

# TREACHERY! WHEN FAIRY CHESS PIECES ATTACK

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ABSTRACT. We introduce a new one-dimensional discrete dynamical system reminiscent of mathematical billiards that arises in the study of two-move riders, a type of fairy chess piece. In this model, particles travel through a bounded convex region along line segments of one of two fixed slopes.

This dynamical system applies to characterize the vertices of the inside-out polytope arising from counting placements of nonattacking chess pieces and also to give a bound for the period of the counting quasipolynomial. The analysis focuses on points of the region that are on trajectories that contain a corner or on cycles of full rank, or are crossing points thereof.

As a consequence, we give a simple proof that the period of the bishops' counting quasipolynomial is 2, and provide formulas bounding periods of counting quasipolynomials for many two-move riders including all partial nightriders. We draw parallels to the theory of mathematical billiards and pose many new open questions.

## 1. INTRODUCTION

The classic  $n$ -Queens Problem asks in how many ways  $n$  nonattacking queens can be placed on an  $n \times n$  chessboard. In a series of six papers [6, 7, 8, 9, 10, 11], Chaiken, Hanusa, and Zaslavsky develop a geometric approach involving lattice point counting to answer a generalization when the board is made up of integer lattice points on the interior of an  $n$ -dilation of a convex polygon  $\mathcal{B}$ , pieces  $\mathbb{P}$  are riders (which means they can travel arbitrarily far in a move's direction like a queen, bishop, or the fairy nightrider), and the number of pieces  $q$  is decoupled from the size of the board. Their main structural result (Theorem 4.1 of [6]) is that the number of nonattacking configurations of  $q$   $\mathbb{P}$ -pieces on the  $(n + 1)$ -dilation of  $\mathcal{B}^\circ$  is always a quasipolynomial in  $n$  of degree  $2q$ .

In this paper we investigate the period of this counting quasipolynomial when the pieces have exactly two moves, on any board and for any number of pieces. (Pieces with only one move are completely understood while pieces with three or more moves are much more complex, as discussed in [9].) We learn that this period is determined by the behavior of a new one-dimensional discrete dynamical system which we present and whose properties we investigate. This discrete dynamical system is similar to that of mathematical billiard theory in that particles travel across a region along line segments and “bounce” when they hit the region's boundary. However, instead of obeying the law of reflection, the line segments have one of two slopes determined by the moves of the fairy chess piece. Compare the diagrams in Figure 1.

The study of mathematical billiards has been a fruitful area of research for over a hundred years; some early papers were written by Artin [1] and Birkhoff [5]. The work of Sinai [18] stimulated interest in the ergodic theory and chaos of billiards, and the connections to geometry, statistical physics, and Teichmüller theory give billiards a wide appeal. We recommend the surveys by Tabachnikov, Masur, and Gutkin [19, 16, 14, 15].

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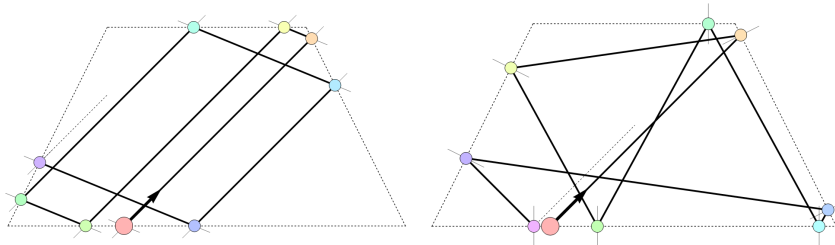


FIGURE 1. A comparison of the behavior of two discrete dynamical systems in a convex region. On the left is our new discrete dynamical system in which the particle bounces off a wall in directions that alternate between the moves of a fairy chess piece. On the right is the classical discrete dynamical system from mathematical billiards in which the particle bounces off a wall by obeying the law of reflection.

We develop our dynamical system for general convex regions. There appear to be some parallels between billiards and our dynamical system, which leads to a number of open questions motivated by our study and by the billiards literature. For example, the particle flows can be periodic, can converge to a limit set, or exhibit ergodicity, and it is not clear when each property occurs. (See Section 7.2.) Furthermore, when we apply our discrete dynamical system to the convex polygons from the  $q$ -Queens Problem, we must explicitly calculate the crossing points of flows; investigating crossing points in the context of billiard theory may lead to further insights there.

