# A combinatorial introduction to reflection groups

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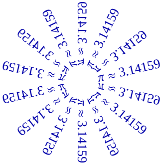


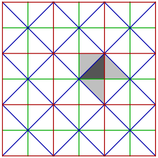

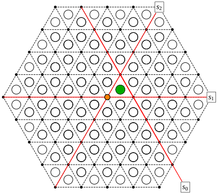


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# Groups

Today, we will discuss the combinatorics of **groups**.

- Made up of a set of elements $W = \{w_1, w_2, \ldots\}$.
- Multiplication of two elements $w_1 w_2$ stays in the group.
  - ALTHOUGH, it might **not** be the case that $w_1 w_2 = w_2 w_1$.
- There is an identity element (id) & Every element has an inverse.
- *Group elements take on the role of both objects and functions.*

**(Non-zero real numbers)**

- We can multiply $a$ and $b$
- It is the case that $ab = ba$
- $1$ is the identity: $a \cdot 1 = a$
- The inverse of $a$ is $1/a$.

**(Invertible $n \times n$ matrices.)**

- We can multiply $A$ and $B$
- *Rarely* is $AB = BA$
- $I_n$ is the identity: $A \cdot I_n = A$
- The inverse of $A$ exists: $A^{-1}$.

# Reflection Groups

More specifically, we will discuss **reflection groups** $W$.

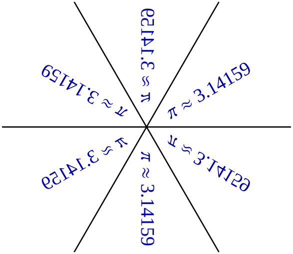
- $W$ is generated by a set of **generators** $S = \{s_1, s_2, \ldots, s_k\}$.
  - Every $w \in W$ can be written as a product of generators.
- Along with a set of **relations**.
  - These are rules to convert between expressions.
  - $s_i^2 = \text{id}$. –and– $(s_i s_j)^{\text{power}} = \text{id}$. (Write down)

For example, $w = s_3 s_2 s_1 s_1 s_2 s_4 = s_3 s_2 \text{id} s_2 s_4 = s_3 \text{id} s_4 = s_3 s_4$

Why should we use **these** rules?

# Pi in the cold of winter

Behold: The perfect wallpaper design for math majors:

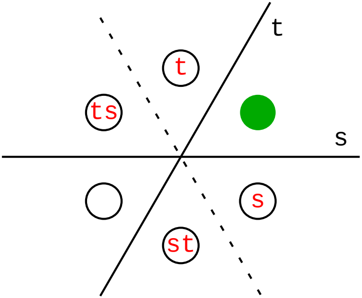



To see the reflections, we insert some **hyperplanes** that act as mirrors.

- In two dimensions, a hyperplane is simply a line.
- In three dimensions, a hyperplane is a plane.

# Reflection Groups

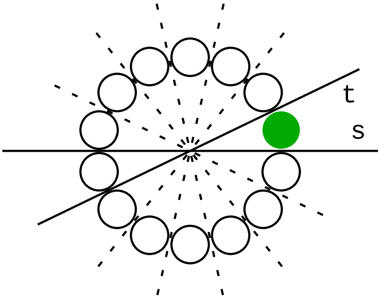
- These regions can be thought of as group elements. Place id.
- The action of multiplying (on the left) by a generator $s$ corresponds to a reflection across a hyperplane $H_s$. $(s^2 = \text{id})$




We see:

- $sts = tst \leftrightarrow \quad ststst = \text{id}$
  Shows $(st)^3 = \text{id}$ is natural.
- Our group has six elements: $\{\text{id}, s, t, st, ts, sts\}$.
- This is the group of symmetries of a hexagon.

# Reflection Groups



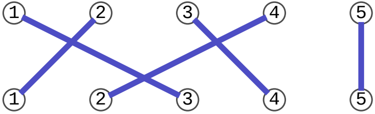

- When the angle between $H_s$ and $H_t$ is $\frac{\pi}{n}$, relation is $(st)^n = \mathrm{id}$.
- The size of the group is $|S| = 2n$.
- All finite reflection groups in the plane are these **dihedral groups**.
- **Two directions:** infinite and higher dimensional.

# Permutations are a group

An $n$-**permutation** is a permutation of $\{1, 2, \ldots, n\}$.

