

Combinatorial interpretations in affine Coxeter groups

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What is a Coxeter group?

A **Coxeter group** is a group with

- ▶ **Generators:** $\{s_1, s_2, \dots, s_n\}$
- ▶ **Relations:** $s_i^2 = 1$, $(s_i s_j)^{m_{ij}} = 1$ where $m_{ij} \geq 2$ or $= \infty$
 - ▶ $m_{ij} = 2$: $(s_i s_j)(s_i s_j) = 1 \implies s_i s_j = s_j s_i$ (they commute)
 - ▶ $m_{ij} = 3$: $(s_i s_j)(s_i s_j)(s_i s_j) = 1 \implies s_i s_j s_i = s_j s_i s_j$ (braid relation)
 - ▶ $m_{ij} = \infty$: s_i and s_j are not related.

Why Coxeter groups?

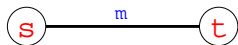
- ▶ They're awesome.
- ▶ **Discrete Geometry:** Symmetries of regular polyhedra.
- ▶ **Algebra:** Symmetric group generalizations. (Kac-Moody, Hecke)
- ▶ **Geometry:** Classification of Lie groups and Lie algebras

Examples of Coxeter groups

A shorthand notation is the **Coxeter graph**:

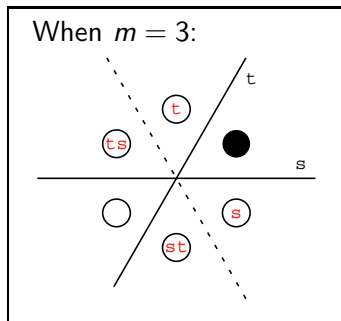
- ▶ **Vertices:** One for every generator i
- ▶ **Edges:** Create an edge between i and j when $m_{i,j} \geq 3$
Label edges with $m_{i,j}$ when ≥ 4 .

Dihedral group



- ▶ Generators: s, t .
- ▶ Relation: $(st)^m = 1$.

Symmetry group of regular m -gon.



Examples of Coxeter groups

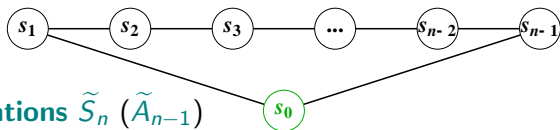
(Finite) n -Permutations S_n (A_{n-1})

- ▶ An n -permutation is a permutation of $\{1, 2, \dots, n\}$. (e.g. **214536**)
- ▶ Every n -permutation is a product of *adjacent transpositions*.
 - ▶ $s_i : (i) \leftrightarrow (i+1)$. (e.g. $s_4 = 123$ **546**).
 - ▶ *Example.* Write **214536** as $s_1 s_3 s_4$.

123	123
213	132
231	312
321	321

These are the generators. What about relations?

- ▶ Only consecutive generators don't commute: $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
- ▶ Visually:



Affine n -Permutations \tilde{S}_n (\tilde{A}_{n-1})

- ▶ Generators: $\{s_0, s_1, \dots, s_{n-1}\}$
- ▶ S_n is a parabolic subgroup of \tilde{S}_n

Minimal length coset representatives

For a Coxeter group \widetilde{W} generated by $\{s_0, s_1, \dots, s_n\}$,

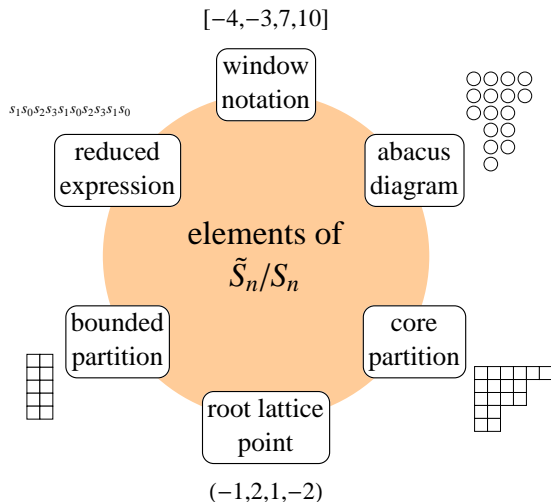
- ▶ An induced subgraph of \widetilde{W} 's Coxeter graph is a subgroup W
 - ▶ Today, we will always have W defined by $\widetilde{W} \setminus \{s_0\}$
- ▶ Every element $\tilde{w} \in \widetilde{W}$ can be written $\tilde{w} = w^0 w$, where $w^0 \in \widetilde{W}/W$ is a coset representative and $w \in W$.

