Let's count: Domino tilings

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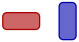
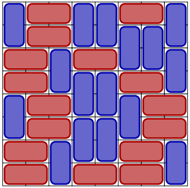
Domino Tilings

Today we'll discuss domino tilings, where:

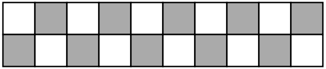
- Our **board** is made up of squares.
- Our **dominoes** have no spots and all look the same.
  - (Although, I will color the dominoes.) 
- One domino covers up two adjacent squares of the board.

A **tiling** is a placement of non-overlapping dominoes which completely covers the board.



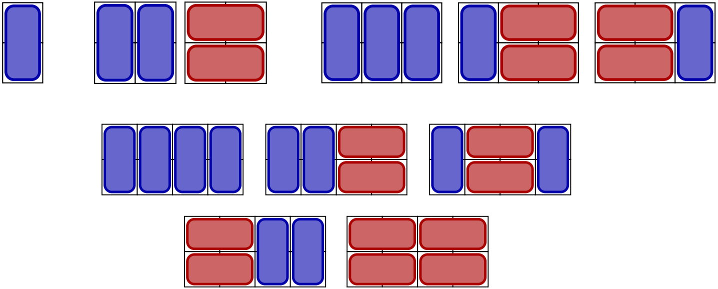
$2 \times n$ board

Question. How many tilings are there on a $2 \times n$ board?



Definition. Let $f_n = \#$ of ways to tile a $2 \times n$ board.

$f_0 = 1$
$f_1 = 1$
$f_2 = 2$
$f_3 = 3$
$f_4 = 5$



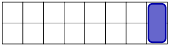
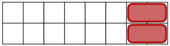
Why Fibonacci?

Fibonacci numbers f_n satisfy

- $f_0 = f_1 = 1$ ✓
- $f_n = f_{n-1} + f_{n-2}$ ✓

There are f_n tilings of a $2 \times n$ board

Every tiling ends in either:

- one vertical domino 
  - **How many?** Fill the initial $2 \times (n-1)$ board in f_{n-1} ways.
- two horizontal dominoes 
  - **How many?** Fill the initial $2 \times (n-2)$ board in f_{n-2} ways.

Total: $f_{n-1} + f_{n-2}$

Fibonacci identities

We have a new definition for Fibonacci:

$$f_n = \text{the number of tilings of a } 2 \times n \text{ board.}$$

This *combinatorial interpretation* of the Fibonacci numbers provides a framework to prove identities.

- Did you know that $f_{2n} = (f_n)^2 + (f_{n-1})^2$?

f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}
1	2	3	5	8	**13**	**21**	34	55	89	144	233	377	**610**

$$f_{14} = f_7^2 + f_6^2$$

$$610 = 441 + 169$$

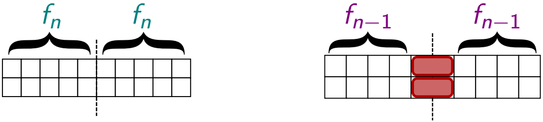
Proof that $f_{2n} = (f_n)^2 + (f_{n-1})^2$

Proof. How many ways are there to tile a $2 \times (2n)$ board?

Answer 1. Duh, f_{2n}.

Answer 2. Ask whether there is a break in the middle of the tiling:

Either there is... Or there isn't...



For a total of $(f_n)^2 + (f_{n-1})^2$ tilings.

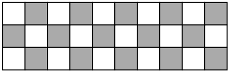
We counted f_{2n} in two different ways, so they must be equal. $\square$

Further reading:

Arthur T. Benjamin and Jennifer J. Quinn
Proofs that Really Count, MAA Press, 2003.

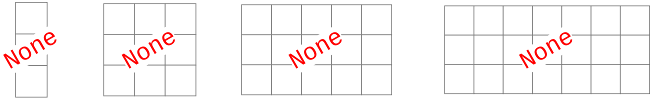
$3 \times n$ board

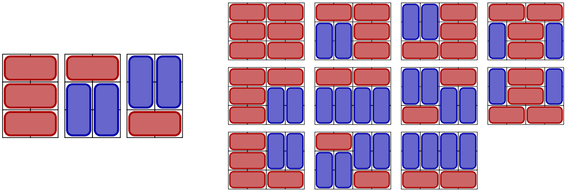
Question. How many tilings are there on a $3 \times n$ board?