- Write in **one-line notation** or use a **string diagram**:

**31425**

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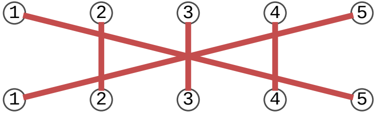
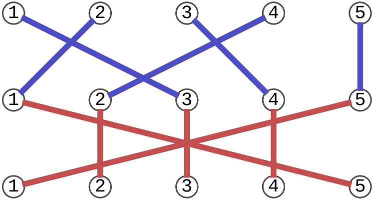



> $n$-Permutations form the **Symmetric group** $S_n$.

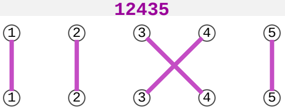
- We can multiply permutations.
- The identity permutation is $\text{id} = 1\,2\,3\,4 \ldots n$.
- Inverse permutation: Flip the string diagram upside down!

**31425 × 52341 = 35421**



# Permutations as a reflection group

A special type of permutation is an **adjacent transposition**, switching two adjacent entries.




- Write $s_i : (i) \leftrightarrow (i+1)$. (e.g. $s_3 = 1\,2\,4\,3\,5$).
- ★ Every $n$-permutation is a product of adjacent transpositions.
  - (Construct any string diagram through individual twists.)
  - *Example.* Write $3\,1\,4\,2\,5$ as $s_1 s_3 s_2$.
- $S = \{s_1, s_2, \ldots, s_{n-1}\}$ are **generators** of $S_n$.

| | |
|---|---|
| 12345 | 12345 |
| **21**345 | 1**32**45 |
| 2**31**45 | **31**245 |
| **32**145 | 3**21**45 |

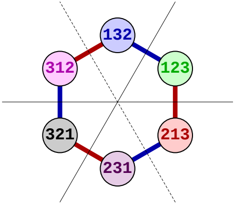
*A reflection group also has* ***relations****:*

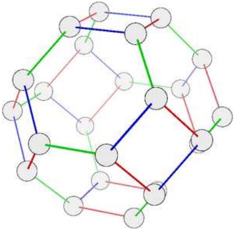
- First, $s_i^2 = \text{id}$.
- Consecutive generators don't commute: $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
- Non-consecutive generators DO commute: $s_i s_j = s_j s_i$

**21345**

# Visualizing symmetric groups

We have already seen $S_3$, generated by $\{s_1, s_2\}$:



We can visualize $S_4$ as a **permutohedron**, generated by $\{s_1, s_2, s_3\}$.

sourceforge.net/apps/trac/groupexplorer/wiki/The First Five Symmetric Groups/

They also give a way to see $S_5$ ...

# Higher-dimension symmetric groups

How can we "see" a reflection group in higher dimensions?

The relation $(s_i s_j)^m$ determines the angle between hyperplanes $H_i$, $H_j$:

- $(s_i s_j)^2 = \text{id} \quad \longleftrightarrow \quad \theta(H_i, H_j) = \pi/2$
- $(s_i s_j)^3 = \text{id} \quad \longleftrightarrow \quad \theta(H_i, H_j) = \pi/3$

For $S_6$, we expect an angle of $60^\circ$ between the hyperplane pairs

$$(H_1, H_2),\ (H_2, H_3),\ (H_3, H_4),\ \text{and}\ (H_4, H_5).$$

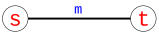
Every other pair will be perpendicular.

# All finite reflection groups

Or see with a **Coxeter diagram**:

- Vertices: One for every generator $i$
- Edges: Between $i$ and $j$ when $m_{i,j} \geq 3$. Label edges with $m_{i,j}$ when $\geq 4$.

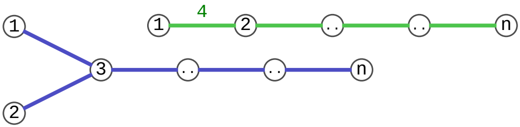
**Dihedral groups**



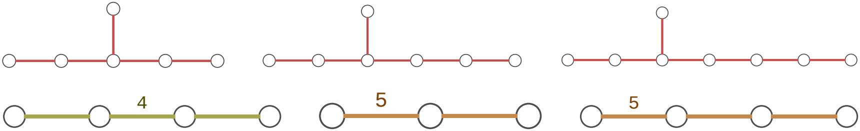

Generators: $s$ and $t$.
Relation: $(st)^m = \text{id}$

**Symmetric groups:**


**Infinite families:**




**Exceptional types:**

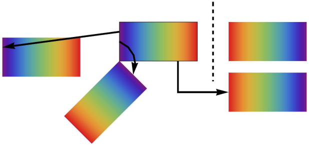


# Wallpaper Groups

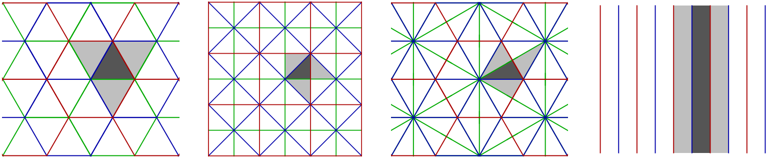
The art of M. C. Escher plays upon symmetries in the plane.