Counting lattice points in polytopes is the subject of a field of mathematics named Ehrhart theory after the work of Eugène Ehrhart [13]. Ehrhart theory has found applications in integer programming, number theory, and algebra, among others [12, 2, 17]; for more background, see the accessible works by De Loera [12] and Beck and Robins [3]. Beck and Zaslavsky [4] count lattice points in a polytope that avoid an arrangement of hyperplanes; the  $q$ -Queens Problem was converted into a counting question in such an *inside-out polytope*. Ehrhart theory tells us that the period of the counting quasipolynomial always divides the denominator of the inside-out polytope—the least common multiple of the denominators of its vertices.

Theorem 5.9 characterizes the vertices of the inside-out polytope for two-move riders as points on flows (trajectories) in our new dynamical system. Vertices either involve trajectories that include corners of the board or cyclical trajectories whose system of defining equations is linearly independent (rigid cycles) or interior crossing points of these trajectories. This characterization allows us to prove a formula for the denominator of the counting quasipolynomial for the number of nonattacking chess piece configurations in Theorem 5.10.

When we analyzed the trajectories to calculate bounds on periods of the counting quasipolynomials we saw some striking behavior. Section 6.1 highlights a case where there are no rigid cycles and the corner trajectories are well behaved. Section 6.2 discusses a case where there is one rigid cycle that serves as an attractor to all other trajectories. In Section 6.3, our dynamical system reduces to that of billiards. In Section 7.2 we show an example where the trajectories appear to behave chaotically.

One of the motivations of this work was to better understand nightriders, riders that move like the knight along slopes of  $\pm 2$  and  $\pm \frac{1}{2}$ , whose behavior was investigated in [10]. The authors suggested that partial nightriders—two-move riders with a subset of the nightrider’s moves—would be fruitful pieces to investigate. Indeed, in Section 6 we are able to determine denominators (and therefore bounds on the period of the counting quasipolynomial) of all two-move partial nightriders. Our work also gives a simple new proof that the period of the counting quasipolynomial for  $q \geq 3$  bishops is 2, avoiding the need to use signed graph theory which was present in the original proof given in [11].

We now share a brief summary of our paper. We recall the necessary background information from the theory of chess piece configurations in Section 2 and explore hyperplanes and rank in Section 3. Section 4 defines the new discrete dynamical system and concepts related to trajectories. In Section 5, we apply the dynamical system to polygonal boards which allows us to characterize vertices of the inside-out polytope in Theorem 5.9 and prove the formula for its denominator in Theorem 5.10. We then restrict to the square board to find explicit formulas for the coordinates of points on trajectories and crossing points in Section 6. We conclude with a wide variety of open problems in Section 7, asking questions about future regions of study, properties of trajectories, generalizations of our dynamical system, among others.

## 2. BACKGROUND

We gather here the necessary Ehrhart and nonattacking chess piece theory background information and notation from [6, 7, 9]. Every  $q$ -Queens Problem involves three parameters, a board  $\mathcal{B}$ , a piece  $\mathbb{P}$ , and a positive integral number of pieces  $q$ .

Our *board*  $\mathcal{B}$  is a convex polygon whose corners have rational coordinates; we use the notation  $\mathcal{B}^\circ$  and  $\partial\mathcal{B}$  for its interior and boundary, respectively. (This is not to be confused with rational polygons, defined in billiard theory whose angles are rational multiples of  $\pi$ .) These boards are dilated by an integer factor of  $(n+1)$ ; pieces are placed on integer lattice points in  $(n+1)\mathcal{B}^\circ \cap \mathbb{Z}^2$ . The *square board* refers to  $\mathcal{B} = [0, 1]^2$ .

A *piece*  $\mathbb{P}$  has a set  $\mathbf{M}$  of non-parallel *basic moves*  $\mathbf{m} = (c, d)$  where  $c$  and  $d$  are relatively prime integers; a piece at position  $(x, y)$  may move to any position  $(x, y) + k\mathbf{m}$  for  $k \in \mathbb{Z}$  and  $\mathbf{m} \in \mathbf{M}$ . (This ability to move arbitrarily far along a basic move is the defining property of a *rider*.) For example, the *bishop* is the piece with basic moves  $(1, 1)$  and  $(-1, 1)$ , while the fairy chess *nightrider* is the rider with the basic moves  $(1, \pm 2)$  and  $(2, \pm 1)$  of the knight.

