Voting Methods and Colluding Voters

Christopher Hanusa

Outline

- Voting Methods
  - Plurality/Majority and refinements
  - Ranked Pairs
  - Borda Count
- Let's vote!
- Mathematics of the Borda Count
- Disorderings of Candidates
- Proofs involving Disorderings

Plurality/Majority

Goal: Ensure that the elected candidate has the support of a majority.

Method: Each person gets one vote. The candidate with the most votes wins.

- Two-candidate Runoff.
  - Keep the top two candidates
  - Hold a runoff election
- Instant Runoff Voting.
  - Rank as many candidates as desired.
  - Redistribute non-winning votes.

Ranked Pairs

Goal: Elect the candidate who would win each head-to-head election. (A **Condorcet winner**)

A	B	C
B	C	A
C	A	B

Careful!
A > B > C > A

Method: Each person ranks all the candidates.

- Determine who wins between c_i and c_j.
- Choose the strongest preference and lock it in.
- Ensure no ambiguity is created.
- Example:

A	A	C
B	C	A
C	B	B

Borda Count

Goal: Choose a consensus candidate.

Method: Each person ranks all n candidates.

Allot n points to the top-ranked candidate.

Allot $n-1$ points to the next-top-ranked candidate.

and so on . . .

The candidate with the most number of points wins.

Let's vote!

Plurality/Majority: Tally the first preferences.

Winner: ___________

Instant Runoff: When a candidate is eliminated, redistribute the votes to the next preferences.

Winner: ___________

Ranked Pairs: Determine and lock in strongest head-to-head preferences.

Winner: ___________

Borda Count: Allot $[n, n-1, n-2, \ldots, 1]$ points based on preferences; determine point winner.

Winner: ___________

Pros, Cons, and Facts

Plurality Refinements:

Pro: Candidate elected by a majority
Pro: Second preferences expressible

Con: Secondary support may be strong

Fact: Favors candidates with strong ideology

Ranked Pairs and Borda Count:

Pro: (RP) Condorcet winner always elected
Pro: (BC) Tries to maximize voter satisfaction
Pro: All preferences influence election

Con: Requires full ranking by voters
Con: Same weight given to each rank
Con: Subject to strategic voting

Fact: Favors consensus building candidates
Fact: Disincentive for candidates to share ideology
Fact: (BC) May not elect candidate favored by majority

Mathematics of the Borda Count

With three candidates, use the *scoring rule*:

[3,2,1]

	Voter 1	Voter 2	Voter 3		
1^{st}	A	A	B	$\rightarrow$	3
2^{nd}	B	C	C	$\rightarrow$	2
3^{rd}	C	B	A	$\rightarrow$	1

Candidate A: $3+3+1=7$ points

Candidate B: $2+1+3=6$ points

Candidate C: $1+2+2=5$ points

Generalization of the Borda Count

In the Borda Count, the scoring rule

$$[\, n \,, n{-}1, n{-}2, \ldots, \; 3 \;, \; 2 \;, \; 1]$$

becomes the *normalized scoring rule*

$$[1, \tfrac{n-2}{n-1}, \tfrac{n-3}{n-1}, \ldots, \tfrac{2}{n-1}, \tfrac{1}{n-1}, 0]$$

Modifying the scoring rule

1999 AL baseball MVP voting:

$$[14, 9, 8, 7, 6, 5, 4, 3, 2, 1]$$

which yields

$$[1, 0.62, 0.54, 0.46, 0.38, 0.31, 0.23, 0.15, 0.08, 0]$$

instead of

$$[1, 0.89, 0.78, 0.67, 0.56, 0.44, 0.33, 0.22, 0.11, 0]$$

$\rightarrow$ Called *positional voting*.

A *normalized scoring rule* is always of the form:

$$[1, x_{n-2}, x_{n-3}, \ldots, x_1, 0],$$

with $1 \geq x_{n-2} \geq \cdots \geq x_1 \geq 0$

Question: If we vary these x's, can different candidates win with the same votes?

YES!