An **isometry** of the plane is a transformation that preserves distance.

**Think:** translations, rotations, reflections, glide reflections.



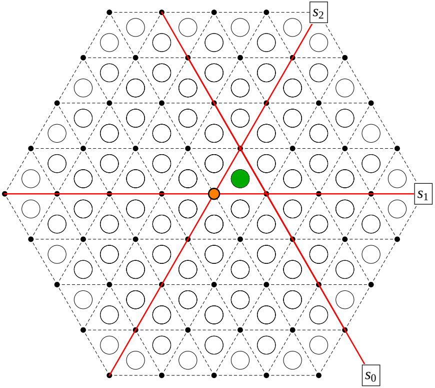
A **wallpaper group** is a group of isometries of the plane with two independent translations. Some are also reflection groups:



# Infinite Reflection Groups

Constructing an infinite reflection group: the **affine permutations** $\widetilde{S}_n$.

- Add a new generator $s_0$ and a new **affine** hyperplane $H_0$.

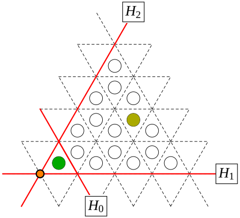



Elements generated by $\{s_0, s_1, s_2\}$ correspond to **alcoves** here.

# Combinatorics of affine permutations

Many ways to reference elements in $\widetilde{S}_n$.

- **Geometry.** Point to the alcove.
- **Alcove coordinates.** Keep track of how many hyperplanes of each type you have crossed to get to your alcove.
- **Word.** Write the element as a (short) product of generators.
- **One-line notation.** Similar to writing finite permutations as 312.
- **Abacus diagram.** Columns of numbers.
- **Core partition.** Hook length condition.
- **Bounded partition.** Part size bounded.
- **Others!** Lattice path, order ideal, etc.




Coordinates:

| | |
|---|---|
| 3 | 1 |
| 1 | |

Word: $s_0 s_1 s_2 s_1 s_0$

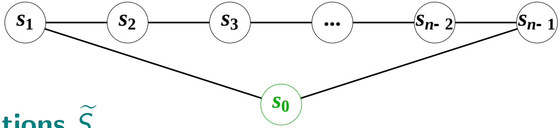
Permutation:

$$(-3, 2, 7)$$

# Affine permutations

**(Finite) $n$-Permutations $S_n$**

- Visually:




**Affine $n$-Permutations $\widetilde{S}_n$**

- Generators: $\{s_0, s_1, \ldots, s_{n-1}\}$
- $s_0$ has a braid relation with $s_1$ and $s_{n-1}$
- How does this impact one-line notation?
    - Perhaps interchanges 1 and $n$?
    - Not quite! (Would add a relation.)

# Window notation

**Affine $n$-Permutations $\widetilde{S}_n$** (G. Lusztig 1983, H. Eriksson, 1994)
Write an element $\widetilde{w} \in \widetilde{S}_n$ in 1-line notation as a permutation of $\mathbb{Z}$.

Generators transpose **infinitely many** pairs of entries:
$s_i$ : **(i)** $\leftrightarrow$ **(i+1)** $\ldots (n+i) \leftrightarrow (n+i+1) \ldots (-n+i) \leftrightarrow (-n+i+1) \ldots$

| In $\widetilde{S}_4$, | $\cdots w(-4)$ | $w(-3)$ $w(-2)$ $w(-1)$ $w(0)$ | $w(1)$ $w(2)$ $w(3)$ $w(4)$ | $w(5)$ $w(6)$ $w(7)$ $w(8)$ | $w(9)\cdots$ |
|---|---|---|---|---|---|
| $s_1$ | … -4 | -2 -3 -1 0 | 2 1 3 4 | 6 5 7 8 | 10 … |
| $s_0$ | … -3 | -4 -2 -1 1 | 0 2 3 5 | 4 6 7 9 | 8 … |
| $s_1 s_0$ | … -2 | -4 -3 -1 2 | 0 1 3 6 | 4 5 7 10 | 8 … |

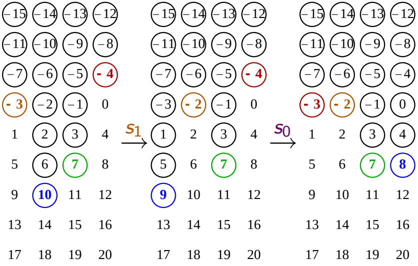
Symmetry: Can think of as integers wrapped around a cylinder.

