Combinatorial interpretations in affine Coxeter groups of types B, C, and D

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What is a Coxeter group?

A **Coxeter group** is a group with

- Generators: $\{s_1, s_2, \ldots, s_n\}$
- Relations: $s_i^2 = 1, \qquad (s_i s_j)^{m_{ij}} = 1$ where $m_{ij} \geq 2$ or $= \infty$
  - $m_{ij} = 2$: $(s_i s_j)(s_i s_j) = 1 \quad \longrightarrow \quad s_i s_j = s_j s_i$ (they commute)
  - $m_{ij} = 3$: $(s_i s_j)(s_i s_j)(s_i s_j) = 1 \rightarrow s_i s_j s_i = s_j s_i s_j$ (braid relation)
  - $m_{ij} = \infty$: s_i and s_j are not related.

Why Coxeter groups?

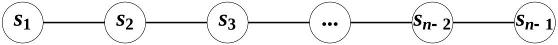
- They're awesome.
- Discrete Geometry: Symmetries of regular polyhedra.
- Algebra: Symmetric group generalizations. (Kac-Moody, Hecke)
- Geometry: Classification of Lie groups and Lie algebras

Examples of Coxeter groups

(Finite) n-Permutations S_n (A_{n-1})

123	123
213	132
231	312
321	321

- Generators $\{s_1, s_2, \ldots, s_{n-1}\}$ are:
  Adjacent transpositions: $s_i : i \leftrightarrow i+1$
- Only consecutive generators don't commute: $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
- See visually with a Coxeter graph:



Affine n-Permutations $\widetilde{S}_n$ ($\widetilde{A}_{n-1}$)

- Generators: $\{s_0, s_1, \ldots, s_{n-1}\}$
- S_n is a parabolic subgroup of $\widetilde{S}_n$

Minimal length coset representatives

For a Coxeter group $\widetilde{W}$ generated by $\{s_0, s_1, \ldots, s_n\}$,

- An induced subgraph of $\widetilde{W}$'s Coxeter graph is a subgroup W
  - Today, we will always have W defined by $\widetilde{W} \setminus \{s_0\}$
- Every element $\widetilde{w} \in \widetilde{W}$ can be written $\widetilde{w} = w^0 w$, where $w^0 \in \widetilde{W}/W$ is a coset representative and $w \in W$.

Simple example: For $\widetilde{w} = s_0 s_1 s_2 s_3 s_0 s_1 s_2 \in \widetilde{S}_4$

$$\widetilde{w} = s_0 s_1 s_2 s_3 s_0 s_1 s_2$$

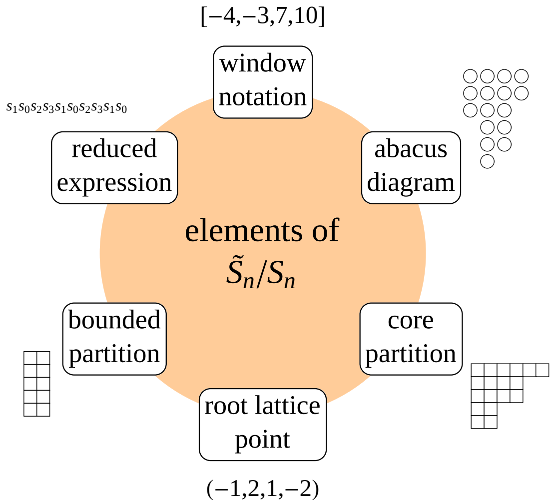
Less simple example: $\widetilde{w} = s_1 s_3 s_2 s_3 s_0 s_1 \in \widetilde{S}_4$

$$\widetilde{w} = s_1 s_2 s_3 s_2 s_0 s_1$$

$$\widetilde{w} = s_1 s_2 s_3 s_0 s_2 s_1$$

★ Combinatorial interpretations are easier to use.★

Combinatorial interpretations of $\widetilde{S}_n/S_n$



Window notation

Affine n-Permutations $\widetilde{S}_n$ (G. Lusztig 1983, H. Eriksson, 1994)

Write an element $\widetilde{w} \in \widetilde{S}_n$ in 1-line notation as a permutation of $\mathbb{Z}$.