Simple example: For $\tilde{w} = s_0 s_1 s_2 s_3 s_0 s_1 s_2 \in \tilde{S}_4$
 $\tilde{w} = s_0 s_1 s_2 s_3 s_0 s_1 s_2$

Less simple example: $\tilde{w} = s_1 s_3 s_2 s_3 s_0 s_1 \in \tilde{S}_4$
 $\tilde{w} = s_1 s_2 s_3 s_2 s_0 s_1$
 $\tilde{w} = s_1 s_2 s_3 s_0 s_2 s_1$

★ Combinatorial interpretations are easier to use. ★

Combinatorial interpretations of \tilde{S}_n/S_n



Window notation

Affine n -Permutations \tilde{S}_n (G. Lusztig 1983, H. Eriksson, 1994)

Write an element $\tilde{w} \in \tilde{S}_n$ in 1-line notation as a permutation of \mathbb{Z} .

Generators transpose **infinitely many** pairs of entries:

$s_i : (\mathbf{i}) \leftrightarrow (\mathbf{i+1}) \dots (n+i) \leftrightarrow (n+i+1) \dots (-n+i) \leftrightarrow (-n+i+1) \dots$

In \tilde{S}_4 ,	$\dots w(-4)$	$w(-3) w(-2) w(-1) w(0)$	$w(1) w(2) w(3) w(4)$	$w(5) w(6) w(7) w(8)$	$w(9) \dots$
s_1	$\dots -4$	$-2 -3 -1 0$	$2 1 3 4$	$6 5 7 8$	$10 \dots$
s_0	$\dots -3$	$-4 -2 -1 1$	$0 2 3 5$	$4 6 7 9$	$8 \dots$
$s_1 s_0$	$\dots -2$	$-4 -3 -1 2$	$0 1 3 6$	$4 5 7 10$	$8 \dots$

\tilde{w} is defined by the window $[\tilde{w}(1), \tilde{w}(2), \dots, \tilde{w}(n)]$. $s_1 s_0 = [0, 1, 3, 6]$

★ For $\tilde{w} = w^0 w$, the window of w^0 is the window of \tilde{w} , **sorted** ↗.

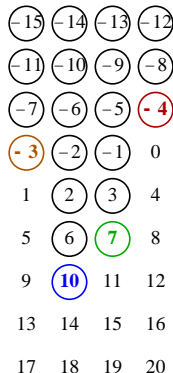
An abacus model for \tilde{S}_n/S_n

(James and Kerber, 1981) Given $w^0 = [w_1, \dots, w_n] \in \tilde{S}_n/S_n$,

- ▶ Place integers in n runners.
- ▶ Circled: *beads*. Empty: *gaps*
- ▶ **Bijection:** Given w^0 , create an abacus where each runner has a lowest bead at w_j .

Example: $[-4, -3, 7, 10]$

The generators act nicely on the abacus.

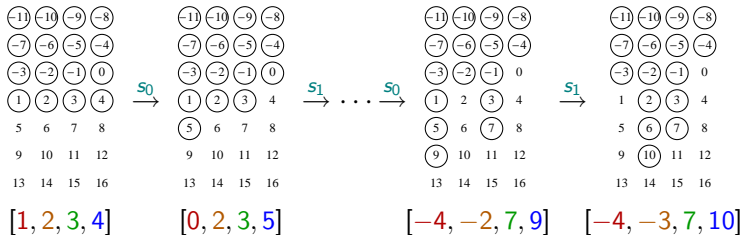


Action of generators on the abacus

- ▶ s_i acts by interchanging runners i and $i + 1$.
- ▶ s_0 acts by interchanging runners 1 and n , with level shifts.