$\widetilde{w}$ is defined by the window $[\widetilde{w}(1), \widetilde{w}(2), \ldots, \widetilde{w}(n)]$. $s_1 s_0 = [0, 1, 3, 6]$

# An abacus model for affine permutations

(James and Kerber, 1981) Given an affine permutation $[w_1, \ldots, w_n]$,

- Place integers in $n$ *runners*.
- Circled: *beads*. Empty: *gaps*
- Create an abacus where each runner has a lowest bead at $w_i$.

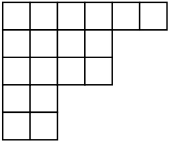



*Example:* $[-4, -3, 7, 10]$

- Generators act nicely.
- $s_i$ interchanges runners $i \leftrightarrow i+1$. $(s_1 : 1 \leftrightarrow 2)$
- $s_0$ interchanges runners 1 and $n$ (with shifts) $(s_0 : 1 \overset{\text{shift}}{\leftrightarrow} 4)$

# Core partitions

For an integer partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ drawn as a Young diagram,



The **hook length** of a box is $\#$ boxes below and to the right.

| | | | | | |
|---|---|---|---|---|---|
| 10 | 9 | 6 | 5 | 2 | 1 |
| 7 | 6 | 3 | 2 | | |
| 6 | 5 | 2 | 1 | | |
| 3 | 2 | | | | |
| 2 | 1 | | | | |

An ***n*-core** is a partition with no boxes of hook length dividing $n$.

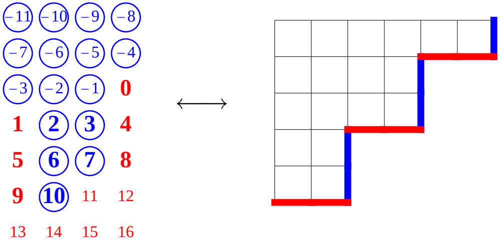
*Example.* $\lambda$ is a 4-core, 8-core, 11-core, 12-core, etc.
$\lambda$ is NOT a 1-, 2-, 3-, 5-, 6-, 7-, 9-, or 10-core.

# Core partition interpretation for affine permutations

**Bijection:** $\{\text{abaci}\} \longleftrightarrow \{n\text{-cores}\}$

Rule: Read the boundary steps of $\lambda$ from the abacus:

- A bead $\leftrightarrow$ vertical step
- A gap $\leftrightarrow$ horizontal step




*Fact:* This is a bijection!

# Action of generators on the core partition

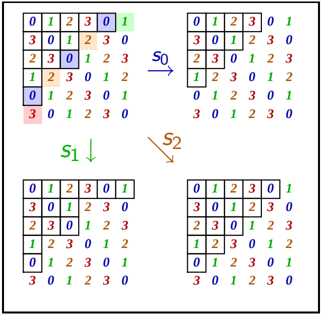
- Label the boxes of $\lambda$ with residues.
- $s_i$ acts by adding or removing boxes with residue $i$.

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 0 | 1 |
| 3 | 0 | 1 | 2 | 3 | 0 |
| 2 | 3 | 0 | 1 | 2 | 3 |
| 1 | 2 | 3 | 0 | 1 | 2 |
| 0 | 1 | 2 | 3 | 0 | 1 |
| 3 | 0 | 1 | 2 | 3 | 0 |

*Example.* $\lambda = (5, 3, 3, 1, 1)$

- has removable 0 boxes
- has addable 1, 2, 3 boxes.

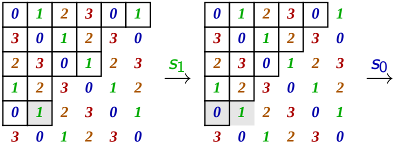
Idea: We can use this to figure out a *word* for $w$.