Consider these candidate preferences of 9 voters:

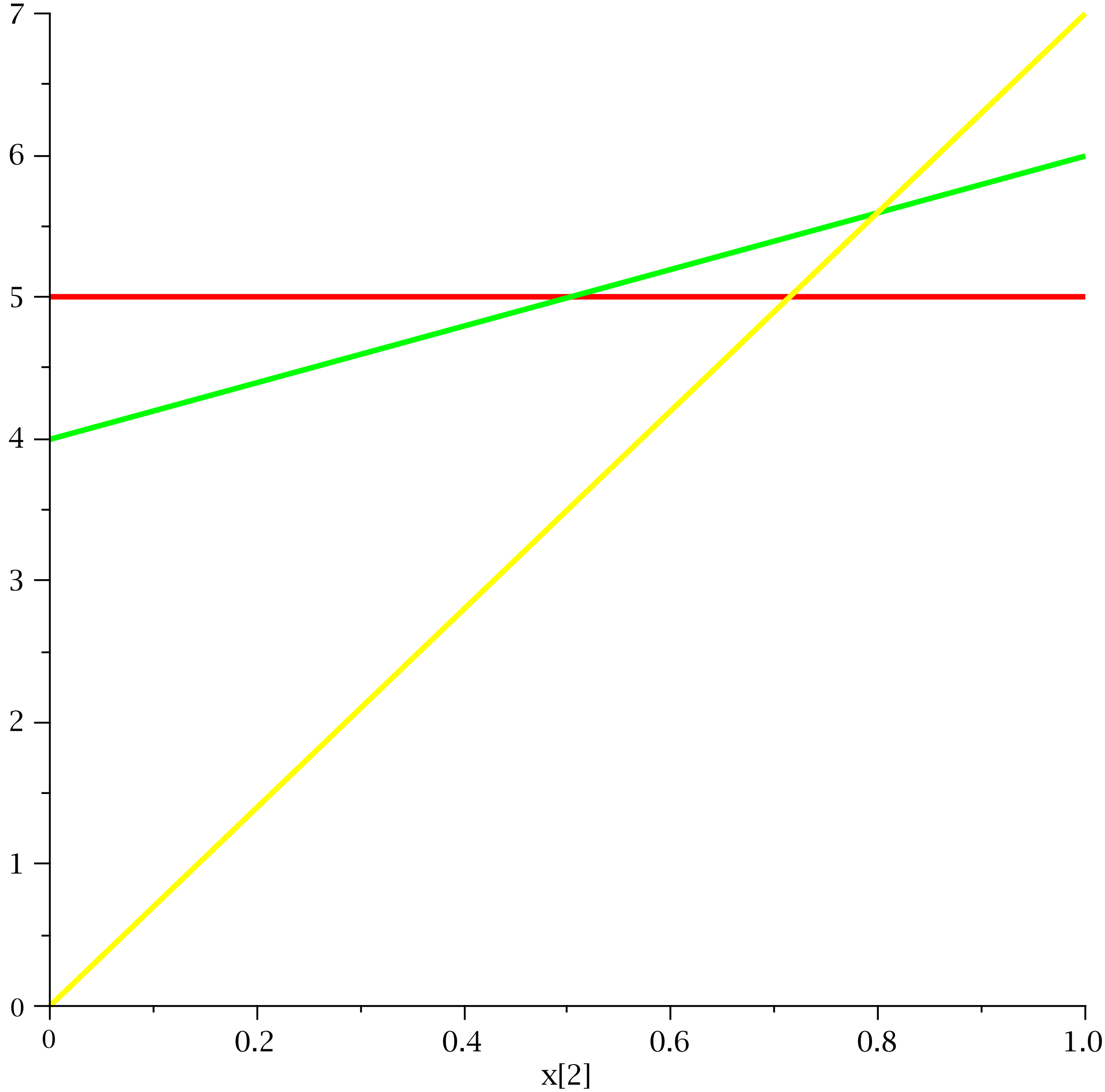
	4 voters	3 voters	2 voters	
1st	B	A	A	$\rightarrow 1$
2nd	C	C	B	$\rightarrow x$
3rd	A	B	C	$\rightarrow 0$

Under the scoring rule $[1, x, 0]$,

A receives 5 points.
B receives $4 + 2x$ points.
C receives $7x$ points.

As x varies, the candidate with the highest point total changes.

Everyone wins!



A set of voters' preferences generates a hyperplane arrangement.

Disordering Candidates

We say that m voters can **disorder** n candidates if there exists a set of preferences such that each of the n candidates can win under some scoring rule.

Such a set of preferences is called a **disordering**.

Disordering Candidates

We saw that 9 voters can disorder 3 candidates.

Question:

For which values of m and n can
m voters disorder n candidates?

Partial answer:

- the minimum m for 3 candidates is $m = 9$.

- Some number of voters can disorder 4 candidates.

Disordering Candidates

9 voters can disorder 3 candidates

6 voters can disorder 4 candidates

only 4 voters are necessary to disorder 5 candidates

and 9 candidates can be disordered by 3 voters!