In this article we consider pieces that are *two-move riders* with basic moves  $\mathbf{m}_1 = (c_1, d_1)$  and  $\mathbf{m}_2 = (c_2, d_2)$ . Three pieces that were proposed in [10] and which motivated our study are the *partial nightriders*: the *lateral nightrider* moves along lines of slope  $\pm 1/2$ , the *inclined nightrider* moves along lines of slope  $1/2$  and  $2$ , and the *orthonightrider* moves along lines of slope  $1/2$  and  $-2$ .

Two pieces are said to *attack* if their positions differ by a multiple of a move. A *configuration* of  $q$  pieces corresponds to an integral point  $\mathbf{z} = (\mathbf{z}_1, \dots, \mathbf{z}_q) \in ((n+1)\mathcal{B})^q \subseteq \mathbb{R}^{2q}$  and is said to be *nonattacking* if no two pieces are attacking. Mathematically, a configuration is nonattacking if it avoids the *hyperplane arrangement*  $\mathcal{A}_{\mathbb{P}}^q$  consisting of all *attack equations of type*  $r$ ,

$$(2.1) \quad (\mathbf{z}_i - \mathbf{z}_j) \cdot (d_r, -c_r) = 0,$$

for  $1 \leq i < j \leq q$  and  $r = 1, 2$ ; we adopt the shorthand notation  $\mathbf{z}_i \sim_r \mathbf{z}_j$  for Equation (2.1). Note that  $\sim_r$  is an equivalence relation.

This construction from [6] converts the question of counting the number of nonattacking configurations of  $q$   $\mathbb{P}$ -pieces on  $(n+1)\mathcal{B}^\circ$ , denoted  $u_{\mathbb{P}}(q; n)$ , into a lattice point counting question in this *inside-out polytope*, denoted  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$ . The boundary equations of  $\mathcal{B}$  are avoided as well, which justifies counting configurations in  $((n+1)\mathcal{B})^q \cap \mathbb{Z}^{2q}$  instead of  $((n+1)\mathcal{B}^\circ)^q \cap \mathbb{Z}^{2q}$ .

A *vertex* of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  is any point of  $\mathcal{B}^q$  that is the intersection of attack equations from  $\mathcal{A}_{\mathbb{P}}^q$  and *fixation equations* (or simply *fixations*) of the form

$$(2.2) \quad (\alpha_1, \alpha_2) \cdot \mathbf{z}_i = \beta,$$

where  $\alpha_1 x + \alpha_2 y = \beta$  is the equation of a side of  $\mathcal{B}$ . The *denominator*  $\Delta(\mathbf{z})$  of a vertex  $\mathbf{z}$  is the least common multiple of the denominators of its coordinates, and the *denominator*  $D(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  of an inside-out polytope is the least common multiple of the denominators of all its vertices. In Theorem 5.9 we determine the structure of all vertices of the inside-out polytope for an arbitrary board  $\mathcal{B}$  and a two-move rider  $\mathbb{P}$ .

As with many counting questions in Ehrhart Theory, the main structural result of [6] is that  $u_{\mathbb{P}}(q; n)$  is always a quasipolynomial in  $n$  of degree  $2q$ . That is, for each fixed  $q$ ,  $u_{\mathbb{P}}(q; n)$  is given by a cyclically repeating sequence of polynomials in  $n$  and its *period*  $p$  is the shortest length of such a cycle. The period of the counting quasipolynomial  $u_{\mathbb{P}}(q; n)$  always divides the denominator  $D(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  [3, Theorem 3.23]. In Ehrhart Theory the period is often difficult to obtain and much smaller than this denominator, but a surprising occurrence in chess counting problems is that the period and denominator always seem to agree, leading to the following conjecture.

**Conjecture 2.1** ([7, Conjecture 8.6]). The period of the counting quasipolynomial  $u_{\mathbb{P}}(q; n)$  equals the denominator  $D([0, 1]^{2q}, \mathcal{A}_{\mathbb{P}}^q)$ .

### 3. HYPERPLANES AND RANK.

We define the following concepts related to the geometry of the inside-out polytope.

**Definition 3.1.** For  $\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_k) \in \mathcal{B}^q$  we define  $\mathcal{H}(\mathbf{z})$ , the *hyperplane arrangement associated to  $\mathbf{z}$* , to be the set of all attack equations and fixations on which  $\mathbf{z}$  lies.