Definition. Let $t_n = \#$ of ways to tile a $3 \times n$ board.

$t_0 = 1$
$t_1 = 0$
$t_2 = 3$
$t_3 = 0$
$t_4 = 11$
$t_5 = 0$
$t_6 = 41$
$t_7 = 0$





Hunting sequences

Question. How many tilings are there on a $3 \times n$ board?

- Our Sequence: 1, 3, 11, 41, ...

Go to the Online Encyclopedia of Integer Sequences (OEIS).

`http://oeis.org/`

- **(Search)** Information on a sequence
  - Formula
  - Other interpretations
  - References
- **(Browse)** Learn new math
- **(Contribute)** Submit your own!

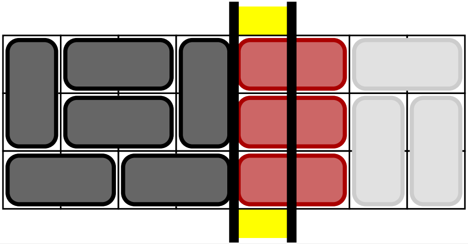
The transfer matrix method

Question. How many tilings are there on a $3 \times n$ board?

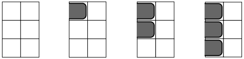
Question. How can we count these tilings intelligently?

Answer. Use the transfer matrix method.

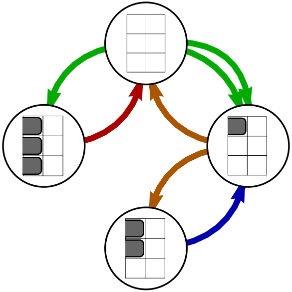
- Like a finite state machine.
- Build the tiling dynamically one column at a time.
- A "state" corresponds to which squares are free in a column.
- Filling the free squares "transitions" to the next state.



The transfer matrix for the $3 \times n$ board

For $3 \times n$ tilings, the possible states are: 

And the possible transitions are:



FROM:

TO:
$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \mathbf{A}$$

Use **a matrix** to keep track of *how many* transitions there are.

The **power** of the transfer matrix

Multiply by **A**. This shows that four steps after ⊞:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{\mathbf{4}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{11} \\ 0 \\ 8 \\ 0 \end{bmatrix}$$

- A complete tiling of $3 \times n$ ↔ ends in ⊞
- \# of tilings of $3 \times n$ ↔ first entry of $\mathbf{A}^n \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

We can calculate values. Is there a formula?

$t_1 = 0$
$t_2 = 3$
$t_3 = 0$
$t_4 = 11$
$t_5 = 0$
$t_6 = 153$
$t_7 = 0$
$t_8 = 571$
$t_9 = 0$
$t_{10} = 2131$

A formula for t_n

Solve by diagonalizing $\mathbf{A}$:

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{D}\mathbf{P}$$
$$\mathbf{A}^n = \mathbf{P}^{-1}\mathbf{D}^n\mathbf{P}$$

$$\mathbf{D} = \begin{bmatrix} -\sqrt{2+\sqrt{3}} & 0 & 0 & 0 \\ 0 & \sqrt{2+\sqrt{3}} & 0 & 0 \\ 0 & 0 & -\sqrt{2-\sqrt{3}} & 0 \\ 0 & 0 & 0 & \sqrt{2-\sqrt{3}} \end{bmatrix}$$

We conclude:

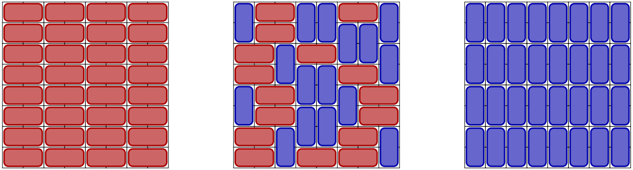
$$t_{2n} = \frac{1}{\sqrt{6}}\left(\sqrt{2-\sqrt{3}}\right)^{2n+1} + \frac{1}{\sqrt{6}}\left(\sqrt{2+\sqrt{3}}\right)^{2n+1}$$

- Method works for rectangular boards of fixed width

On a chessboard

Back to our original question:

How many domino tilings are there on an 8×8 board?



