Self-conjugate core partitions: It's storytime!

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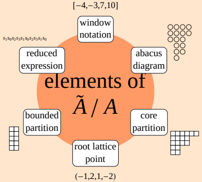
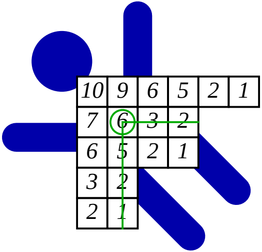
Meet Mr. Core Partition

Coxeter groups:

t-cores biject with min. wt. coset reps in $\widetilde{A}_t/A_t$. (action)





Let $c_t(n)$ be the number of t-core partitions of n.

Representation Theory:

t-cores label the t-blocks of irreducible characters of S_n.

Mock theta functions

The **Young diagram** of $\lambda = (\lambda_1, \ldots, \lambda_k)$ has λ_i boxes in row i.

The **hook length** of a box = # boxes below + # boxes to right + box

λ is a **t-core** if no boxes have hook length t.

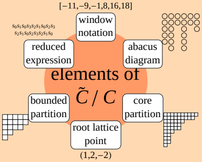
Example: **Mr. Core** is not 3-, 5-, 6-core; is a 4-, 8-, 11-core.

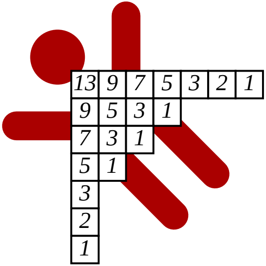
Meet Mrs. Core Partition

Coxeter groups:

s-c t-cores biject with min. wt. coset reps in $\widetilde{C}_t/C_t$.

(Hanusa, Jones '12)





Let $sc_t(n)$ be the number of self-conjugate t-core partitions of n.

Representation Theory:

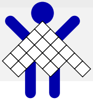
s-c t-cores label defect zero t-blocks of A_n that arise from splitting t-blocks of S_n.

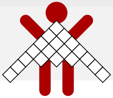
(Ask Rishi)

A partition is **self-conjugate** if it is symmetric about its main diagonal.

In this talk: Understanding self-conjugate core partitions.

Beauty contest





Core partitions

Generating function:
(Olsson, 1976)

$$\sum_{n\geq 0} c_t(n)q^n = \prod_{n\geq 1} \frac{(1-q^{nt})^t}{1-q^n}$$

Positivity. (Granville, Ono, '96)
$c_t(n) > 0$ when $t \geq 4$.

Monotonicity. (Stanton '99)
Conjecture: $c_{t+1}(n) \geq c_t(n)$
(Craven '06) (Anderson '08)

Self-conjugate core partitions

Generating function:
(Olsson, 1990) $\sum_{n\geq 0} sc_t(n)q^n =$

$$\begin{cases} \prod_{n\geq 1} \frac{(1+q^{2n-1})(1-q^{2tn})^{(t-1)/2}}{1+q^{t(2n-1)}} & t \text{ odd} \\ \prod_{n\geq 1} (1-q^{2tn})^{t/2}(1+q^{2n-1}) & t \text{ even} \end{cases}$$

Positivity? ✓ (Baldwin et al, '06)
$sc_t(n) > 0$ fot $t = 8, \geq 10,\ n > 2$.
Monotonicity?
What else can we say?

Understanding Monotonicity

Self-conjugate partitions of 22

									Total
6-core	×	×	✓	×	×	✓	×	×	2
7-core	×	×	×	×	✓	×	×	×	**1**
8-core	×	✓	×	✓	✓	×	✓	×	4
9-core	×	×	✓	✓	×	×	×	×	**2**
10-core	✓	✓	✓	✓	✓	✓	✓	✓	8
11-core	×	×	×	×	×	✓	×	✓	**2**
12-core	✓	✓	✓	✓	✓	✓	✓	✓	8
13-core	✓	✓	✓	✓	×	×	✓	✓	**6**
14-core	✓	✓	✓	✓	✓	✓	✓	✓	8
15-core	✓	✓	✓	×	✓	✓	✓	✓	**7**

- **Much variability!**
- Self-conjugate cores do not satisfy $sc_{t+1}(n) \geq sc_t(n)$.
- **Most** partitions are t-cores (t large)
- Self-conjugate cores might satisfy $sc_{t+2}(n) \geq sc_t(n)$.

Monotonicity Conjectures & Theorems

Monotonicity Conjecture. (Stanton '99)
$c_{t+1}(n) \geq c_t(n)$ when $4 \leq t \leq n-1$.