In other words,  $\mathcal{H}(\mathbf{z})$  will include the attack equation  $\mathbf{z}_i \sim_r \mathbf{z}_j$  if pieces  $i$  and  $j$  attack and will include the fixation  $(\alpha_1, \alpha_2) \cdot \mathbf{z}_i = \beta$  if and only if  $\mathbf{z}_i$  lies on the edge of  $\mathcal{B}$  defined by  $\alpha_1 x + \alpha_2 y = \beta$ .

The rank of hyperplane arrangements, equations, and sets of points will help determine when  $\mathbf{z} \in \mathcal{B}^q$  is a vertex of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$ .

**Definition 3.2.** The *rank* of a hyperplane arrangement  $\mathcal{H}$  in  $\mathbb{R}^d$  is the rank of the system of equations given by its hyperplanes.  $\mathcal{H}$  has *full rank* if it has rank  $d$ . We say the *rank* of a point  $\mathbf{z} \in \mathbb{R}^{2q}$  is the rank of  $\mathcal{H}(\mathbf{z})$ , and  $\mathbf{z}$  has *full rank* if  $\mathcal{H}(\mathbf{z})$  has full rank. We say the *rank* of a set  $S = \{\mathbf{z}_1, \dots, \mathbf{z}_k\} \subseteq \mathbb{R}^2$  is the rank of the point  $\mathbf{z} = (\mathbf{z}_1, \dots, \mathbf{z}_k)$ , and  $S$  has *full rank* if  $\mathbf{z}$  has full rank.

**Definition 3.3.** A set  $\mathcal{H}$  of hyperplanes in  $\mathbb{R}^d$  is said to be *linearly independent* if the rank of  $\mathcal{H}$  is equal to its size, or equivalently, if the set of normal vectors to these hyperplanes is linearly independent.

**Lemma 3.4.**  $\mathbf{z} \in \mathcal{B}^q$  has full rank if and only if  $\mathbf{z}$  is a vertex of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$ .

*Proof.* Suppose  $\mathbf{z}$  (and therefore  $\mathcal{H}(\mathbf{z})$ ) has full rank. By removing redundant hyperplanes,  $\mathcal{H}(\mathbf{z})$  can be reduced to a linearly independent set of hyperplanes  $\mathcal{H}$  of full rank of which  $\mathbf{z}$  is the intersection point, so  $\mathbf{z}$  is a vertex of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$ . If  $\mathbf{z}$  is a vertex,  $\mathcal{H}(\mathbf{z})$  contains this  $\mathcal{H}$ , so  $\mathcal{H}(\mathbf{z})$  (and therefore  $\mathbf{z}$ ) has full rank.  $\square$

**Example 3.5.** Consider the orthonightrider on the square board with moves  $\mathbf{m}_1 = (1, 2)$  and  $\mathbf{m}_2 = (2, -1)$ .

When  $\mathbf{z} = (0, 0, 1, 1/2)$ ,  $\mathcal{H}(\mathbf{z})$  contains the fixations  $x_1 = 0$ ,  $y_1 = 0$ , and  $x_2 = 1$  and the attack equation  $\mathbf{z}_1 \sim_1 \mathbf{z}_2$ . These four equations form a system of full rank; we conclude  $\mathcal{H}(\mathbf{z})$  and  $\mathbf{z}$  have full rank and  $\mathbf{z}$  is a vertex of  $([0, 1]^4, \mathcal{A}_{\mathbb{P}}^2)$ .

When  $\mathbf{z} = (0, 0, 0, 0, 1, 1)$ ,  $\mathcal{H}(\mathbf{z})$  consists of the fixations  $x_1 = 0$ ,  $y_1 = 0$ ,  $x_2 = 0$ ,  $y_2 = 0$ ,  $x_3 = 1$ , and  $y_3 = 1$  and the attack equations  $\mathbf{z}_1 \sim_1 \mathbf{z}_2$  and  $\mathbf{z}_1 \sim_2 \mathbf{z}_2$  since  $\mathbf{z}_1 = \mathbf{z}_2$ .  $\mathcal{H}(\mathbf{z})$  contains eight equations; the attack equations are redundant because the fixations uniquely determine  $\mathbf{z}$ ; those six equations form a system of full rank, so  $\mathcal{H}(\mathbf{z})$  and  $\mathbf{z}$  have full rank, and  $\mathbf{z}$  is a vertex of  $([0, 1]^6, \mathcal{A}_{\mathbb{P}}^3)$ .

