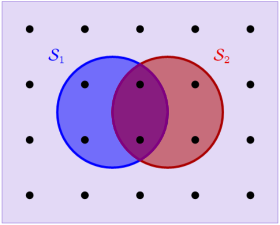

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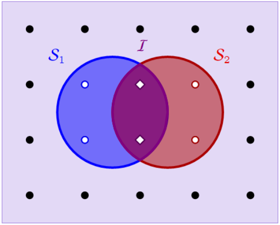

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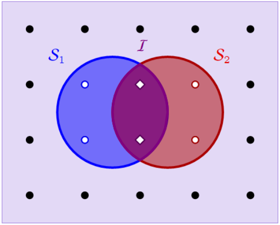

$20 - 4 - 4 + 2 = 14$

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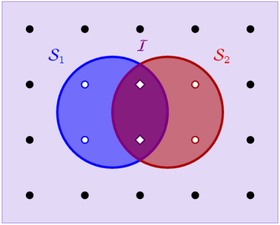

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- In general, alternate signs: $|P| - \sum_i |S_i| + \sum_{i,j} |S_i \cap S_j| - \sum_{ijk} |S_i \cap S_j \cap S_k| + \sum_{ijkl} \cdots$

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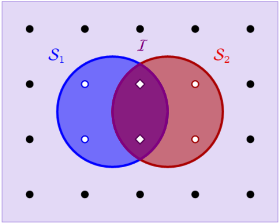

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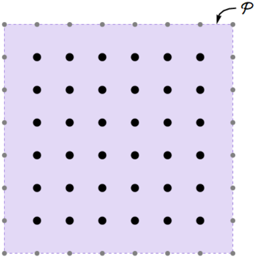

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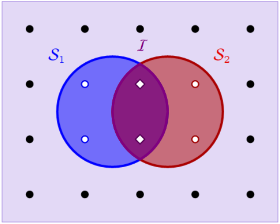

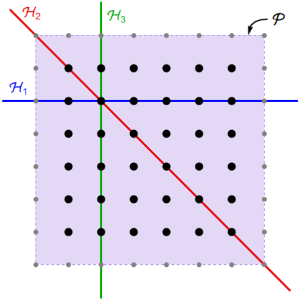

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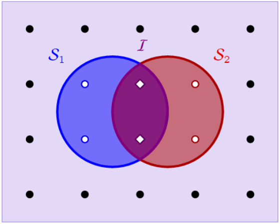

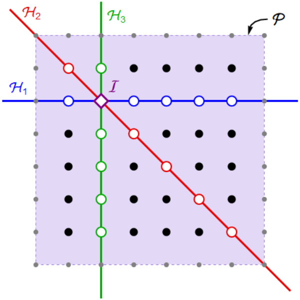

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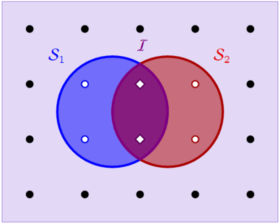

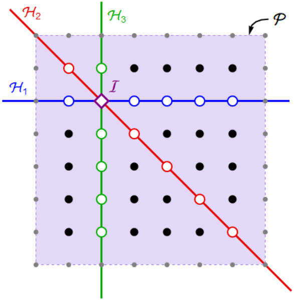

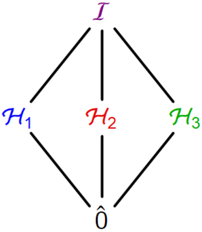

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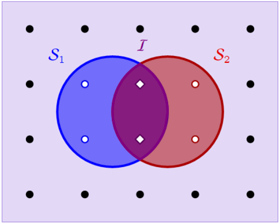

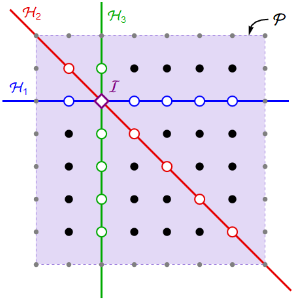

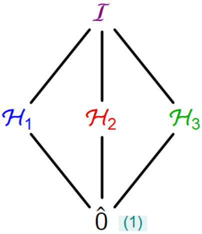

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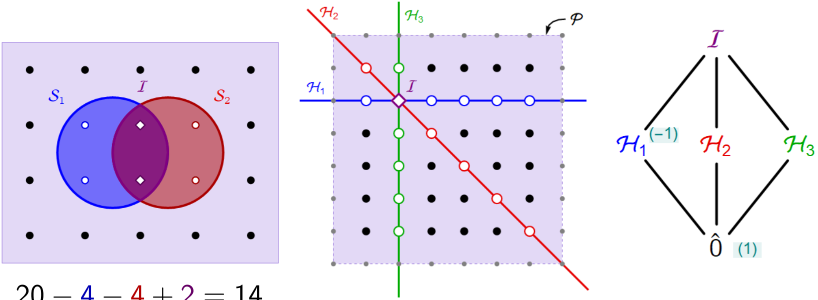

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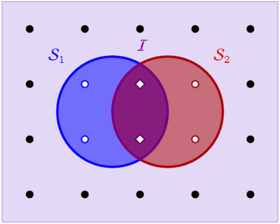