Generators transpose **infinitely many** pairs of entries:

s_i : **(i)** $\leftrightarrow$ **(i+1)** $\ldots (n+i) \leftrightarrow (n+i+1) \ldots (-n+i) \leftrightarrow (-n+i+1) \ldots$

In $\widetilde{S}_4$,	$\cdots w(-4)$	$w(-3)$	$w(-2)$	$w(-1)$	$w(0)$	$w(1)$	$w(2)$	$w(3)$	$w(4)$	$w(5)$	$w(6)$	$w(7)$	$w(8)$	$w(9) \cdots$
s_1	... -4	-2	-3	-1	0	2	1	3	4	6	5	7	8	10 ...
s_0	... -3	-4	-2	-1	1	0	2	3	5	4	6	7	9	8 ...
$s_1 s_0$	... -2	-4	-3	-1	2	0	1	3	6	4	5	7	10	8 ...

$\widetilde{w}$ is defined by the window $[\widetilde{w}(1), \widetilde{w}(2), \ldots, \widetilde{w}(n)]$. $s_1 s_0 = [0, 1, 3, 6]$

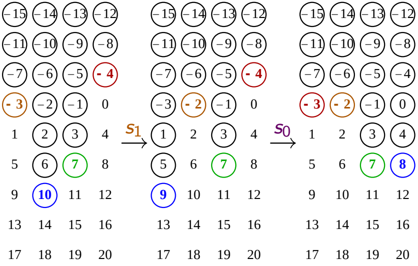
★ For $\widetilde{w} = w^0 w$, the window of w^0 is the window of $\widetilde{w}$, **sorted** $\nearrow$.

An abacus model for $\widetilde{S}_n/S_n$

(James and Kerber, 1981) Given $w^0 = [w_1, \ldots, w_n] \in \widetilde{S}_n/S_n$,

- Place integers in n *runners*.
- Circled: *beads*. Empty: *gaps*
- Bijection: Given w^0, create an abacus where each runner has a lowest bead at w_i.

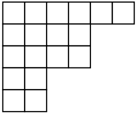
Example: $[-4, -3, 7, 10]$



- Generators act nicely.
- s_i interchanges runners $i \leftrightarrow i+1$. ($s_1 : 1 \leftrightarrow 2$)
- s_0 interchanges runners 1 and n (with shifts) ($s_0 : 1 \overset{\text{shift}}{\leftrightarrow} 4$)

Integer partitions and n-core partitions

For an integer partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ drawn as a Ferrers diagram,



The *hook length* of a box is # boxes below and to the right.

10	9	6	5	2	1
7	6	3	2		
6	5	2	1		
3	2				
2	1				

An *n-core* is a partition with no boxes of hook length dividing n.

Example. λ is a 4-core, 8-core, 11-core, 12-core, etc.
λ is NOT a 1-, 2-, 3-, 5-, 6-, 7-, 9-, or 10-core.

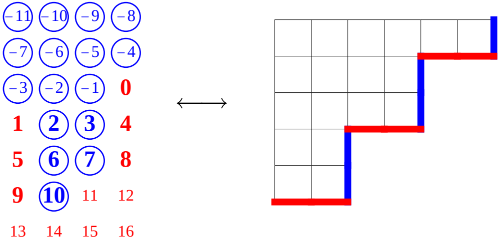
Core partitions for $\widetilde{S}_n/S_n$

Elements of $\widetilde{S}_n/S_n$ are in bijection with n-cores.

Bijection: {abaci} $\longleftrightarrow$ {n-cores}

Rule: Read the boundary steps of λ from the abacus:

- A bead $\leftrightarrow$ vertical step
- A gap $\leftrightarrow$ horizontal step



Fact: Abacus flush with n-runners $\leftrightarrow$ partition is n-core.