When  $\mathbf{z} = (1, 1/2, 3/4, 0)$ ,  $\mathcal{H}(\mathbf{z}) = \{x_1 = 1, y_2 = 0, \mathbf{z}_1 \sim_2 \mathbf{z}_2\}$ , which has rank at most 3, so  $\mathcal{H}(\mathbf{z})$  is not of full rank and  $\mathbf{z}$  is not a vertex of  $([0, 1]^4, \mathcal{A}_{\mathbb{P}}^2)$ .

**Lemma 3.6.** Suppose  $\mathcal{H}$  is a hyperplane arrangement consisting of hyperplanes in  $\mathbb{R}^{2k}$ , and

$$\mathbf{z} = (x_1, y_1, x_2, y_2, \dots, x_k, y_k) \in \mathbb{R}^{2k}$$

is the unique intersection point of the elements of  $\mathcal{H}$ . Then, for all  $i$  between 1 and  $k$ ,  $\mathcal{H}$  contains at least 2 hyperplanes whose equations involve either  $x_i$  or  $y_i$ .

*Proof.* Suppose there exists  $i$  between 1 and  $k$  such that  $\mathcal{H}$  contains at most one hyperplane with equation involving  $(x_i, y_i)$ . Take  $\mathcal{H}$  to contain  $2k$  linearly independent hyperplanes (removing redundant hyperplanes as necessary).

Since  $\mathcal{H}$  only contains one equation involving  $x_i$  or  $y_i$ ,  $\mathcal{H}$  contains at least  $2k - 1$  hyperplanes whose equations only involve the other  $2k - 2$  variables, which contradicts the linear independence of  $\mathcal{H}$ .  $\square$

The rank of a point depends only on the set its constituent coordinate pairs.

**Proposition 3.7.**  $\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_q) \in \mathcal{B}^q$  has full rank if and only if  $\mathbf{z}' = (\mathbf{z}_1, \dots, \mathbf{z}_q, \mathbf{z}_q) \in \mathcal{B}^{q+1}$  has full rank.

*Proof.* First, suppose  $\mathbf{z}$  has rank  $2q$ . Then there is a hyperplane arrangement  $\mathcal{H}$  with rank  $2q$ , whose members are attack equations and fixations involving  $\mathbf{z}_1, \dots, \mathbf{z}_q$  and whose set  $\mathcal{N}$  of normal vectors forms a basis of  $\mathbb{R}^{2q}$ . Therefore the hyperplane arrangement

$$\mathcal{H} \cup \{\mathbf{z}_{q+1} \sim_1 \mathbf{z}_q, \mathbf{z}_{q+1} \sim_2 \mathbf{z}_q\}$$

is also linearly independent because the set

$$\mathcal{N} \cup \{(0, 0, \dots, d_1, -c_1, -d_1, c_1), (0, 0, \dots, d_2, -c_2, -d_2, c_2)\}$$

forms a basis of  $\mathbb{R}^{2(q+1)}$ .

Now, suppose  $\mathbf{z}'$  has full rank, so that it is the unique intersection point of a linearly independent hyperplane arrangement  $\mathcal{H}'$ , consisting of  $2q + 2$  attack equations and fixations. Without loss of generality, we can assume  $\mathcal{H}'$  contains the hyperplanes

$$\mathbf{z}_{q+1} \sim_1 \mathbf{z}_q \text{ and } \mathbf{z}_{q+1} \sim_2 \mathbf{z}_q$$

If not, we can add these to  $\mathcal{H}'$  and remove two redundant hyperplanes.

We can ensure that  $\mathcal{H}'$  has at most two attack equations involving  $\mathbf{z}_{q+1}$  and no fixations involving  $\mathbf{z}_{q+1}$  by replacing all other occurrences of  $\mathbf{z}_{q+1}$  by  $\mathbf{z}_q$ . Then, this equivalent system of equations has exactly two equations involving  $\mathbf{z}_{q+1}$ ; removing these two equations leaves  $2q$  linearly independent equations involving  $\mathbf{z}_1$  through  $\mathbf{z}_q$ , so  $\mathbf{z}$  has rank  $2q$ .  $\square$

The following observation is straightforward but helpful to state explicitly.

