A q-Queens Problem

Christopher R. H. Hanusa

Queens College, CUNY

Joint work with

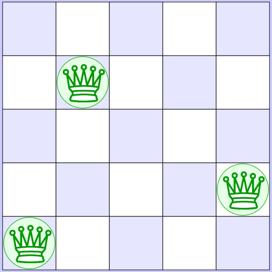
Thomas Zaslavsky, Binghamton University (SUNY)

and Seth Chaiken, University at Albany (SUNY)

qc.edu/chanusa > Research > Talks

When Queens Attack!

A **queen** is a chess piece that can move horizontally, vertically, and diagonally.



- Two pieces are **attacking** when one piece can move to the other's square.
- A **configuration** is a placement of chess pieces on a chessboard.
- A configuration is **nonattacking** if no two pieces are attacking.

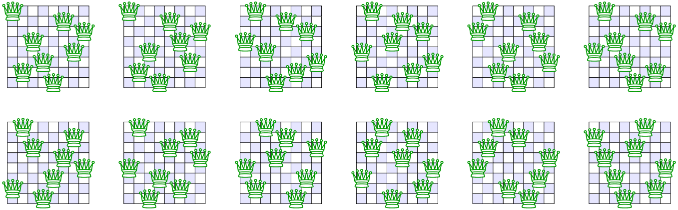
Question: How many nonattack'g queens MIGHT fit on a chessboard?

The 8-Queens Problem

Q: In how many ways

Q: Can you place 8 nonattacking queens on an 8×8 chessboard?

A: 92



The *n*-Queens Problem: Find a formula for the number of nonattacking configurations of n queens on an $n \times n$ chessboard.

n	1	2	3	4	5	6	7	8	9	10
#	1	0	0	2	10	4	40	92	352	724

From *n*-Queens to *q*-Queens

The *n*-Queens Problem:

nonatt. configs of n queens
on a $n \times n$ square board

A *q*-Queens Problem:

nonatt. configs of q pieces $\mathbb{P}$
on dilations of a polygonal board $\mathcal{B}$

- A number q.
  # of pieces in config.
- A piece $\mathbb{P}$.
  A set of basic moves.
- A board $\mathcal{B}$.
  A convex polygon and its dilations.

A **piece** $\mathbb{P}$ is defined by its moves $(c, d) \in \mathbf{M}$.

$$(x, y) \longrightarrow (x, y) + \alpha(c, d) \text{ for } \alpha \in \mathbb{Z}$$

Queen:

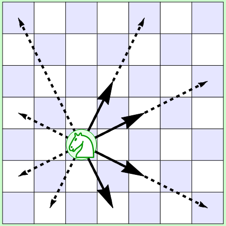
$$\mathbf{M} = \{(1,0),(0,1),(1,1),(1,-1)\}$$

Bishop:

$$\mathbf{M} = \{(1,1),(1,-1)\}$$

Nightrider:

$$\mathbf{M} = \{(1,2),(1,-2),(2,1),(2,-1)\}$$



From *n*-Queens to *q*-Queens

The *n*-Queens Problem:

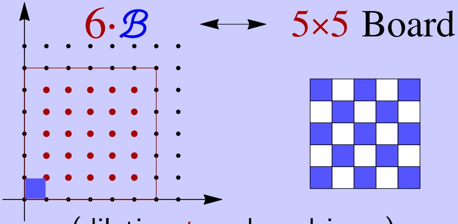
nonatt. configs of n queens
on a $n \times n$ square board

A *q*-Queens Problem:

nonatt. configs of q pieces $\mathbb{P}$
on dilations of a polygonal board $\mathcal{B}$

- A number q.
  # of pieces in config.
- A piece $\mathbb{P}$.
  A set of basic moves.
- A board $\mathcal{B}$.
  A convex polygon
  and its dilations.

A **board** is the set of integral points
on the *interior* of a dilation
of a rational convex polygon $\mathcal{B} \subset \mathbb{R}^2$



(dilation t vs. boardsize n)

A q-Queens Problem

Our Quest: Find a formula for the number of nonattacking configurations of q pieces $\mathbb{P}$ inside dilations of $\mathcal{B}$.

Theorem: (CZ'05, CHZ'14)
Given q, $\mathbb{P}$, and $\mathcal{B}$, the number of nonattacking configurations of q pieces $\mathbb{P}$ inside $t\mathcal{B}$ is a quasipolynomial function of t.

Definition: A **quasipolynomial** is a function $f(t)$ on $t \in \mathbb{Z}_+$ s.t. $f(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_0$, where each c_i is periodic in t.

