Matrix Types and Operations Arising in Matching Theory

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Current Problems Seminar

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Determinants

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$= a_{11}a_{12}a_{13} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Permutation expansion of the determinant:

$$\det A = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, a_{1\,\sigma(1)} a_{2\,\sigma(2)} \cdots a_{n\,\sigma(n)}$$

Recall: A permutation σ is a member of the symmetric group on n letters if $\{i\}_{1 \le i \le n} = \{\sigma(i)\}_{1 \le i \le n}$.

Recall: The sign $\mathrm{sgn}(\sigma)$ of a permutation σ is (-1) to the number of transpositions that it takes to make up σ.

Have title slide on projector.

Thanks for coming. Today I'm going to talk about matrix operations and special types of matrices and how they relate to the combinatorics I do. My goal by the end of the talk is threefold − to introduce you to at least one new matrix operation, at least one new type of matrix, and to show you how these ideas come up in matching theory.

Put "Determinants" slide on projector.

We'll start off with something everyone is familiar with, determinants. When you think of determinants, you probably think back to linear algebra and the definition of the determinant as an expansion about rows or columns — basically some sort of recursive definition where we take an $n \times n$ determinant and break it into n $n-1 \times n-1$ determinants.

As combinatorialists, we carry out this recursion to its end, where we end up with a sum over all products of n entries that cover each row and each column exactly once. The way we write this is as a sum in terms of permutations.

Remember that a permutation is a way to take a function of the entries 1 to n and rearrange them so that each number appears once.

A Determinant Calculus

$\det AB = \det A \det B$

$\det A = \det LU = \prod_{i=1}^{n} \ell_{ii}$

Dodgson's Condensation:

$$\det A \cdot \det A_{1,n}^{1,n} = \det A_1^1 \cdot \det A_n^n - \det A_1^n \cdot \det A_n^1$$

$$\det A = \det \begin{pmatrix} \det A_1^1 & \det A_1^n \\ \det A_n^1 & \det A_n^n \end{pmatrix} / \det A_{1,n}^{1,n}$$

Example:

$$\det \begin{pmatrix} 1 & 2 & 6 \\ 2 & 6 & 22 \\ 6 & 22 & 90 \end{pmatrix} = \det \begin{pmatrix} 56 & 8 \\ 8 & 2 \end{pmatrix} / 6 = 8$$

See *Advanced Determinant Calculus* by Christian Krattenthaler — arXiv:math.CO/0503507

Put Slide "A Determinant Calculus" on projector.

We like determinants because they satisfy nice properties (or the "determinant calculus" is nice). They have properties like, which when combined with the nice decomposition of a matrix in its LU-decomposition gives a fast way to calculate it.

Also, we can use a sneaky rule called condensation, invented by Charles Dodgson aka Lewis Carroll in 1866. It is a way to break down a large determinant into many smaller determinants.

Write on board $I = \{i_1, \ldots, i_k\} \subset [n]$, $J = \{j_1, \ldots, j_k\} \subset [n]$, A_J^I, draw picture of $n \times n$ matrix and strike out rows $i_1, \ldots, i_k$, columns $j_1, \ldots, j_k$.

Define two sets of k integers that are subsets of the integers between 1 and n. Define the notation A_J^I to be the matrix where we strike rows i and columns j from A. What condensation says is that we can replace the computation of an $n \times n$ determinant with four $n-1 \times n-1$ determinants and one $n-2 \times n-2$ determinant; compare this with the previous way to expand the determinant about rows/columns. Here is an example of how we could use condensation to calculate the determinant of a 3 × 3 matrix.

Go through the example of how we could use condensation.

If you would like to learn more advanced methods in determinantal calculus, see the article by Christian Krattenthaler.

Permanents

$$\det A = \sum_{\sigma \in S_n} \mathsf{sgn}(\sigma)\, a_{1\,\sigma(1)} a_{2\,\sigma(2)} \cdots a_{n\,\sigma(n)}$$

$$\mathsf{perm}\, A = \sum_{\sigma \in S_n} a_{1\,\sigma(1)} a_{2\,\sigma(2)} \cdots a_{n\,\sigma(n)}$$

$$\mathsf{perm}\, AB \neq \mathsf{perm}\, A \cdot \mathsf{perm}\, B$$

$$
\begin{aligned}
\mathsf{perm} \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix} &= 1 \cdot 7 + 3 \cdot 5 \\
&= 7 + 15 \\
&= 22
\end{aligned}
$$

Domino Tilings

Question: How many ways are there to completely cover a chessboard with dominoes? (2×1 or 1×2 rectangles)

Answer:

New Question: How can we count how many ways there are to completely cover a chessboard with dominoes?