$m \backslash^{n}$	3	4	5	6	7	8	9
3	×	×	×	×	×	×	·
4	×	×	×	·	·	·	·
5	×	·	·	·	·	·	·
6	×	·	·	·	·	·	·
7	×	·	·	·	·	·	·
8	×	·	·	·	·	·	·
9	·	·	·	·	·	·	·

for larger m and n,

m **voters can always disorder** n **candidates**

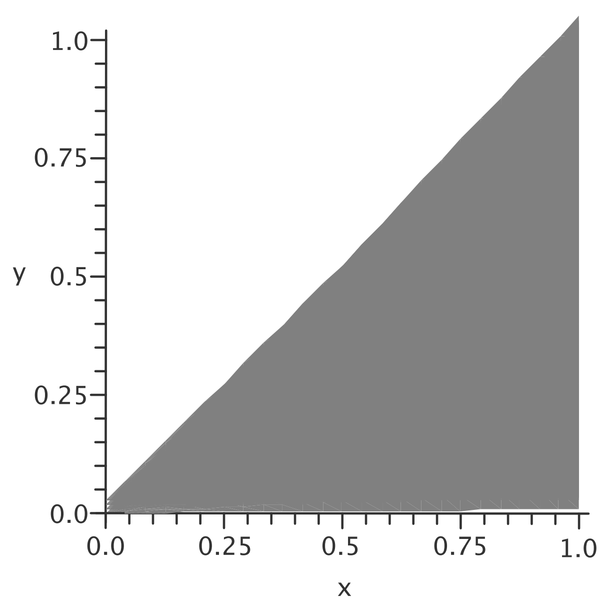
Why?

Analyze the 4-candidate situation:

A scoring rule is now of the form $[1, x, y, 0]$,
with $1 \geq x \geq y \geq 0$

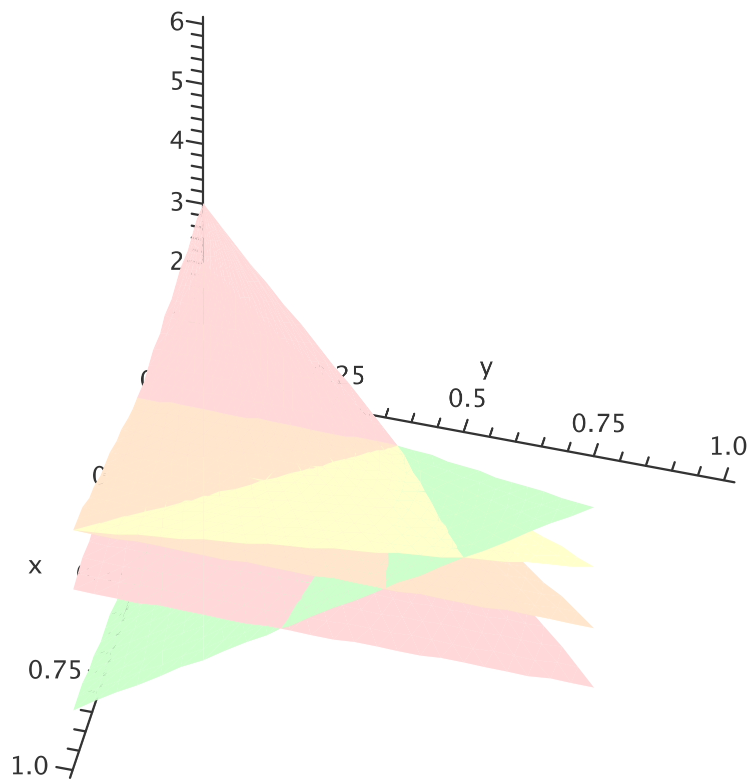
More degrees of freedom!

A set of voter preferences is now represented by a 3-D hyperplane arrangement over the triangular region