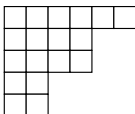
Example: Consider $[-4, -3, 7, 10] = s_1 s_0 s_2 s_1 s_3 s_2 s_0 s_3 s_1 s_0$.

Start with $\text{id} = [1, 2, 3, 4]$ and apply the generators one by one:

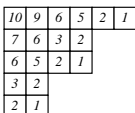


Integer partitions and n -core partitions

For an integer partition $\lambda = (\lambda_1, \dots, \lambda_k)$ drawn as a Ferrers diagram,



The *hook length* of a box is # boxes below and to the right.



An n -core is a partition with no boxes of hook length dividing n .

Example. λ is a 4-core, 8-core, 11-core, 12-core, etc.

λ is NOT a 1-, 2-, 3-, 5-, 6-, 7-, 9-, or 10-core.

Core partitions for \tilde{S}_n/S_n

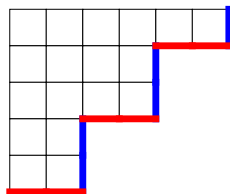
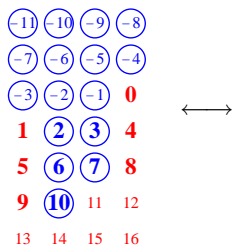
Elements of \tilde{S}_n/S_n are in bijection with n -cores.

Bijection: $\{\text{abaci}\} \longleftrightarrow \{n\text{-cores}\}$

Rule: Read the boundary steps of λ from the abacus:

► A bead \leftrightarrow vertical step

► A gap \leftrightarrow horizontal step



Fact: Abacus with n -runners \leftrightarrow partition is n -core.

Action of generators on the core partition

- ▶ Label the boxes of λ with residues.
- ▶ s_i acts by adding or removing boxes with residue i .

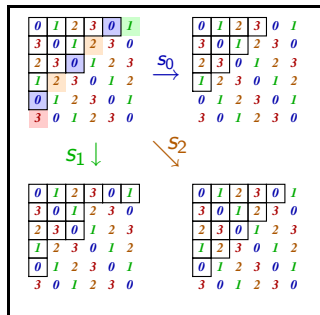
0	1	2	3	0	1
3	0	1	2	3	0
2	3	0	1	2	3
1	2	3	0	1	2
0	1	2	3	0	1
3	0	1	2	3	0

Example. $\lambda = (5, 3, 3, 1, 1)$

- ▶ has removable 0 boxes (s_0 is a descent)
- ▶ has addable 1, 2, 3 boxes (s_1, s_2, s_3 are ascents)

Idea: Use to determine a canonical reduced expression for w^0 .

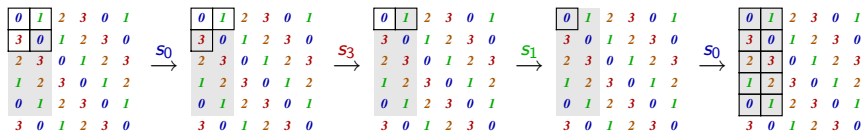
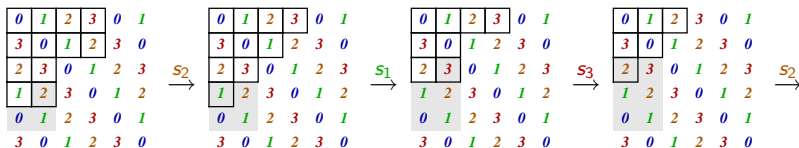
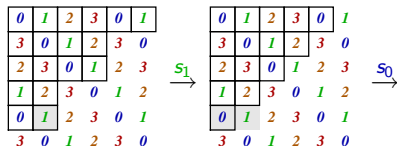
- ▶ Tally residues from bottom to top, from right to left.