Even Monotonicity Conjecture. (Hanusa, Nath '12)
$sc_{2t+2}(n) > sc_{2t}(n)$ for all $n \geq 20$ and $6 \leq 2t \leq 2\lfloor n/4 \rfloor - 4$

Odd Monotonicity Conjecture.
$sc_{2t+3}(n) > sc_{2t+1}(n)$ for all $n \geq 56$ and $9 \leq 2t+1 \leq n-17$

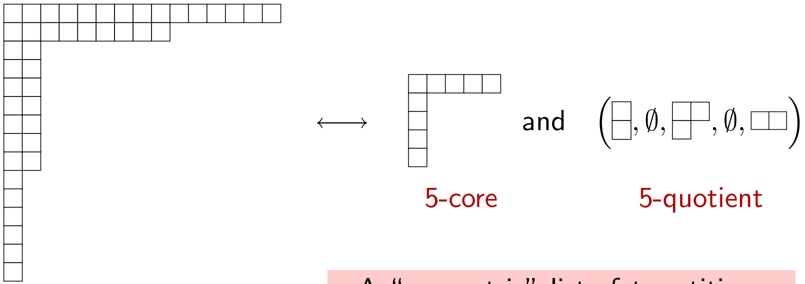
Some progress:

Theorem. $sc_{2t+2}(n) > sc_{2t}(n)$ when $n/4 < 2t \leq 2\lfloor n/4 \rfloor - 4$.

And: $sc_{2t+3}(n) > sc_{2t+1}(n)$ for all $n \geq 48$ and $n/3 \leq 2t+1 \leq n-17$.

Key idea: The t-quotient of λ

!!! We can define the ***t*-core** λ^0 of any partition λ. Successively remove hooks of hooklength t and keep track in λ's ***t*-quotient**.



A self-conjugate **non-t-core** partition of n $\overset{\textit{unique}}{\longleftrightarrow}$ A "symmetric" list of t partitions (i boxes total) $+$ a self-conj t-core partition of $n - i \cdot t$

Key idea: The t-quotient of λ

Since $sc_t(n) = sc(n) - nsc_t(n)$, we can prove results like:

Proposition. For $n/3 < 2t + 1 \leq n/2$,
$$sc_{2t+1}(n) = sc(n) - sc(n - 2t - 1) - (t - 1)\, sc(n - 4t - 2).$$

Proposition. For $n/4 < 2t \leq n/2$,
$$sc_{2t}(n) = sc(n) - t\, sc(n - 4t).$$

Consequence: For $n/4 < 2t \leq n/2$,

$$sc_{2t+2}(n) > sc_{2t}(n) \quad \longleftrightarrow \quad t\, sc(n - 4t - 4) > (t + 1) sc(n - 4t).$$

or instead

$$\frac{sc(n-4t-4)}{sc(n-4)} \leq \frac{t}{t+1}.$$

Look Ma, No cores!

Positivity for small t

We found some holes in the literature:

$sc_2(n) = 0$ except when n triangular.

$sc_4(n) = 0$ when $\left\{ \right.$ factorization of $\mathbf{8n+5}$ contains a $(4k+3)$-prime to an odd power. (Ono, Sze, '97)

$sc_6(n) = 0$ when $n \in \{2, 12, 13, 73\}$.

$sc_3(n) = 0$ except when $n = 3d^2 \pm 2d$

$sc_5(n) = 0$ when $\left\{ \right.$ factorization of n contains a $(4k+3)$-prime to an odd power. (Garvan, Kim, Stanton '90)

$sc_7(n) = 0$ when $n = (8m+1)4^k - 2$

$sc_9(n) = 0$ when $n = (4^k - 10)/3$ (Baldwin et al + Montgomery '06)

Sums of squares

Theorem. If $n = (8m+1)4^k - 2$ for $m,k > 0$, then $sc_7(n) = 0$.

Proof. (Garvan, Kim, Stanton '90) shows that

$$sc_7(n) = \begin{array}{c}\#\text{ triples } (x_1, x_2, x_3) \text{ satisfying} \\ n = 7{x_1}^2 + 2x_1 + 7{x_2}^2 + 4x_2 + 7{x_3}^2 + 6x_3\end{array}$$

Consider a minimal n of the above type. After substituting, rewriting:

$$7(8m+1)4^k = (7x_1+1)^2 + (7x_2+2)^2 + (7x_3+3)^2$$

$\equiv 0$ or $4 \mod 8$ ↑ So these are all even.↑

Choosing $(\frac{x_2}{2}, -\frac{x_3+1}{2}, -\frac{x_1+1}{2})$ gives a smaller n. □

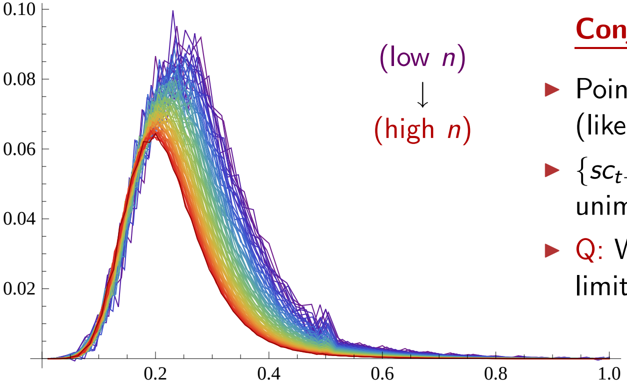
Legendre: The only integers NOT sum of 3 squares: $n = (8m+7)4^k$.