Answer: Use the dual graph.

Put Slide "Permanents" on projector.

Compare determinants to an apparently more simple matrix operation, the permanent. It's just like the determinant in that you take the sum over all permutations of a product of n entries of the matrix but with no sign term stuck in.

Permanents do not have a nice Permanent Calculus. In particular, they do not satisfy perm AB = perm A perm B. (Notice that if we had this, then we could use the LU decomposition of a matrix to calculate the permanent.

Here is an example.

Whereas computing the determinant of a matrix can be done in polynomial time (using Gauss-Jordan elimination (row echelon form)), computing the permanent of a matrix with only $0 - 1$ entries is $\#P$ complete (the time to enumerate solutions to existence problems in NP), which is stronger than NP-complete.

So why do I even bring up permanents? Because the permanent is indeed a simpler formula, it does appear; it has appeared in my own studies. Let me explain how.

Put Slide "Domino Tilings" on projector.

We're going to talk about domino tilings of regions. The idea you should have in your head is of a chessboard that we want to cover completely with 32 non-overlapping dominoes. We call these dominoes tiles, the chessboard itself a region, and any collection of 32 dominoes we call a tiling of the region.

So a question we can ask of this game is "How many ways are there to completely cover a chessboard with dominoes?"

Depending on how many people have seen the presentation before, play the guessing game.

Think in your head of what you think the answer to this question should be. How many people think that there are more than 10 possible tilings? 100? 1,000? 10,000? 100,000? 1,000,000? 10,000,000?

The correct answer is 12,988,816. (I just realized that this is very close to my UW student ID number!)

Fill in answer.

We've got an answer, but then we need to ask how we might actually calculate this number not using brute force methods.

We'll be using a construction called the dual graph.

Dual Graph

A *graph* $G = (V, E)$: V are the *vertices*, E are the *edges*.

A *dual graph*: Given a region, place a vertex v in every square, and place an edge $e = v_1 v_2$ if square v_1 is adjacent to v_2.

A *bipartite* graph: black vertex set V and white vertex set W; no edges between vertices of the same color.

A *perfect matching* of a graph G: choice of edges $e_1, \cdots, e_n$ that cover each of the $2n$ vertices.

Domino Tilings
$\updownarrow$
Perfect Matchings

Put Slide "Dual Graph" on projector.

To review, the idea of a graph is one of abstracting connections of related objects, like roads connecting cities. We have a set of points, or vertices, where two vertices can be connected by edges.

The idea of a dual graph of a region is easiest to think of as a picture. Let's consider a mini-chessboard example of a 4 × 4 square.

Draw 4 × 4 square on blackboard.

We place a vertex in the center of every square, and connect two vertices with an edge if the two squares that they correspond to are adjacent. Just like in the chessboard itself, we can give the vertices colors — in this example the dual graph has 8 black and 8 white vertices.

Notice that no black vertex is connected to a black vertex or a white vertex to a white vertex. This is the idea of a bipartite graph — that we can break the vertices into two sets and there are no edges between vertices of the same color.

The last definition we need for now is of a perfect matching. If there are $2n$ vertices, we need to pick n edges so that every vertex is covered by some edge.

Show a perfect matching on the 4 × 4 example.

The key observation that we can make is that for every domino tiling of a region, there is a perfect matching of the dual graph and vice versa.

Show the correspondence on the board example, using perfect matching.

So that means that instead of counting domino tilings of a region, we can count perfect matchings of a dual graph, and this is where the permanent comes in.

Counting Perfect Matchings
(Kasteleyn)

Given a bipartite graph, create a matrix A:

$$a_{ij} = \begin{cases} 1 & \text{if } v_i w_j \text{ is an edge} \\ 0 & \text{if } v_i w_j \text{ is not an edge} \end{cases}$$

Non-zero term in permutation expansion of perm A

n non-zero entries in A covering each row, column

Assignment of n edges covering each vertex.

$$\begin{aligned} \text{perm } A &= \text{\# Perfect matchings of the graph} \\ &= \text{\# Domino tilings of the region} \end{aligned}$$

4×4 Permanent Example

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

perm $A = 36$

Put Slide "Counting Perfect Matchings" on projector.

The following method was used in the early 1960's by a physicist named Kasteleyn to find a formula for the number of matchings in a rectangular grid like our chessboard.