Canonical reduced expression for \tilde{S}_n/S_n

Example: Reduced expression corresponding to $\lambda = (6, 4, 4, 2, 2)$:

$$\mathcal{R}(\lambda) = s_1 s_0 s_2 s_1 s_3 s_2 s_0 s_3 s_1 s_0$$



Bounded partitions for \tilde{S}_n/S_n

A partition $\beta = (\beta_1, \dots, \beta_k)$ is *b-bounded* if $\beta_i \leq b$ for all i .

Elements of \tilde{S}_n/S_n are in bijection with $(n-1)$ -bounded partitions.

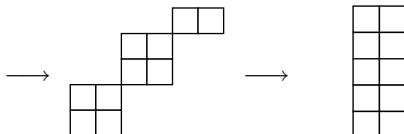
Bijection: (Lapointe, Morse, 2005)

$$\{n\text{-cores } \lambda\} \leftrightarrow \{(n-1)\text{-bounded partitions } \beta\}$$

- ▶ Remove all boxes of λ with hook length $\geq n$
- ▶ Left-justify remaining boxes.

10	9	6	5	2	1
7	6	3	2		
6	5	2	1		
3	2				
2	1				

$$\lambda = (6, 4, 4, 2, 2)$$



$$\beta = (2, 2, 2, 2, 2)$$

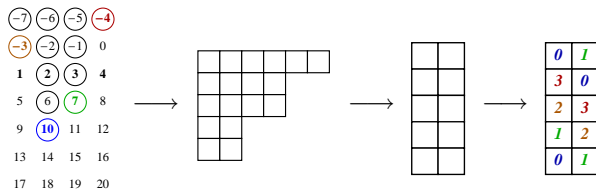
Canonical reduced expression for \tilde{S}_n/S_n

Given the bounded partition, read off the reduced expression:

Method: (Berg, Jones, Vazirani, 2009)

- ▶ Fill β with residues i
- ▶ Tally s_i reading right-to-left in rows from bottom-to-top

Example. $[-4, -3, 7, 10] = s_1 s_0 s_2 s_1 s_3 s_2 s_0 s_3 s_1 s_0.$



- ▶ The Coxeter length of w^0 is the number of boxes in β .

Summary for \tilde{S}_n/S_n

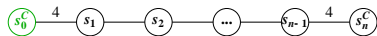
- ▶ See S_n as parabolic subgroup of \tilde{S}_n
- ▶ **Window notation**
 - ▶ \tilde{S}_n elements can be written as a permutation of \mathbb{Z}
 - ▶ Min. len. coset rep's correspond to **sorted** \mathbb{Z} -permutations.
- ▶ **Abacus models**
 - ▶ Define the abacus by reading the entries from a window
 - ▶ Generators act nicely: They interchange runners
- ▶ **Core partitions**
 - ▶ Define the core by reading beads and gaps
 - ▶ Generators act nicely: They add and remove boxes using residues
- ▶ **Bounded partitions**
 - ▶ Define by collapsing the core partition
 - ▶ Reading the residues gives a reduced expression!

Summary for \widetilde{W}/W

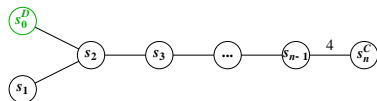
- ▶ See W as parabolic subgroup of \widetilde{W}
- ▶ **Window notation**
 - ▶ \widetilde{W} elements can be written as a permutation of \mathbb{Z}
 - ▶ Min. len. coset rep's correspond to **sorted** \mathbb{Z} -permutations.
- ▶ **Abacus models**
 - ▶ Define the abacus by reading the entries from a window
 - ▶ Generators act nicely: They interchange runners
- ▶ **Core partitions**
 - ▶ Define the core by reading beads and gaps
 - ▶ Generators act nicely: They add and remove boxes using residues
- ▶ **Bounded partitions**
 - ▶ Define by collapsing the core partition
 - ▶ Reading the residues gives a reduced expression!