4-candidate example

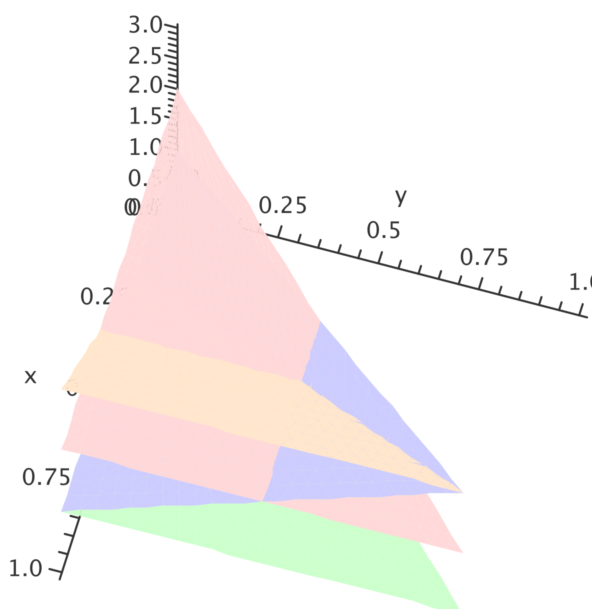
c_3	c_1	c_1	c_2	c_2	c_1
c_2	c_2	c_3	c_3	c_3	c_4
c_4	c_4	c_4	c_4	c_4	c_3
c_1	c_3	c_2	c_1	c_1	c_2



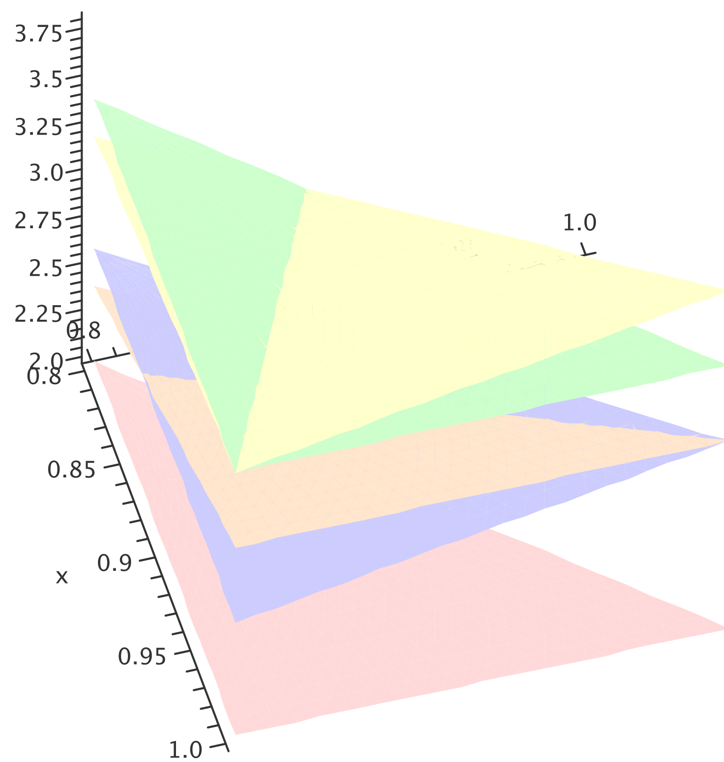
5-candidate example

$$
\begin{array}{cccc}
c_1 & c_1 & c_5 & c_4 \\
c_2 & c_3 & c_2 & c_2 \\
c_3 & c_5 & c_3 & c_5 \\
c_4 & c_4 & c_4 & c_3 \\
c_5 & c_2 & c_1 & c_1
\end{array}
$$

$[1, x, y, 0, 0]$



$[1, x, y, 0.8, 0]$



Theorem

Claim: A collection of m voters can disorder n candidates whenever $m \geq 3$ and $n \geq 3$, **except**

– when $m = 3$ and $n \leq 8$,

– when $n = 3$ and $m \leq 8$, and

– when $n = 4$ and $m = 4, 5$.

$m \backslash n$	3	4	5	6	7	8	9	10	11	12
3	×	×	×	×	×	×	·	·	·	·
4	×	×	×	·	·	·	·	·	·	·
5	×	·	·	·	·	·	·	·	·	·
6	×	·	·	·	·	·	·	·	·	·
7	×	·	·	·	·	·	·	·	·	·
8	×	·	·	·	·	·	·	·	·	·
9	·	·	·	·	·	·	·	·	·	·
10	·	·	·	·	·	·	·	·	·	·
11	·	·	·	·	·	·	·	·	·	·
12	·	·	·	·	·	·	·	·	·	·

Proof of Theorem

- $m \neq 2$

- $n \neq 2$

- Prove $\times$'s

- Create infinite families of disorderings.
  Lemma: From special (m, n): more voters
  Lemma: From special (m, n): more candidates

- Generate the special disorderings.