How many people think there are more than:

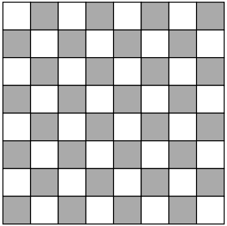
$$1{,}000$$

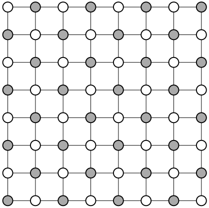
How to determine?

A Chessboard Graph

A *graph* is a collection of *vertices* and *edges*.
A *perfect matching* is a selection of edges which pairs all vertices.

Create a graph from the chessboard:





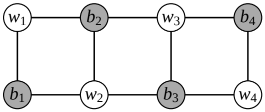
A tiling of the chessboard $\longleftrightarrow$ A perfect matching of the graph.

Chessboard Graph

Question. How many perfect matchings on the chessboard graph?

Create G's adjacency matrix. (Rows: white w_i, Columns: black b_j)

$$\text{Define } m_{i,j} = \begin{cases} 1 & \text{if } w_i b_j \text{ is an edge} \\ 0 & \text{if } w_i b_j \text{ is not an edge} \end{cases}$$



$$\begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

A perfect matching: Choose one in each row and one in each column.
Sound familiar?

Counting domino tilings

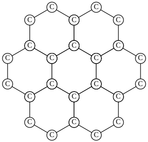
- To count domino tilings,
  - Take a determinant of a matrix
- To find a formula for the determinant,
  - Analyze the structure of the matrix.

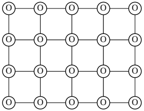
Answer: For a $2m \times 2n$ chessboard,

$$\#R_{2m\times 2n} = \prod_{j=1}^{n}\prod_{k=1}^{m}\left(4\cos^2\frac{\pi j}{2n+1} + 4\cos^2\frac{\pi k}{2m+1}\right)$$

History:

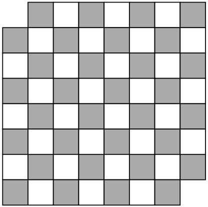
- 1930's: Chemistry and Physics
- 1960's: Determinant method of Kasteleyn and Percus





HOLeY Chessboard!

One last question: How many domino tilings on this board?

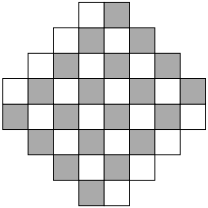


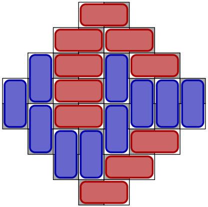
We've removed two squares — but there are now 0 tilings!

- Every domino covers two squares (1 black and 1 white)
- There are now **32** black squares and **30** white squares.

Aztec diamonds

This board is called an Aztec diamond (AZ_4)



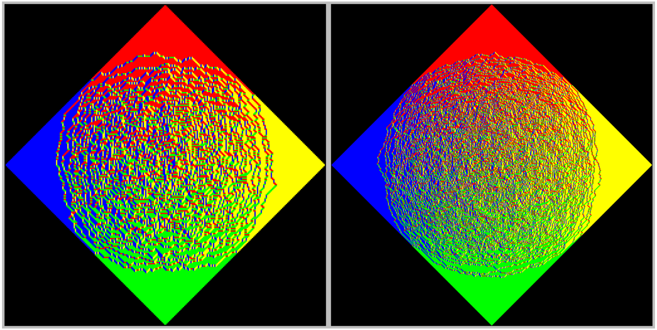


How many domino tilings are there on AZ_n?

$$2^{\binom{n+1}{2}} = 2^{\frac{n(n+1)}{2}}$$

Random tiling of an Aztec diamond

A random tiling has a surprising structure:



Pictures from: `http://tuvalu.santafe.edu/~moore/`

The *arctic circle phenomenon*.

Thank you!

Slides available: `people.qc.cuny.edu/chanusa` > `Talks`

Arthur T. Benjamin and Jennifer J. Quinn
Proofs that Really Count, MAA Press, 2003.

Online Encyclopedia of Integer Sequences
http://oeis.org

Random Tilings (James Propp)
http://faculty.uml.edu/jpropp/tiling/