Coxeter Graphs for Types \tilde{B} , \tilde{C} , \tilde{D}

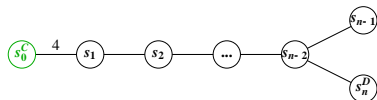
Type \tilde{C}/C :



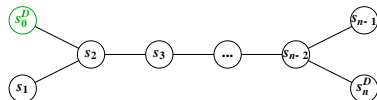
Type \tilde{B}/B :



Type \tilde{B}/D :



Type \tilde{D}/D :



Window notation for \widetilde{W}

Write $\widetilde{w} \in \widetilde{W}$ as a **mirrored permutation of \mathbb{Z}** with period $N = 2n + 1$.

- ▶ Satisfies $\widetilde{w}(i + N) = \widetilde{w}(i) + N$ and $\widetilde{w}(-i) = -\widetilde{w}(i)$.
- ▶ Define action of generators on $[\widetilde{w}(1), \widetilde{w}(2), \dots, \widetilde{w}(n)]$; extend:
 - ▶ s_i : switch $\widetilde{w}(i) \leftrightarrow \widetilde{w}(i + 1)$
 - ▶ s_0^C : switch $\widetilde{w}(-1) \leftrightarrow \widetilde{w}(1)$ ▶ s_n^C : switch $\widetilde{w}(n) \leftrightarrow \widetilde{w}(n + 1)$

In \widetilde{C}_4	...	$w(-4)$	$w(-3)$	$w(-2)$	$w(-1)$	$w(0)$	$w(1)$	$w(2)$	$w(3)$	$w(4)$	$w(5)$	$w(6)$	$w(7)$	$w(8)$	$w(9)$	$w(10)$...
s_1	...	-4	-3	-1	-2	0	2	1	3	4	5	6	8	7	9	11	...
s_0	...	-4	-3	-2	1	0	-1	2	3	4	5	6	7	10	9	8	...
s_4	...	-5	-3	-2	-1	0	1	2	3	5	4	6	7	8	9	10	...
$s_1 s_0$...	-4	-3	-1	2	0	-2	1	3	4	5	6	8	11	9	7	...

\widetilde{w} is defined by the **window** $[\widetilde{w}(1), \widetilde{w}(2), \dots, \widetilde{w}(n)]$. $s_1 s_0 = [-2, 1, 3, 4]$

Window notation for \widetilde{W}

In type \widetilde{B} (version 1) and type \widetilde{D} , the s_0 generator is:

- ▶ s_0^D : switch $\{\widetilde{w}(-2), \widetilde{w}(-1)\} \leftrightarrow \{\widetilde{w}(1), \widetilde{w}(2)\}$
- ▶ Therefore, $|\{i < 0 : w(i) > 0\}|$ is even.

In \widetilde{D}_4	...	$w(-4)$	$w(-3)$	$w(-2)$	$w(-1)$	$w(0)$	$w(1)$	$w(2)$	$w(3)$	$w(4)$	$w(5)$	$w(6)$	$w(7)$	$w(8)$	$w(9)$	$w(10)$...
s_0^D	...	-4	-3	1	2	0	-2	-1	3	4	5	6	10	11	9	7	...
s_4^D	...	-6	-5	-2	-1	0	1	2	5	6	3	4	7	8	9	10	...

In type \widetilde{B} (version 2) and type \widetilde{D} , the s_n generator is:

- ▶ s_n^D : switch $\{\widetilde{w}(n-1), \widetilde{w}(n)\} \leftrightarrow \{\widetilde{w}(n+1), \widetilde{w}(n+2)\}$
- ▶ Therefore, $|\{i \geq n+1 : w(i) \leq n\}|$ is even.