We restrict ourselves to the case when the dual graph is bipartite. This covers most enumeration problems that combinatorialists consider at this time. If there is a perfect matching of the dual graph, then there must be the same number of black vertices as white vertices. So label the black vertices v_1 through v_n and label the white vertices w_1 through w_n.

We create an $n \times n$ matrix A. Let entry a_{ij} of A be 1 if there is an edge between v_i and w_j, and 0 if there is not. Consider the permutation expansion of the permanant that we talked about earlier. What does a non-zero entry correspond to?

Remember that this is a collection of n entries of the matrix that covers each row and each column. If this choice of n entries has a non-zero product, then they are all 1's. Going back to the combinatorial interpretation of the matrix, this means there is an edge between v_i and $w_{\sigma(i)}$ for all i, or in other words that there are n edges that completely cover all the vertices.

This means that taking the permanent of the matrix gives exactly the number of perfect matchings of the dual graph. Which by our previous correspondence implies that we can count the number of tilings of a region by taking the permanent of a matrix.

Put Slide "4 × 4 Permanent Example" on projector.

Back to De Terminants
(Percus)

Permanent $\longrightarrow$ Determinants?

On a square grid, the rule is simple.
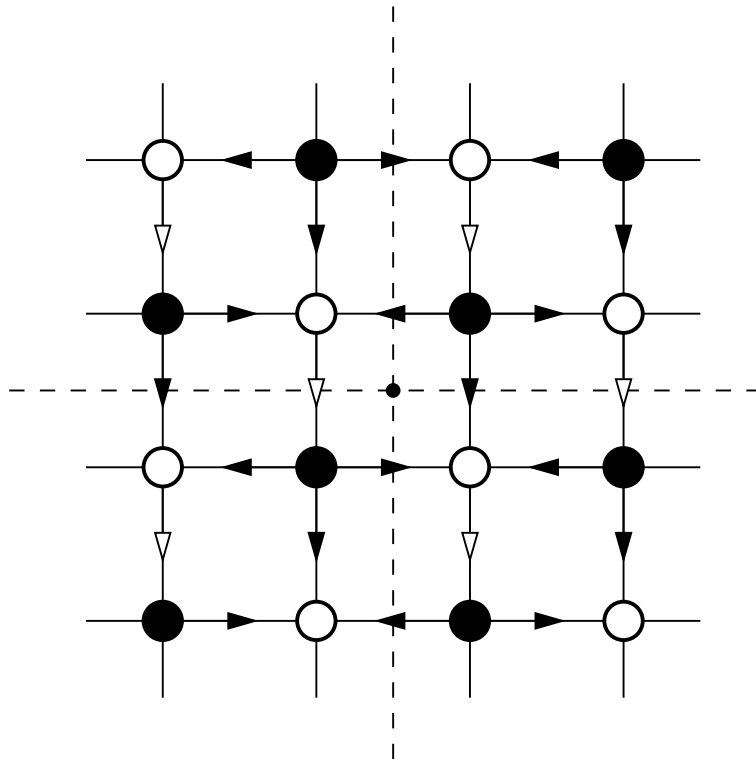
4×4 Determinant Example

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{pmatrix}$$

$$\det A' = 36$$

Put Slide "Back to De Terminants" on projector.

Now, we already decided that determanants are much nicer than permanents. What goes wrong in terms of counting our matchings if we take a determinant? Terms cancel. Is there some way we can put signs into the matrix to turn the permanent into a determinant?

I wouldn't be asking this question of you if there weren't. In terms of a graph on a square grid, the rule for placing signs was introduces in 1963 by a physicist named Percus. It's actually quite simple.

Put Slide "4 × 4 Determinant Example" on projector.

The resulting matrix is called a Kasteleyn-Percus matrix, and here is our modified example.

Pfaffians

Given a $2n \times 2n$ skew-symmetric matrix B ($B^T = -B$)
the Pfaffian of B, $\mathrm{Pf}(B)$ is defined to be

$$\sum_{\beta} \mathrm{sgn}(\beta) b_{i_1, j_1} b_{i_2, j_2} \cdots b_{i_n, j_n},$$

where β is some pairwise partition of $\{1, \ldots, 2n\}$
into n pairs $\{(i_1, j_1), \ldots, (i_n, j_n)\}$
and $\mathrm{sgn}(\beta) = \mathrm{sgn}(i_1, j_1, i_2, \cdots, i_n, j_n)$ ($\in S_{2n}$).