Action of generators on the core partition

0	1	2	3	0	1
3	0	1	2	3	0
2	3	0	1	2	3
1	2	3	0	1	2
0	1	2	3	0	1
3	0	1	2	3	0

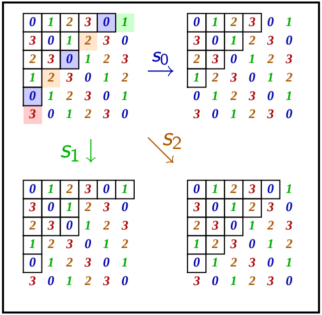
- Label the boxes of λ with residues.
- s_i acts by adding or removing boxes with residue i.

Example. $\lambda = (5, 3, 3, 1, 1)$

- has removable 0 boxes (s_0 is a descent)
- has addable 1, 2, 3 boxes. (s_1, s_2, s_3 are ascents)

Idea: Use to determine a canonical reduced expression for w^0.

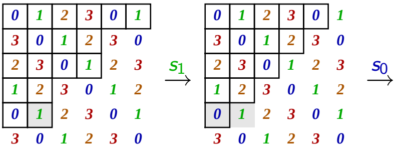
- Tally residues from bottom to top, from right to left.

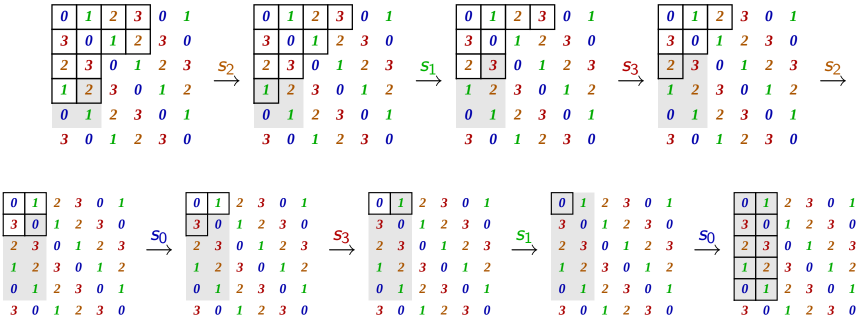


Canonical reduced expression for $\widetilde{S}_n/S_n$

Example: Reduced expression corresponding to $\lambda = (6,4,4,2,2)$:

$$\mathcal{R}(\lambda) = s_1 s_0 s_2 s_1 s_3 s_2 s_0 s_3 s_1 s_0$$





Bounded partitions for $\widetilde{S}_n/S_n$

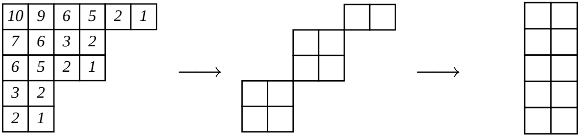
A partition $\beta = (\beta_1, \ldots, \beta_k)$ is *b-bounded* if $\beta_i \leq b$ for all i.

Elements of $\widetilde{S}_n/S_n$ are in bijection with $(n-1)$-bounded partitions.

Bijection: (Lapointe, Morse, 2005)

$$\{n\text{-cores } \lambda\} \leftrightarrow \{(n-1)\text{-bounded partitions } \beta\}$$

- Remove all boxes of λ with hook length $\geq n$
- Left-justify remaining boxes.



$\lambda = (6, 4, 4, 2, 2)$ $\quad\quad$ $\beta = (2, 2, 2, 2, 2)$

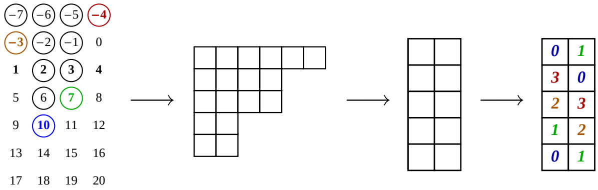
Canonical reduced expression for $\widetilde{S}_n/S_n$

Given the bounded partition, read off the reduced expression:

Method: (Berg, Jones, Vazirani, 2009)

- Fill β with residues i
- Tally s_i reading right-to-left in rows from bottom-to-top

Example. $[-4, -3, 7, 10] = s_1 s_0 s_2 s_1 s_3 s_2 s_0 s_3 s_1 s_0$.