Simple Cases

Two voters can disorder no number of candidates

No number of voters can disorder two candidates

A Necessary Condition for Disorderings

What must be true in a disordering?

$$\begin{array}{cccccc} c_1 & c_1 & c_5 & c_4 & \rightarrow & 1 \\ c_2 & c_3 & c_2 & c_2 & \rightarrow & x_3 \\ c_3 & c_5 & c_3 & c_5 & \rightarrow & x_2 \\ c_4 & c_4 & c_4 & c_3 & \rightarrow & x_1 \\ c_5 & c_2 & c_1 & c_1 & \rightarrow & 0 \end{array}$$

For candidate c_1 to be able to win over c_2:

For candidate c_2 to be able to win over c_1:

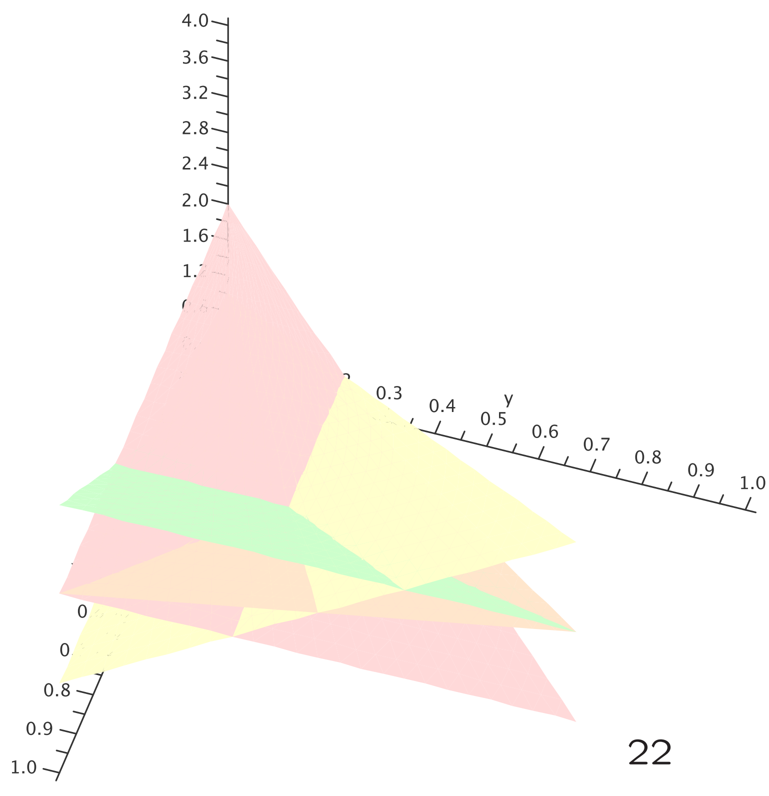
Necessary condition: If two candidates c_1 and c_2 are disordered, then there must exist integers j and k such that $R_j(c_1) > R_j(c_2)$ and $R_k(c_1) < R_k(c_2)$.

Computer Assistance

- Choose m and n
- Generate all sets of voter preferences.
- Check the necessary condition for each.
- If n.c. satisfied, verify whether disordering.

This condition is not sufficient!

c_1	c_1	c_2	c_3
c_2	c_4	c_4	c_4
c_3	c_3	c_3	c_2
c_4	c_2	c_1	c_1



A New Disordering from an Old

Whenever m voters disorder n candidates,
$m + n$ voters can disorder n candidates as well.

$$(m, n) \qquad \rightarrow \qquad (m+n, n)$$

Splittable Disorderings

Sometimes it is possible to add a candidate to an existing disordering in a simple fashion.

If so, we call the disordering *splittable*.

Not only can we add one candidate, we can add n' candidates.

$m, n \neq 2$ $\times$'s ∞-fam special

Generated Disorderings

$m \backslash n$	3	4	5	6	7	8	9	10	11	12
3	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\odot$	$\odot$	$\odot$	$\cdot$
4	$\times$	$\times$	$\times$	$\odot$	$\odot$	$\odot$	$\bullet$	$\bullet$	$\bullet$	$\cdot$
5	$\times$	$\odot$	$\odot$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\cdot$
6	$\times$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
7	$\times$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
8	$\times$	$\bullet$	$\bullet$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
9	$\odot$	$\bullet$	$\bullet$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
10	$\bullet$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
11	$\bullet$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
12	$\bullet$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
13	$\bullet$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
14	$\bullet$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
15	$\bullet$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
16	$\bullet$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
17	$\bullet$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
18	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Thanks!

I am: Christopher Hanusa

http://qc.edu/~chanusa/

Additional reading:

Electoral Process: ACE Encyclopaedia (UN)
`http://aceproject.org/ace-en`

Geometry of the Borda Count:
Millions of election outcomes from a single profile,
by Donald Saari

Preprint of this research:
Ensuring every candidate wins under
positional voting, available on the above website.