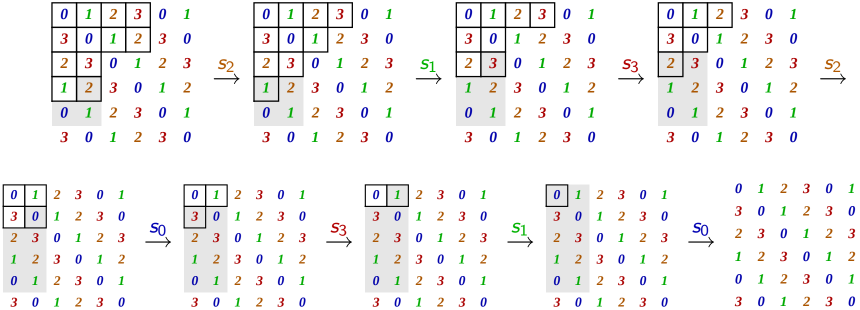


# Finding a word for an affine permutation.

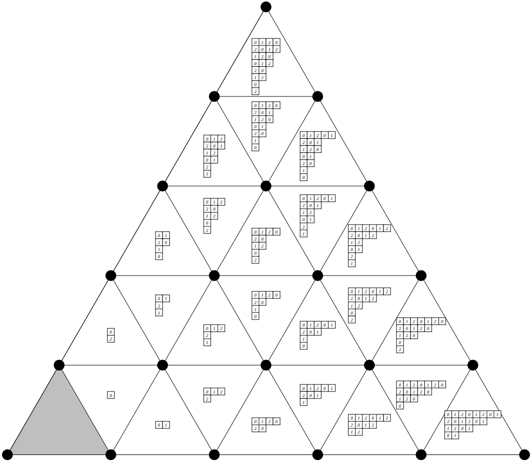
*Example:* The word in $S_4$ corresponding to $\lambda = (6,4,4,2,2)$:

$s_1 s_0 s_2 s_1 s_3 s_2 s_0 s_3 s_1 s_0$



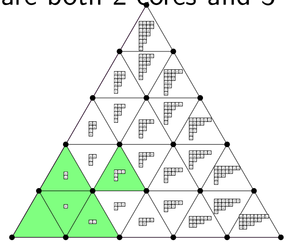



# The bijection between cores and alcoves



# Simultaneous core partitions

How many partitions are both 2-cores and 3-cores? **2**.



How many partitions are both 3-cores and 4-cores? **5**.

How many simultaneous 4/5-cores? **14**.

How many simultaneous 5/6-cores? **42**.

How many simultaneous $n/(n+1)$-cores? $C_n$!

Jaclyn Anderson proved that the number of $s/t$-cores is $\frac{1}{s+t}\binom{s+t}{s}$.

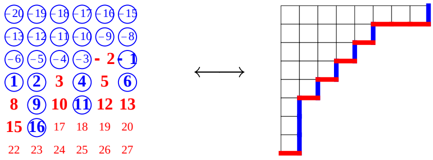
The number of 3/7-cores is $\frac{1}{10}\binom{10}{3} = \frac{1}{10}\frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1} = 12$.

Fishel–Vazirani proved an alcove interpretation of $n/(mn+1)$-cores.

# Research Questions

★ Can we extend combinatorial interps to other reflection groups?

- Yes! Involves self-conjugate partitions. arχiv:1105.5333
- Joint with Brant Jones, James Madison University.



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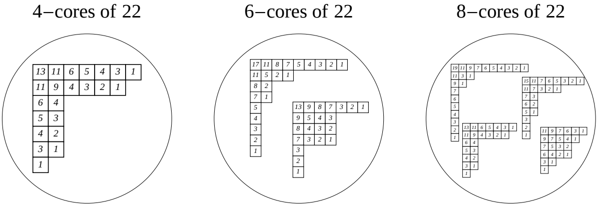
- ★ What numerical properties do self-conjugate core partitions have?
  - Joint with Rishi Nath, York College. ar$\chi$iv:1201.6629
  - We found & proved some impressive numerical conjectures.
  - There are more (s.c. $t+2$-cores of $n$) than (s.c. $t$-cores of $n$).



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★ What is the average size of an $s/t$-core partition?
- In progress. We “know” the answer, but we have to prove it!
- Working with Drew Armstrong, University of Miami.

# Thank you!

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**Interact:** people.qc.cuny.edu/chanusa > Animations

M. A. Armstrong.
*Groups and symmetry*. Springer, 1988.
Easy-to-read introduction to groups, (esp. reflection)

James E. Humphreys
*Reflection groups and Coxeter groups*. Cambridge, 1990.
More advanced and the reference for reflection groups.

http://www.mcescher.com/

http://www.math.ubc.ca/∼cass/coxeter/crm1.html

http://sourceforge.net/apps/trac/groupexplorer/wiki/
The First Five Symmetric Groups/