- The Coxeter length of w^0 is the number of boxes in β.

Summary for $\widetilde{S}_n/S_n$

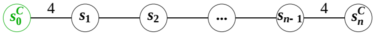
- See S_n as parabolic subgroup of $\widetilde{S}_n$
- **Window notation**
  - $\widetilde{S}_n$ elements can be written as a permutation of $\mathbb{Z}$
  - Min. len. coset rep's correspond to **sorted** $\mathbb{Z}$-permutations.
- **Abacus models**
  - Define the abacus by reading the entries from a window
  - Generators act nicely: They interchange runners
- **Core partitions**
  - Define the core by reading beads and gaps
  - Generators act nicely: They add and remove boxes using residues
- **Bounded partitions**
  - Define by collapsing the core partition
  - Reading the residues gives a reduced expression!

Summary for $\widetilde{W}/W$

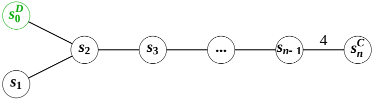
- See W as parabolic subgroup of $\widetilde{W}$
- **Window notation**
  - $\widetilde{W}$ elements can be written as a permutation of $\mathbb{Z}$
  - Min. len. coset rep's correspond to **sorted** $\mathbb{Z}$-permutations.
- **Abacus models**
  - Define the abacus by reading the entries from a window
  - Generators act nicely: They interchange runners
- **Core partitions**
  - Define the core by reading beads and gaps
  - Generators act nicely: They add and remove boxes using residues
- **Bounded partitions**
  - Define by collapsing the core partition
  - Reading the residues gives a reduced expression!

Coxeter Graphs for Types $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$

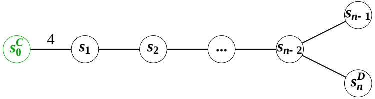
Type $\widetilde{C}/C$:



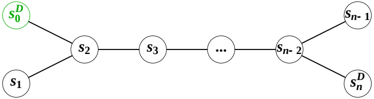
Type $\widetilde{B}/B$:



Type $\widetilde{B}/D$:



Type $\widetilde{D}/D$:



Window notation for $\widetilde{W}$

Write $\widetilde{w} \in \widetilde{W}$ as a mirrored permutation of $\mathbb{Z}$ with period $N = 2n + 1$.

- Satisfies $\widetilde{w}(i+N) = \widetilde{w}(i) + N$ and $\widetilde{w}(-i) = -\widetilde{w}(i)$.
- Define action of generators on $[\widetilde{w}(1), \widetilde{w}(2), \ldots, \widetilde{w}(n)]$; extend:
  - s_i : switch $\widetilde{w}(i) \leftrightarrow \widetilde{w}(i+1)$
  - s_0^C : switch $\widetilde{w}(-1) \leftrightarrow \widetilde{w}(1)$
  - s_n^C : switch $\widetilde{w}(n) \leftrightarrow \widetilde{w}(n+1)$

In $\widetilde{C}_4$	...	w(-4)	w(-3)	w(-2)	w(-1)	w(0)	w(1)	w(2)	w(3)	w(4)	w(5)	w(6)	w(7)	w(8)	w(9)	w(10)	...
s_1	...	-4	-3	-1	-2	0	2	1	3	4	5	6	8	7	9	11	...
s_0	...	-4	-3	-2	1	0	-1	2	3	4	5	6	7	10	9	8	...
s_4	...	-5	-3	-2	-1	0	1	2	3	5	4	6	7	8	9	10	...
$s_1 s_0$	...	-4	-3	-1	2	0	-2	1	3	4	5	6	8	11	9	7	...