Example. The number of ways to place two nightriders on an $n \times n$ chessboard is:

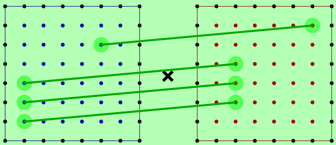
$$u_{♘}(2; n) = \begin{cases} \frac{n^4}{2} - \frac{5n^3}{6} + \frac{3n^2}{2} - \frac{2n}{3} & \text{for even } n \\ \frac{n^4}{2} - \frac{5n^3}{6} + \frac{3n^2}{2} - \frac{7n}{6} & \text{for odd } n \end{cases}$$

Proof uses Inside-out polytopes

Two pieces $\mathbb{P}$ in positions (x_i, y_i) and (x_j, y_j) inside $t\mathcal{B}$ are attacking if:

$$(x_i, y_i) - (x_j, y_j) = \alpha(c, d) \quad \overset{\text{move eqn.}}{\longleftrightarrow} \quad d(x_i - x_j) = c(y_i - y_j)$$

With two pieces, a move equation defines a *forbidden hyperplane* in $\mathcal{B}^2 \subset \mathbb{R}^4$.



Our quest becomes: Count lattice points inside $\mathcal{B}^q$ that avoid forbidden hyperplanes.

Inside-out polytope! Apply theory of Beck and Zaslavsky.

- Answer is a quasipolynomial • degree $2q$ • $\text{vol}(\mathcal{B}^q) \rightsquigarrow$ initial term
- Inclusion-Exclusion for exact formula (later!)

Computing formulas experimentally

Restatement: The number of ways to place q $\mathbb{P}$-pieces inside a t dilation of $\mathcal{B}$ is a quasipolynomial:

$$u_{\mathbb{P}}(q;t) = \begin{cases} c_{2q,0}\, t^{2q} + \cdots + c_{1,0}\, t + c_{0,0} & t \equiv 0 \mod p \\ c_{2q,1}\, t^{2q} + \cdots + c_{1,1}\, t + c_{0,1} & t \equiv 1 \mod p \\ \vdots & \\ c_{2q,p-1} t^{2q} + \cdots + c_{1,p-1} t + c_{0,p-1} & t \equiv p-1 \mod p \end{cases}$$

Consequence: If we can prove what the period is (or a bound), then with enough data we can solve for the coefficients!

Gives a proof of correctness for $u_{\mathbb{P}}(q;t)$!

Enough data?

Let me introduce *Václav Kotěšovec*:

- Comprensive Book
- Tables of Data
- Conjectured Formulas
- Essential check to our theory

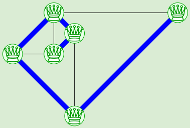
⚠ Collecting enough data is HARD for a large period. ⚠

Imp. Q. What is the period?

Thm. (qq.VI) Bishops' period is 2.

Conj. (qq.IV, K.) Queens' period is $\mathrm{lcm}(\{1, \ldots, \mathrm{fibonacci}_q\})$**!?!** 5:60

Discrete Fibonacci spiral!

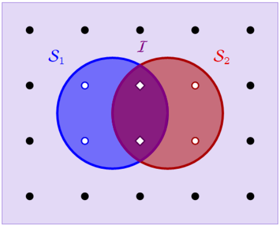


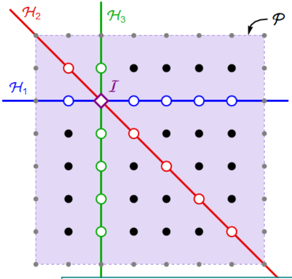
Upper Bound: LCM of denoms of facet/hyperplane intersection pts.

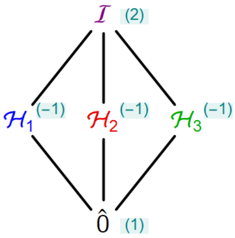
Deriving formulas theoretically

Our Quest: Count lattice points inside $\mathcal{P}$ avoiding hyperplanes.

Use **Möbius Inversion**, an extension of Inclusion/Exclusion:







$20 - 4 - 4 + 2 = 14$

$1\cdot 36 - 1\cdot 6 - 1\cdot 6 - 1\cdot 6 + 2\cdot 2 = 20$

- Hyperplane intersections are subspaces w/complex interactions
- Form the poset of subspace inclusion. $\mu(\mathcal{U}) = -\sum_{\mathcal{T}<\mathcal{U}} \mu(\mathcal{T})$
- Find # lattice points in each subspace, calculate $\sum_{\mathcal{U}} \mu(\mathcal{U})|\mathcal{U}|$

Deriving formulas theoretically

Derive exact formulas for *leading coeffs* of quasipolynomial:

Interior integer points **NOT** in the hyperplane arrangement is given by Möbius inversion on points **IN** the arrangement.

Calculate poset of multiway intersections of hyperplanes

Each corresponds to placements of k attacking pieces

For each $\mathcal{U} \cap \mathcal{B}^q$, count number of lattice points

We end up counting number of ways k pieces attack

Apply Möbius Inversion!

(And place the other $q-k$ pieces!)