**Lemma 3.8.** Let  $\mathbf{z} = (\mathbf{z}_1, \dots, \mathbf{z}_q) \in \mathcal{B}^q$ . If there exists a point  $\mathbf{z}' = (\mathbf{z}'_1, \dots, \mathbf{z}'_q) \in \mathcal{B}^q$  such that  $\mathcal{H}(\mathbf{z}) = \mathcal{H}(\mathbf{z}')$  and the sets  $\{\mathbf{z}_i\}_{1 \leq i \leq q}$  and  $\{\mathbf{z}'_i\}_{1 \leq i \leq q}$  are different, then  $\mathbf{z}$  is not of full rank.

*Proof.* Because there are two points  $\mathbf{z}, \mathbf{z}' \in \mathbb{R}^{2q}$  that satisfy the system of equations,  $\mathcal{H}(\mathbf{z})$  (and therefore  $\mathbf{z}$ ) is not of full rank.  $\square$

#### 4. A DISCRETE DYNAMICAL SYSTEM FOR FAIRY CHESS

In this section we introduce a new discrete dynamical system that arises naturally in our study of attacking chess piece configurations. It originated from the idea of trajectories in attacking chess piece configurations that were introduced by Hanusa in a preliminary version of [9]. Our construction has been informed by surveys on the billiard model by Gutkin [15] and Tabachnikov [19]. Open problems related to this system have been gathered in Section 7.

We start with any bounded convex region  $\mathcal{R}$  (our board) and any nonparallel pair of vectors  $\mathbf{m}_1$  and  $\mathbf{m}_2$  (our basic moves). We let  $\mathcal{M} \subseteq \mathbb{S}^1$  consist of the four unit vectors parallel to  $\mathbf{m}_1$  or  $\mathbf{m}_2$ . We investigate the movement of a particle, determined by its position  $\mathbf{r} \in \mathcal{R}$  and its velocity  $\mathbf{v}$ , restricted to be an element of  $\mathcal{M}$ . The particle moves along the ray starting at  $\mathbf{r}$  in the direction  $\mathbf{v}$  until it hits a point  $\mathbf{b}$  on the boundary of  $\mathcal{R}$ , denoted  $\partial\mathcal{R}$ .

In this discrete dynamical system, the particle “bounces” differently from billiards. The convexity of  $\mathcal{R}$  implies  $\mathbf{b}$  has at most two vectors from  $\mathcal{M}$  pointing toward the interior of  $\mathcal{R}$ , including  $-\mathbf{v}$ . When there is a second vector  $\mathbf{v}'$ , the particle “bounces” and leaves  $\mathbf{b}$  in that direction, as



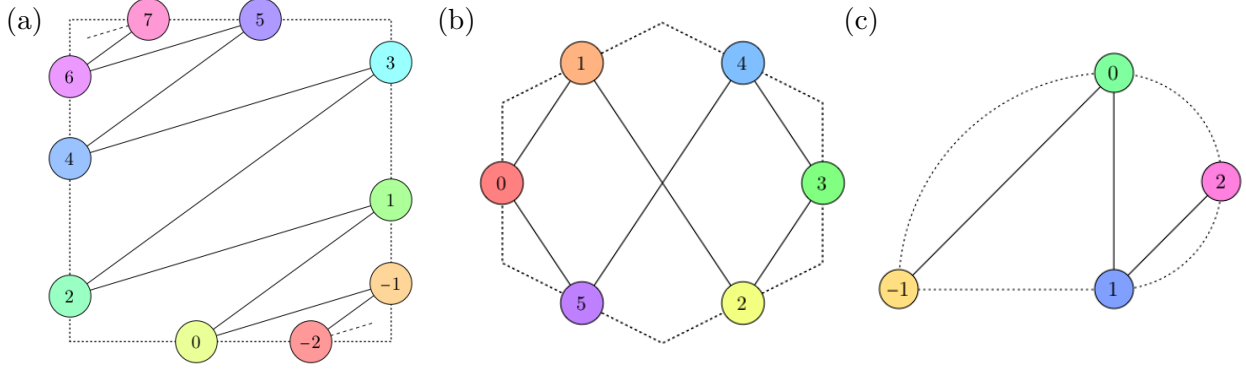


FIGURE 3. The behavior of three extended trajectories for the dynamical systems discussed in Example 4.1.

Figure 3(b) shows a hexagonal board with basic moves  $(1, 2)$  and  $(2, 1)$ . The chosen extended trajectory overlaps itself infinitely many times; its six points are periodic and form a complete orbit.