$\widetilde{w}$ is defined by the window $[\widetilde{w}(1), \widetilde{w}(2), \ldots, \widetilde{w}(n)]$. $s_1 s_0 = [-2, 1, 3, 4]$

Window notation for $\widetilde{W}$

In type $\widetilde{B}$ (version 1) and type $\widetilde{D}$, the s_0 generator is:

- s_0^D : switch $\{\widetilde{w}(-2), \widetilde{w}(-1)\} \leftrightarrow \{\widetilde{w}(1), \widetilde{w}(2)\}$
- Therefore, $\big|\{i < 0 : w(i) > 0\}\big|$ is even.

In $\widetilde{D}_4$	...	w(-4)	w(-3)	w(-2)	w(-1)	w(0)	w(1)	w(2)	w(3)	w(4)	w(5)	w(6)	w(7)	w(8)	w(9)	w(10)	...
s_0^D	...	-4	-3	1	2	0	-2	-1	3	4	5	6	10	11	9	7	...
s_4^D	...	-6	-5	-2	-1	0	1	2	5	6	3	4	7	8	9	10	...

In type $\widetilde{B}$ (version 2) and type $\widetilde{D}$, the s_n generator is:

- s_n^D : switch $\{\widetilde{w}(n-1), \widetilde{w}(n)\} \leftrightarrow \{\widetilde{w}(n+1), \widetilde{w}(n+2)\}$
- Therefore, $\big|\{i \geq n+1 : w(i) \leq n\}\big|$ is even.

Window notation for $\widetilde{W}/W$

Theorem. Given an element $\widetilde{w} \in \widetilde{W}$ written as a mirrored permutation of $\mathbb{Z}$, we obtain its minimal length coset representative $w^0 \in \widetilde{W}/W$ by sorting the entries in the base window:

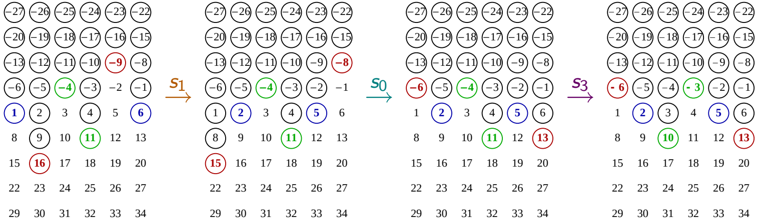
Type	Sorting conditions
$\widetilde{C}/C$	$w(1) < w(2) < \cdots < w(n) < w(n+1)$
$\widetilde{B}/B$	$w(1) < w(2) < \cdots < w(n) < w(n+1)$ Elements of $\widetilde{B}_n/B_n$ are elements of $\widetilde{C}_n/C_n$.
$\widetilde{B}/D$	$w(1) < w(2) < \cdots < w(n) < w(n+2)$ Elements of $\widetilde{B}_n/D_n$ are not necessarily elements of $\widetilde{C}_n/C_n$.
$\widetilde{D}/D$	$w(-2) < w(1) < w(2) < \cdots < w(n) < w(n+2)$ Elements of $\widetilde{D}_n/D_n$ are also elements of $\widetilde{B}_n/D_n$.

- It makes sense to define abaci for $\widetilde{W}/W$!

Abacus models for $\widetilde{W}/W$

(Hanusa and Jones, 2011) Given $w^0 = [w_1, \ldots, w_{2n}] \in \widetilde{W}/W$, create an abacus with $2n$ runners with lowest beads in positions w_i.

Example: $[-9, -4, 1, 6, 11, 16] \in \widetilde{C}_3/C_3$



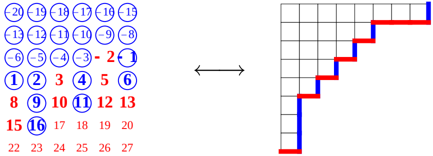
Again, generators interchange runners:

- s_i: $(i) \leftrightarrow (i+1)$ & $(2n-i) \leftrightarrow (2n-i+1)$. $(s_1 : 1 \leftrightarrow 2$ & $5 \leftrightarrow 6)$
- s_0^C: $1 \overset{\text{shift}}{\leftrightarrow} 2n$ $(s_0 : 1 \overset{\text{shift}}{\leftrightarrow} 6)$
- s_n^C: $n \leftrightarrow n+1$ $(s_3 : 3 \leftrightarrow 4)$
- s_0^D: $\{1, 2\} \overset{\text{shift}}{\leftrightarrow} \{2n-1, 2n\}$
- s_n^D: $\{n-1, n\} \leftrightarrow \{n+1, n+2\}$

Structure of abaci and cores in $\widetilde{W}/W$

In abaci:

- Symmetry: bead in position $i \leftrightarrow$ gap in position $2n+1-i$.
- If s_0^D: number of gaps $< 2n+1$ is even. (**even** abacus)



Under the bijection between abaci and core partitions,

- Symmetry: Abaci in $\widetilde{W}/W \leftrightarrow$ *Self-conjugate* $(2n)$-cores
- If s_0^D: even number of boxes on the main diagonal (**even** core)
- Know the action of generators on cores.

Residue Structure in $\widetilde{W}/W$

In $\widetilde{C}_n/C_n$, we have fixed residue structure.

The residues increase from 0 up to n and back down to 0:

0	1	2	3	2	1	0	1	2	3
1	0	1	2	3	2	1	0	1	2
2	1	0	1	2	3	2	1	0	1
3	2	1	0	1	2	3	2	1	0
2	3	2	1	0	1	2	3	2	1
1	2	3	2	1	0	1	2	3	2
0	1	2	3	2	1	0	1	2	3
1	0	1	2	3	2	1	0	1	2
2	1	0	1	2	3	2	1	0	1
3	2	1	0	1	2	3	2	1	0

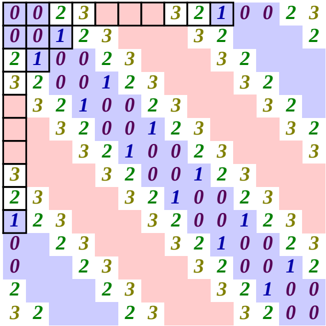
- s_i acts by adding or removing boxes with residue i.

Example: $[-9, -4, 1, 6, 11, 16] = s_1\, s_0\, s_3\, s_2\, s_1\, s_0\, s_2\, s_3\, s_2\, s_1\, s_0$.

Residue Structure in $\widetilde{W}/W$

In other types, residues involving $\{n-1, n\}$ and $\{0, 1\}$ depend on λ.

In type $\widetilde{D}/D$, there are both escalators and descalators.



- s_i adds or removes boxes with residue i (in contiguous groups).

Example in $\widetilde{D}_5/D_5$: $s_1\ s_2\ s_3\ s_4\ s_5\ s_3\ s_2\ s_0$.

Properties of abaci and cores for $\widetilde{W}/W$

Theorem. Minimal length coset representatives in $\widetilde{W}/W$ are in bijection with the following sets of abaci and self-conjugate partitions:

Theorem. The residues in the partitions have the following structure:

Type	Abaci	Partitions	Residues
$\widetilde{C}/C$	all abaci	all self-conj $(2n)$-cores	fixed
$\widetilde{B}/B$	even abaci	even self-conj $(2n)$-cores	fixed w/descalators
$\widetilde{B}/D$	all abaci	all self-conj $(2n)$-cores	fixed w/escalators
$\widetilde{D}/D$	even abaci	even self-conj $(2n)$-cores	fixed w/descalators and escalators

Canonical reduced expression for $\widetilde{W}/W$

Peel a core to obtain a *canonical reduced expression* for w^0.

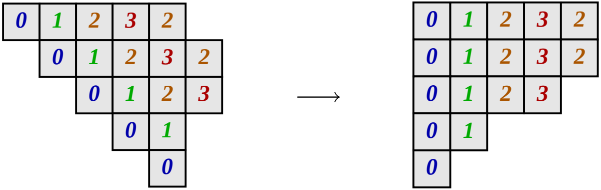
Remove boxes from the center and record the residues at each step.