On a square board, $u_{\mathbb{P}}(q;n) = \frac{1}{q!} \sum_{\mathcal{U} \in \mathscr{L}(\mathscr{A}_{\mathbb{P}})} \mu(\mathcal{U})\ \alpha(\mathcal{U};n)\ n^{2q-2k}$.

Subspaces from two hyperplanes (Codimension 2)

How might two attack equations interact?
And how do we count them?

Four pieces
$\mathbb{P}_1$ attacks $\mathbb{P}_2$ on any slope.
$\mathbb{P}_3$ attacks $\mathbb{P}_4$ on any slope.

[No interaction.]
(Count # ways two in a row)^2.

Three pieces
$\mathbb{P}_1$ attacks $\mathbb{P}_2$ on any slope.
$\mathbb{P}_2$ attacks $\mathbb{P}_3$ on **another** slope.

[No restriction on $\mathbb{P}_1$ vs. $\mathbb{P}_3$.]
Cases based on actual slopes.

Two pieces.
$\mathbb{P}_1$ attacks $\mathbb{P}_2$ on any slope.
$\mathbb{P}_1$ attacks $\mathbb{P}_2$ on **another** slope.

[$\Rightarrow$ $\mathbb{P}_1$ and $\mathbb{P}_2$ share a point.]
Count # of points on board.

Three pieces
$\mathbb{P}_1$ attacks $\mathbb{P}_2$ on any slope.
$\mathbb{P}_2$ attacks $\mathbb{P}_3$ on **same** slope.

[$\Rightarrow$ $\mathbb{P}_1$ and $\mathbb{P}_3$ also attack.]
Count # of ways three in a row.

✓ **Codim 3 for Partial Queens $\mathbb{P} = \mathbb{Q}^{hk}$:**

- explicit $u_{\mathbb{P}}(3; n)$
- leading 4 coeffs of $u_{\mathbb{P}}(q; n)$; period of 5–7.

A (not-very-useful) formula for n-Queens

Set $q = n$ to give the first closed-form formula for the n-Queens Problem:

Theorem

The number of ways to place n unlabelled copies of a rider piece $\mathbb{P}$ on a square $n \times n$ board so that none attacks another is

$$\frac{1}{n!}\sum_{i=1}^{2n} n^{2n-i}\sum_{\kappa=2}^{2i}(n)_\kappa \sum_{\nu=\lceil \kappa/2\rceil}^{\min(i,2\kappa-2)} \sum_{[\mathcal{U}_\kappa^\nu]:\mathcal{U}_\kappa^\nu \in \mathscr{L}(\mathscr{A}_{\mathbb{P}}^{\infty})} \mu(\hat{0},\mathcal{U}_\kappa^\nu)\frac{\bar{\gamma}_{i-\nu}(\mathcal{U}_\kappa^\nu)}{|\operatorname{Aut}(\mathcal{U}_\kappa^\nu)|}.$$

This formula is very complicated but **it is** explicitly computable.

Brief Aside

I've never used so many variables!

- Blackboard letters: $\mathbb{BNPQRZ}$
- Bold letters: $\mathbf{abcdxyzILM}\boldsymbol{\beta}$
- Callig. letters: $\mathscr{A}\mathcal{BCDEFGHIJK}\mathscr{L}\mathscr{M}\mathcal{NOPQ}\mathscr{R}\mathscr{S}\mathcal{TUWXYZ}$
- Greek letters: $\alpha\beta\gamma\delta\varepsilon\zeta\theta\kappa\lambda\mu\nu\xi\pi\varphi\omega$ ABΔΓΗΛΠΣΨ
- upper case: $ABCDEFGHIJKLMNOPQRSTUVWXYZ$
- lower case: $abcdefghijklmnopqrstuvwxyz$

(That's 102 variables!!! Plus the reuse of indices!)

What is next?

What Questions Are Interesting?

- **Fun** test case for Ehrhart Theory (lattice point) questions.
  - Period of quasipolynomial $\neq$ LCM of denominators
- Special pieces
  - One-move riders show that period of quasip. depends on move
  - Other fairy pieces (Progress made with Arvind Mahankali)
- Special boards
  - Rook placement theory on other boards
  - Nice pieces on nice boards (Angles of 45, 90, 135 degrees)
- Determining all subspaces $\mathcal{U}$; What is structure of posets?
- Discrete Geometry: Fibonacci spiral.

Thank you!

Chaiken, Hanusa, Zaslavsky:

Our "A *q*-Queens Problem" Series:

I. General theory. Electronic J Comb 2014
II. The square board. J Alg Comb 2015
III. Partial queens. Australasian J Comb 2019
IV. Attacking config's and their denom's. Discrete Math 2020
V. A few of our favorite pieces. J Korean Math Soc 202?
VI. The bishops' period. Ars Math Contemp 2019
VII. Combinatorial types of riders. Australasian J Comb. 2020

Slides available: `qc.edu/chanusa` > `Research` > `Talks`

3D Printed Mathematical Jewelry: `hanusadesign.com`