A matrix operation that works on all graphs
(not just bipartite graphs).

Given $G = (V, E)$, define a matrix B.
Put an arbitrary direction on the edges.

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \to v_j \text{ is an edge} \\ -1 & \text{if } v_j \to v_i \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

$\mathrm{Pf}(B)$ counts the number of matchings of G.

Bipartite graph yields $\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}$

Pfaffian Properties

Let B be $2n \times 2n$ skew-symmetric, C be $2n \times 2n$.

$$[\mathsf{Pf}(B)]^2 = \det(B)$$

$$\mathsf{Pf}(CBC^T) = \det(C)\mathsf{Pf}(B)$$

$$\mathsf{Pf}\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \mathsf{Pf}(A_1)\mathsf{Pf}(A_2)$$

$$\mathsf{Pf}\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} = (-1)^{n(n-1)/2}\det(A)$$

It's time for me to start living up to my goals I stated at the beginning of the talk.

Put Slide "Pfaffians" on projector.

Even if you've seen the determinant and the permanent before, it's unlikely you've come across the Pfaffian before. This permanent result of Kasteleyn was actually first formulated as a Pfaffian.

What is a Pfaffian? How does it work?

Draw a matrix on the board, highlighting entry $(1, 2)$, and motioning that rows 1,2 and columns 1,2 would not be used.

A Pfaffian acts on a $2n \times 2n$ matrix. It takes a set of n entries out of the matrix so that no entry is used more than once. For example, if you choose entry $(1, 2)$, you would not include any other entries from rows 1 or 2 or from columns 1 or 2

Even when graphs are not bipartite, we can still calculate its number of perfect matchings using a Pfaffian.

Here you go. Ignoring the sign for a minute, notice that this is plausible for counting matchings. We are checking to see if some set of n edges exists, and we do not count a vertex twice.

In the case of a bipartite graph, the ordering of black vertices then white vertices yields $[[0, A], [A^T, 0]]$, where A is the Kasteleyn-Percus matrix with an arbitrary sign on each entry.

Put Slide "Pfaffian Properties" on projector.

What should you take away from my telling you about Pfaffians? When you hear Pfaffian, you should think "square-root of determinant". This is because if we square the Pfaffian, we get the determinant of the matrix.

These have actually been studied since the early 1800's, and were named after the German mathematician named Pfaff, who worked on first-order PDEs.

This is the end of the diversion to Pfaffians.

4×4 Determinant Example

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{pmatrix}$$

$$\det A' = 36$$

Pattern Hunting

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

Put Slide "4 × 4 Determinant Example" on projector.

Let me go back to that Kasteleyn-Percus matrix in our example. If you stare at anything for a long enough period of time, patterns start to appear. You may be hallucinating, but sometimes there actually are pattern.

For example, look at this K-P matrix. There are 1's in some positions and -1's in others; is there a pattern? What do you notice?

(wait for response?)

Anything else? Sure there's a simple pattern in the -1's. And the matrix is not symmetric, but it is rotationally symmetric (by a rotation of 180 degrees)

In general, the pattern of where the -1's appear is a tad more mysterious. I'm going to need to rearrange the rows and columns suggestively.

Put Slide "Pattern Hunting" on projector.

We might start to see that when we rotate the matrix 180 degrees NOW, certain values of the matrix switch sign. Which ones?

Alternating Centrosymmetric Matrices

A is *alternating centrosymmetric* if the entries of A satisfy $a_{i,j} = (-1)^{i+j} a_{n+1-i, n+1-j}$.

Another way to say this is $A = KAK$, where

$$
K = \begin{pmatrix}
0 & & \cdots & & & 0 & 1 \\
 & & & & & -1 & 0 \\
\vdots & & & & 1 & & \\
 & & & -1 & & & \vdots \\
0 & 1 & & & & & \\
-1 & 0 & & & \cdots & & 0
\end{pmatrix}
$$

Example:

$$
A = \begin{pmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 8 & 9 \\
9 & -8 & 6 & -5 \\
-4 & 3 & -2 & 1
\end{pmatrix}
$$

I had to make up a name for this type of matrix, since I couldn't find it in the literature anywhere. I used as my model the type of matrix called a centrosymmetric matrix.

The conditions on matrices look similar, but one might think less confusing. As an example, you can see here:

They also satisfy lots of nice properties.

Centrosymmetric matrices come up in many various applications, so I wanted to say something briefly about them.