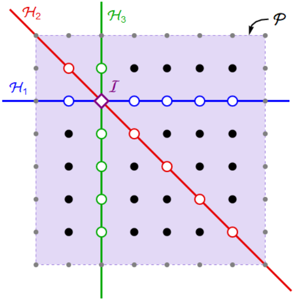

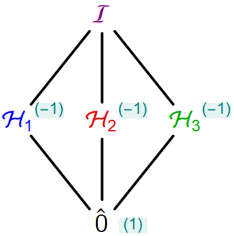

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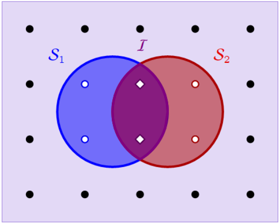

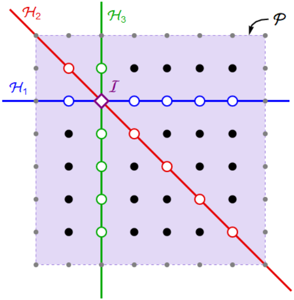

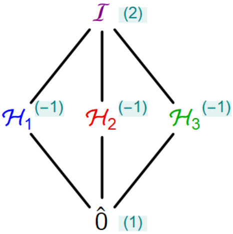

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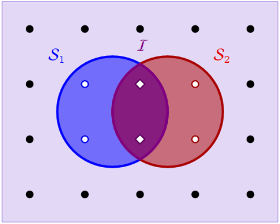
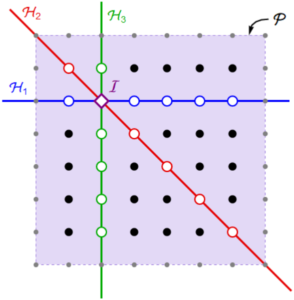
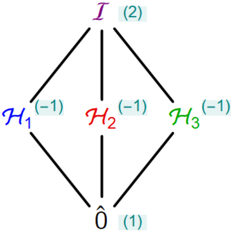






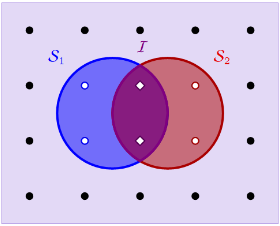
$20 - 4 - 4 + 2 = 14$

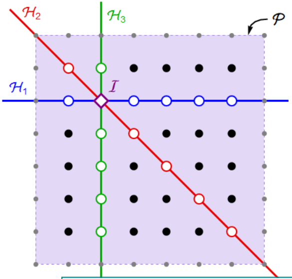
- Hyperplane intersections are subspaces w/complex interactions
- Form the poset of subspace inclusion. $\mu(\mathcal{U}) = -\sum_{\mathcal{T}<\mathcal{U}} \mu(\mathcal{T})$
- Find # lattice points in each subspace, calculate $\sum_{\mathcal{U}} \mu(\mathcal{U})|\mathcal{U}|$

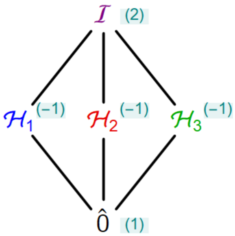
Deriving formulas theoretically

Our Quest: Count lattice points inside $\mathcal{P}$ avoiding hyperplanes.

Use **Möbius Inversion**, an extension of Inclusion/Exclusion:







$20 - 4 - 4 + 2 = 14$

$1 \cdot 36 - 1 \cdot 6 - 1 \cdot 6 - 1 \cdot 6 + 2 \cdot 2 = 20$

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(And place the other $q - k$ pieces!)

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How might two attack equations interact?
And how do we count them?

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✓ **Codim 3 for Partial Queens $\mathbb{P} = \mathbb{Q}^{hk}$:**

- explicit $u_{\mathbb{P}}(3; n)$
- leading 4 coeffs of $u_{\mathbb{P}}(q; n)$; period of 5–7.

A (not-very-useful) formula for *n*-Queens

Set $q = n$ to give the first closed-form formula for the *n*-Queens Problem:

Theorem

The number of ways to place n unlabelled copies of a rider piece $\mathbb{P}$ *on a square* $n \times n$ *board so that none attacks another is*

$$\frac{1}{n!}\sum_{i=1}^{2n} n^{2n-i}\sum_{\kappa=2}^{2i}(n)_\kappa \sum_{\nu=\lceil \kappa/2\rceil}^{\min(i,2\kappa-2)} \sum_{[\mathcal{U}_\kappa^\nu]:\mathcal{U}_\kappa^\nu\in\mathscr{L}(\mathscr{A}_{\mathbb{P}}^\infty)} \mu(\hat{0},\mathcal{U}_\kappa^\nu)\frac{\bar{\gamma}_{i-\nu}(\mathcal{U}_\kappa^\nu)}{|\operatorname{Aut}(\mathcal{U}_\kappa^\nu)|}.$$

This formula is very complicated but **it is** explicitly computable.

Brief Aside

I've never used so many variables!

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- upper case: $ABCDEFGHIJKLMNOPQRSTUVWXYZ$
- lower case: $abcdefghijklmnopqrstuvwxyz$

(That's 102 variables!!! Plus the reuse of indices!)

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- Determining all subspaces $\mathcal{U}$; What is structure of posets?
- Discrete Geometry: Fibonacci spiral.

Thank you!

Chaiken, Hanusa, Zaslavsky:

Our "A *q*-Queens Problem" Series:

I. General theory. Electronic J Comb 2014
II. The square board. J Alg Comb 2015
III. Partial queens. Australasian J Comb 2019
IV. Attacking config's and their denom's. Discrete Math 2020
V. A few of our favorite pieces. J Korean Math Soc 202?
VI. The bishops' period. Ars Math Contemp 2019
VII. Combinatorial types of riders. Australasian J Comb. 2020

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