Window notation for \widetilde{W}/W

Theorem. Given an element $\tilde{w} \in \widetilde{W}$ written as a mirrored permutation of \mathbb{Z} , we obtain its minimal length coset representative $w^0 \in \widetilde{W}/W$ by sorting the entries in the base window:

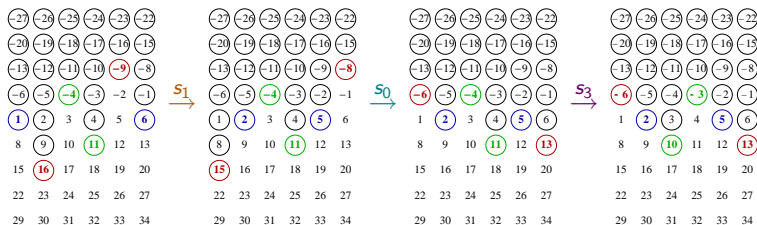
Type	Sorting conditions
\widetilde{C}/C	$w(1) < w(2) < \dots < w(n) < w(n+1)$
\widetilde{B}/B	$w(1) < w(2) < \dots < w(n) < w(n+1)$ Elements of \widetilde{B}_n/B_n are elements of \widetilde{C}_n/C_n .
\widetilde{B}/D	$w(1) < w(2) < \dots < w(n) < w(n+2)$ Elements of \widetilde{B}_n/D_n are not necessarily elements of \widetilde{C}_n/C_n .
\widetilde{D}/D	$w(1) < w(2) < \dots < w(n) < w(n+2)$ Elements of \widetilde{D}_n/D_n are also elements of \widetilde{B}_n/D_n .

- It makes sense to define abaci for \widetilde{W}/W !

Abacus models for \widetilde{W}/W

(Hanusa and Jones, 2011) Given $w^0 = [w_1, \dots, w_{2n}] \in \widetilde{W}/W$, create an abacus with $2n$ runners with lowest beads in positions w_i .

Example: $[-9, -4, 1, 6, 11, 16] \in \widetilde{C}_3/C_3$



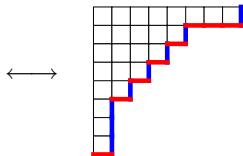
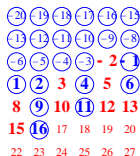
Again, generators interchange runners:

- ▶ $s_i: (i) \leftrightarrow (i+1) \ \& \ (2n-i) \leftrightarrow (2n-i+1)$. ($s_1: 1 \leftrightarrow 2 \ \& \ 5 \leftrightarrow 6$)
- ▶ $s_0^C: 1 \overset{\text{shift}}{\leftrightarrow} 2n$ ($s_0: 1 \overset{\text{shift}}{\leftrightarrow} 6$)
- ▶ $s_0^D: \{1, 2\} \overset{\text{shift}}{\leftrightarrow} \{2n-1, 2n\}$
- ▶ $s_n^C: n \leftrightarrow n+1$ ($s_3: 3 \leftrightarrow 4$)
- ▶ $s_n^D: \{n-1, n\} \leftrightarrow \{n+1, n+2\}$

Structure of abaci and cores in \widetilde{W}/W

In abaci:

- ▶ **Symmetry:** bead in position $i \leftrightarrow$ gap in position $2n + 1 - i$.
- ▶ If s_0^D : number of gaps $< 2n + 1$ is even. (**even** abacus)



Under the bijection between abaci and core partitions,

- ▶ **Symmetry:** Abaci in $\widetilde{W}/W \leftrightarrow$ *Self-conjugate* $(2n)$ -cores
- ▶ If s_0^D : even number of boxes on the main diagonal (**even** core)
- ▶ Know the action of generators on cores.

Residue Structure in \widetilde{W}/W

In \widetilde{C}_n/C_n , we have fixed residue structure.

The residues increase from 0 up to n and back down to 0 :

0	1	2	3	2	1	0	1	2	3
1	0	1	2	3	2	1	0	1	2
2	1	0	1	2	3	2	1	0	1
3	2	1	0	1	2	3	2	1	0
2	3	2	1	0	1	2	3	2	1
1	2	3	2	1	0	1	2	3	2
0	1	2	3	2	1	0	1	2	3
1	0	1	2	3	2	1	0	1	2
2	1	0	1	2	3	2	1	0	1
3	2	1	0	1	2	3	2	1	0

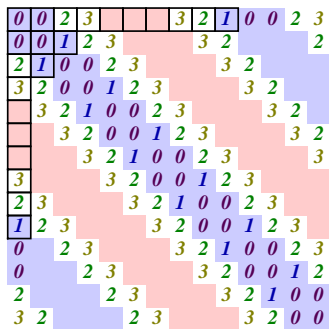
► s_i acts by adding or removing boxes with residue i .