0	1	2	3	2	1	0	1	2	3
1	0	1	2	3	2	1	0	1	2
2	1	0	1	2	3	2	1	0	1
3	2	1	0	1	2	3	2	1	0
2	3	2	1	0	1	2	3	2	1
1	2	3	2	1	0	1	2	3	2
0	1	2	3	2	1	0	1	2	3
1	0	1	2	3	2	1	0	1	2
2	1	0	1	2	3	2	1	0	1
3	2	1	0	1	2	3	2	1	0

$$\mathcal{R}(\lambda) = s_0\, s_1\, s_0\, s_3\, s_2\, s_1\, s_0\, s_2\, s_3\, s_2\, s_1\, s_0\, s_2\, s_3\, s_2\, s_1\, s_0$$

Bounded partitions for $\widetilde{W}/W$

Left-justifying this **upper diagram** gives a **bounded partition** (satisfying type-dependent conditions).

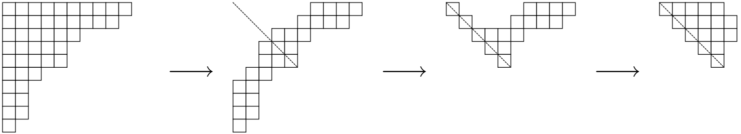


Bounded partitions:

- Encode a reduced expression for the element
- Have Coxeter length number of boxes
- Have appeared in crystal basis theory (as Young walls), in work of Eriksson–Eriksson, and in work of Billey–Mitchell (as affine colored partitions)

Lapointe–Morse-like bijection for bounded partitions

- Remove all boxes of λ with hook length $\geq 2n$
- Reinsert the boxes on the main diagonal, remove those below.
- Left-justify remaining boxes to diagonal.
- (When not $\widetilde{C}/C$, remove boxes on main and/or n-th diagonal.)
- Result: Upper diagram.



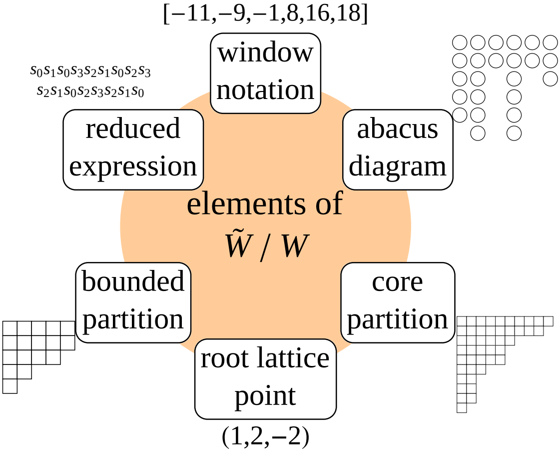
$\lambda = (10, 9, 6, 5, 5, 3, 2, 2, 2, 1)$ $\beta = (5, 5, 4, 2, 1)$

Conditions on bounded partitions

Theorem. We have a bijection of $\widetilde{W}/W$ with these bounded partitions:

Type	Bounded partition structure
$\widetilde{C}/C$	parts $\leq 2n$, where $1, \ldots, n$ occur at most once.
$\widetilde{B}/B$	parts $\leq 2n-1$, where $1, \ldots, n-1$ occur at most once.
$\widetilde{B}/D$	parts $\leq 2n-1$, where $1, \ldots, n-1$ occur at most once, and one n part may be starred.
$\widetilde{D}/D$	parts $\leq 2n-2$, where $1, \ldots, n-2$ occur at most once, and one $n-1$ part may be starred.

Combinatorial interpretations in $\widetilde{W}/W$



Future Work

- More combinatorial interpretations for $\widetilde{W}/W$
  - Learn more about the alcove model
  - What do you know?
- Fully commutative elements in types $\widetilde{B}$, $\widetilde{C}$, and $\widetilde{D}$
  - Investigation in $\widetilde{A}$ required combinatorial interpretations ✓
  - Find a 321-avoiding characterization?
- Self-conjugate core partitions
  - Related to $\widetilde{C}_n/C_n$.
  - Related to the alternating group.

Thank you!

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