Centrosymmetric Matrices

A is *centrosymmetric* if the entries of A satisfy $a_{i,j} = a_{n+1-i,n+1-j}$.

Another way to say this is $A = JAJ$, where

$$J = \begin{pmatrix} 0 & & \cdots & & 0 & 1 \\ & & & & 1 & 0 \\ \vdots & & & 1 & & \\ & & 1 & & & \vdots \\ 0 & 1 & & & & \\ 1 & 0 & & \cdots & & 0 \end{pmatrix}$$

is the exchange matrix.

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 8 & 9 \\ 9 & 8 & 6 & 5 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Centrosymmetric Matrices

Centrosymmetric matrices have the form

$$A = \begin{pmatrix} B & CJ \\ JC & JBJ \end{pmatrix},$$

which has the same determinant as the matrix

$$A' = \begin{pmatrix} B + C & CJ \\ 0 & J(B - C)J \end{pmatrix}.$$

All eigenvectors $\mathbf{v}$ of A are either

symmetric ($\mathbf{v} = J\mathbf{v}$) or
anti-symmetric ($\mathbf{v} = -J\mathbf{v}$).

Centrosymmetric Matrices

Applications of Centrosymmetric Matrices

- Wavelets / Signal Processing

- Numerical methods for PDE

  - Radial Basis Function
    (Chen, Tanaka)

  - (Harmonic) Differential Quadrature
    (Chen, He, Zhong)

- Least Squares (Generalized Centrosymmetric)

I wish I understood the following applications of centrosymmetric matrices better.

Wavelets: When you are constructing the orthonormal wavelet basis, centrosymmetric matrices come up often. (article by Zhong-Yun Liu)

Numerical Methods for PDE: Noticing that key matrices are centrosymmetric allows computation effort to be reduced by 75%. This is because it simplifies the calculations of determinants, inverses, and eigenvalues.

The least-squares part is really interesting. See paper by Hsin-Chu Chen, Generalized Reflexive Matrices: Special Properties and Applications. It has a great example of how to reduce a least squares problem to smaller dimension.

Simplifying Determinants

$K_2 = $ upper-right quarter-matrix of K

Alternating Centrosymmetric matrices have the form

$$A = \begin{pmatrix} B & CK_2 \\ K_2^{-1}C & -K_2^{-1}BK_2 \end{pmatrix}$$

which has the same determinant as the matrix

$$A' = \begin{pmatrix} B - iC & CK_2 \\ 0 & -K_2^{-1}(B + iC)K_2 \end{pmatrix}.$$

Theorem. The determinant of a $2k \times 2k$ alternating centrosymmetric matrix is (up to sign) a sum of two squares.

$$\begin{aligned} \det A &= (-1)^k \det(B + iC)\det(B - iC) \\ &= (-1)^k (x^2 + y^2) \end{aligned}$$

Sum of Squares Example

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 8 & 9 \\ 9 & -8 & 6 & -5 \\ -4 & 3 & -2 & 1 \end{pmatrix}$$

$$\det A = \det \begin{pmatrix} 1 - 4i & 2 + 3i \\ 5 - 9i & 6 + 8i \end{pmatrix} \det \begin{pmatrix} 1 + 4i & 2 - 3i \\ 5 + 9i & 6 - 8i \end{pmatrix}$$

$$= (1 - 13i)(1 + 13i) = 1^2 + 13^2 = 170.$$

Stuff here

I wanted to give you a flavor of the type of proof that we use when proving this sum of squares formula, but it was too boring for me to say in front of an audience. It's basically just matrix manipulation to mold A into block diagonal form, where the two blocks have determinant $(B + iC)$ and $(-1)^k(B - iC)$.

Revisiting Jockusch

A 2-even-symmetric graph is a graph with a symmetry of 180 degrees and such that the length of a path between a vertex and its antipode is even.

Theorem (Jockusch). The number of perfect matchings of a 2-even symmetric graph is a sum of two squares.

Theorem (H, 2004). If G is a graph that can be embedded symmetrically into the square grid with its point of rotation the origin, then its Kasteleyn matrix is alternating centrosymmetric.

Corollary (H, 2004). If G is a graph that can be embedded symmetrically into the square grid, the number of perfect matchings of G is a sum of two squares.

(Should be more general.)

Talk about these theorems.

The key is again noting that the inherent structure in a centrosymmetric matrix allows us to simplify calculations and gives us nice results.

Cycle Systems

Given a graph G, a *cycle system* is a union of vertex-disjoint cycles.