Example: $[-9, -4, 1, 6, 11, 16] = s_1 s_0 s_3 s_2 s_1 s_0 s_2 s_3 s_2 s_1 s_0$.

Residue Structure in \widetilde{W}/W

In other types, residues involving $\{n-1, n\}$ and $\{0, 1\}$ depend on λ .

In type \widetilde{D}/D , there are both **escalators** and **descalators**.



- s_i adds or removes boxes with residue i (in contiguous groups).

Example in \widetilde{D}_5/D_5 : $s_1 s_2 s_3 s_4 s_5 s_3 s_2 s_0$.

Properties of abaci and cores for \widetilde{W}/W

Theorem. Minimal length coset representatives in \widetilde{W}/W are in bijection with the following sets of abaci and self-conjugate partitions:

Theorem. The residues in the partitions have the following structure:

Type	Abaci	Partitions	Residues
\widetilde{C}/C	all abaci	all self-conj $(2n)$ -cores	fixed
\widetilde{B}/B	even abaci	even self-conj $(2n)$ -cores	fixed w/descalators
\widetilde{B}/D	all abaci	all self-conj $(2n)$ -cores	fixed w/escalators
\widetilde{D}/D	even abaci	even self-conj $(2n)$ -cores	fixed w/descalators and escalators

Canonical reduced expression for \widetilde{W}/W

Peel a core to obtain a *canonical reduced expression* for w^0 .

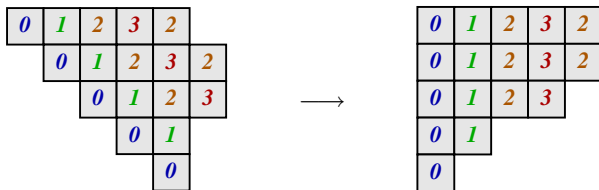
Remove boxes from the center and record the residues at each step.

0	1	2	3	2	1	0	1	2	3
1	0	1	2	3	2	1	0	1	2
2	1	0	1	2	3	2	1	0	1
3	2	1	0	1	2	3	2	1	0
2	3	2	1	0	1	2	3	2	1
1	2	3	2	1	0	1	2	3	2
0	1	2	3	2	1	0	1	2	3
1	0	1	2	3	2	1	0	1	2
2	1	0	1	2	3	2	1	0	1
3	2	1	0	1	2	3	2	1	0

$$\mathcal{R}(\lambda) = s_0 s_1 s_0 s_3 s_2 s_1 s_0 s_2 s_3 s_2 s_1 s_0 s_2 s_3 s_2 s_1 s_0$$

Bounded partitions for \widetilde{W}/W

Left-justifying this **upper diagram** gives a **bounded partition** (satisfying type-dependent conditions).

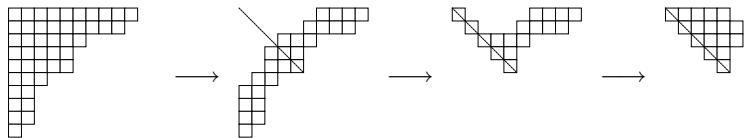


Bounded partitions:

- ▶ Encode a reduced expression for the element
- ▶ Have Coxeter length number of boxes
- ▶ Have appeared in crystal basis theory (as Young walls), in work of Eriksson–Eriksson, and in work of Billey–Mitchell (as affine colored partitions)

Lapointe–Morse-like bijection for bounded partitions

- ▶ Remove all boxes of λ with hook length $\geq 2n$
- ▶ Reinsert the boxes on the main diagonal, remove those below.
- ▶ Left-justify remaining boxes to diagonal.
- ▶ (When not \tilde{C}/C , remove boxes on main and/or n -th diagonal.)
- ▶ **Result:** Upper diagram.