Here: The only integers NOT sum of 3 squares of diff. residues mod 7: $n = (56m+7)4^k$.

Unimodality and Asymptotics

We conjecture $sc_{t+2}(n) > sc_t(n)$; structure of increase?

Plot Normalized increase for different n: $\big(sc_{t+2}(n) - sc_t(n)\big)/sc(n)$



Conjectures:

- Pointwise limit → 0. (like Craven '06)
- $\{sc_{t+2}(n) - sc_t(n)\}$ unimodal for n large
- Q: What is the limiting distribution?

Other peculiarities

Conjecture: There are infinitely many n such that $sc_9(n) < sc_7(n)$.

Includes many (but not all) values of $n \equiv 82 \mod 128$:

$\{9, 18, 21, 82, 114, 146, 178, 210, 338, 402, 466, 594, 658, 722, 786, 850, 978, 1106, 1362, 1426, 1618, 1746, 1874, 2130, 2386, 2514, 2642, 2770, 2898, 3154, 3282, 3410, 3666, 3922, 4050, 4178, 4306, 4434, 4690, 4818, 4946, 5202, 5458, 5586, 5970, 6226, 6482, 6738, 6994, 7250, 7506, 8018, 8274, 8530, 8786, 9042, 9298, 9554, 9810\}$.

Conjecture: For $n \geq 0$, $sc_7(4n+6) = sc_7(n)$.

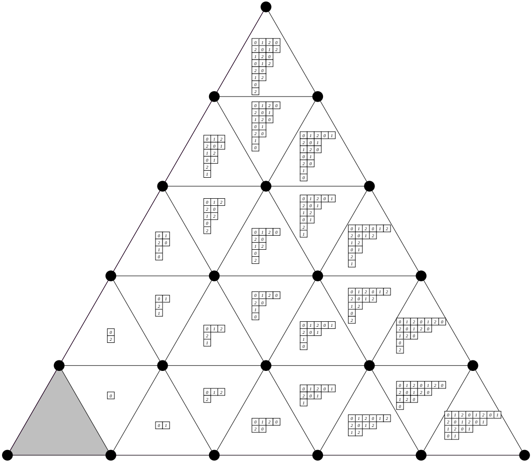
Conjecture: Let n be a non-negative integer.

1. Suppose $n \geq 49$. Then $sc_9(4n) > 3\, sc_9(n)$.
2. Suppose $n \geq 1$. Then $sc_9(4n+1) > 1.9\, sc_9(n)$.
3. Suppose $n \geq 17$. Then $sc_9(4n+3) > 1.9\, sc_9(n)$.
4. Suppose $n \geq 1$. Then $sc_9(4n+4) > 2.6\, sc_9(n)$.

What's next?

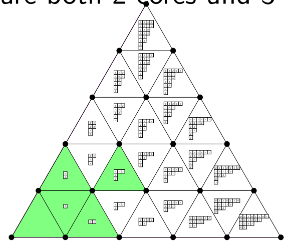
- Core survey
  - Coxeter Gp. POV: Fix t, let n vary.
    Rep. Theory POV: Fix n, let t vary.
  - Can they be unified? Can we help each other?
  - Gathering sources stage — What do you know?
- Simultaneous core partitions (λ is both an s-core and a t-core)
  - Geometrical interpretation of cores:

The bijection between 3-cores and alcoves



Simultaneous core partitions

How many partitions are both 2-cores and 3-cores? **2**.



How many partitions are both 3-cores and 4-cores? **5**.

How many simultaneous 4/5-cores? **14**.

How many simultaneous 5/6-cores? **42**.

How many simultaneous $n/(n+1)$-cores? C_n!

Jaclyn Anderson proved that the number of s/t-cores is $\frac{1}{s+t}\binom{s+t}{s}$.

The number of 3/7-cores is $\frac{1}{10}\binom{10}{3} = \frac{1}{10}\frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 12$.

Fishel–Vazirani proved an alcove interpretation of $n/(mn+1)$-cores.

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  - Can they be unified? Can we help each other?
  - Gathering sources — What do you know?
- Simultaneous core partitions (λ is an s-core and a t-core)
  - Geometrical interpretation of cores.
- **Question:** What is the average size of an s/t-core partition?
  - In progress (on pause).
    We "know" the answer, but we have to prove it!
  - Working with Drew Armstrong, University of Miami.

Thank you!

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Interact: `people.qc.cuny.edu/chanusa` > `Animations`

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