$$\lambda = (10, 9, 6, 5, 5, 3, 2, 2, 2, 1)$$

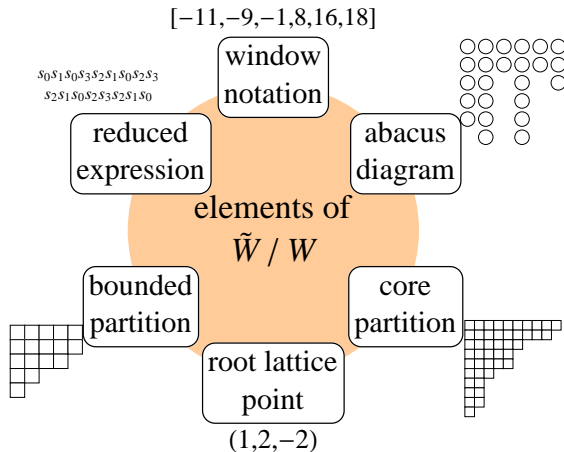
$$\beta = (5, 5, 4, 2, 1)$$

Conditions on bounded partitions

Theorem. We have a bijection of \widetilde{W}/W with these bounded partitions:

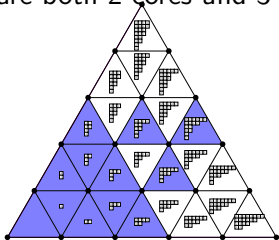
Type	Bounded partition structure
\widetilde{C}/C	parts $\leq 2n$, where $1, \dots, n$ occur at most once.
\widetilde{B}/B	parts $\leq 2n - 1$, where $1, \dots, n - 1$ occur at most once.
\widetilde{B}/D	parts $\leq 2n - 1$, where $1, \dots, n - 1$ occur at most once, and one n part may be starred.
\widetilde{D}/D	parts $\leq 2n - 2$, where $1, \dots, n - 2$ occur at most once, and one $n - 1$ part may be starred.

Combinatorial interpretations in \tilde{W}/W



Simultaneous core partitions

How many partitions are both 2-cores and 3-cores? **2.**



How many partitions are both 3-cores and 4-cores? **5.**

How many simultaneous 4/5-cores? **14.**

How many simultaneous 5/6-cores? **42.**

How many simultaneous $n/(n+1)$ -cores? $C_n!$

Jaclyn Anderson proved that the number of s/t -cores is $\frac{1}{s+t} \binom{s+t}{s}$.

The number of 3/7-cores is $\frac{1}{10} \binom{10}{3} = \frac{1}{10} \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 12$.

Fishel–Vazirani proved an alcove interpretation of $n/(mn+1)$ -cores.

Future Work

- ▶ More (!) combinatorial interpretations for \widetilde{W}/W
 - ▶ Alcoves, lattice paths, order filters, ...
 - ▶ What do you know?
- ▶ Properties of core partitions
 - ▶ Simultaneous core partitions & Catalan combinatorics
 - ▶ Numerical properties of self-conjugate core partitions
- ▶ Fully commutative elements in types \widetilde{B} , \widetilde{C} , and \widetilde{D}
 - ▶ Investigation in \widetilde{A} required combinatorial interpretations ✓
 - ▶ Find a 321-avoiding characterization?

Thank you!

Slides available: people.qc.cuny.edu/chanusa > Talks

Interact: people.qc.cuny.edu/chanusa > Animations



Christopher R. H. Hanusa and Brant C. Jones.

Abacus models for parabolic quotients of affine Coxeter groups

To appear, *Journal of Algebra*. arXiv:1105.5333



Christopher R. H. Hanusa and Rishi Nath.

The number of self-conjugate core partitions

Submitted, 2012. arXiv:1201.6629



Christopher R. H. Hanusa and Brant C. Jones.

The enumeration of fully commutative affine permutations

European Journal of Combinatorics. Vol 31, 1342–1359. (2010)