The non-polygonal board in Figure 3(c) is made up of two circular curves and one line segment. The basic moves are  $(1, 1)$  and  $(0, 1)$ . We show an example of a finite extended trajectory in the corresponding dynamical system—the board has a vertical tangent at  $\mathbf{b}_2$  and the point  $\mathbf{b}_{-1}$  is located at a corner with no points of the board accessible vertically.

It will be useful to also describe the chess attack map using the following *antipode maps*, which are involutions on  $\partial\mathcal{R}$ , and originate from the case of one-move riders in [9].

**Definition 4.2.** For a bounded convex region  $\mathcal{R}$  and a pair of vectors  $\mathbf{m}_1 = (c_1, d_1)$  and  $\mathbf{m}_2 = (c_2, d_2)$ , define  $s_r : \partial\mathcal{R} \rightarrow \partial\mathcal{R}$  for  $r = 1, 2$  as follows. Suppose  $\mathbf{b} \in \partial\mathcal{R}$ , and consider the line

$$\ell = \{\mathbf{b} + \lambda(c_r, d_r) \mid \lambda \in \mathbb{R}\}.$$

If  $\ell \cap \mathcal{B}^\circ = \emptyset$ , define  $s_r \mathbf{b} = \mathbf{b}$ . Otherwise, since  $\mathcal{R}$  is convex,  $\ell \cap \partial\mathcal{R}$  has exactly 2 elements and we define  $s_r \mathbf{b}$  to be the other element.

The chess attack map for a point  $\mathbf{b} \in \partial\mathcal{R}$  and a velocity  $\mathbf{v}$  pointing toward the interior of  $\mathcal{R}$  can then be described as  $\varphi(\mathbf{b}) = s_r \mathbf{b}$ , where  $\mathbf{v}$  is parallel to  $\mathbf{m}_r$ .

For a point  $\mathbf{b} \in \partial\mathcal{R}$  and a direction  $\mathbf{v} \in \mathcal{M}$  pointing into or out of  $\mathcal{R}$ , we define a *trajectory* to be a finite sequence  $T = [\mathbf{b}, \varphi(\mathbf{b}), \dots, \varphi^{l-1}(\mathbf{b})]$  of *distinct* points. We say  $T$  has *length*  $l$ . Equivalently, a trajectory is a consecutive subsequence of an extended trajectory. Note that if  $\mathbf{b}_1$  is periodic of period  $p$ , then the longest trajectory  $[\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_l]$  is of length  $p$  and satisfies  $\varphi(\mathbf{b}_p) = \mathbf{b}_1$ . We call such a trajectory a *cyclical trajectory*; it necessarily contains all points in  $\text{Orb}(\mathbf{b}_1)$ . As an example, the trajectory  $[\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_5]$  from Figure 3(b) is a cyclical trajectory.

We see that any trajectory in  $\mathcal{R}$  can be obtained by alternately applying  $s_1$  and  $s_2$  to an initial point  $\mathbf{b}$ . In other words, every trajectory is of the form

$$[\mathbf{b}, s_1 \mathbf{b}, s_2 s_1 \mathbf{b}, s_1 s_2 s_1 \mathbf{b}, \dots] \quad \text{or} \quad [\mathbf{b}, s_2 \mathbf{b}, s_1 s_2 \mathbf{b}, s_2 s_1 s_2 \mathbf{b}, \dots].$$

Critical to our study of periods of counting quasipolynomials are both the points on trajectories  $T = [\mathbf{b}_1, \dots, \mathbf{b}_l]$  and points on the interior of  $\mathcal{R}$  where flows that extend a bit on either side of  $\mathbf{b}_1$  and  $\mathbf{b}_l$  cross.

**Definition 4.3.** Let  $T_a = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k]$  and  $T_b = [\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_l]$  be consecutive subsequences of extended trajectories in  $\mathcal{R}$ . We say  $\mathbf{c}$  is a **crossing point** of  $T_a$  and  $T_b$  if  $\mathbf{c} \in \mathcal{R}^\circ$  and there exist some  $i$  and  $j$  such that  $\mathbf{c}$  is contained in the line segments from  $\mathbf{a}_i$  to  $\mathbf{a}_{i+1}$  and from  $\mathbf{b}_j$  to  $\mathbf{b}_{j+1}$  for

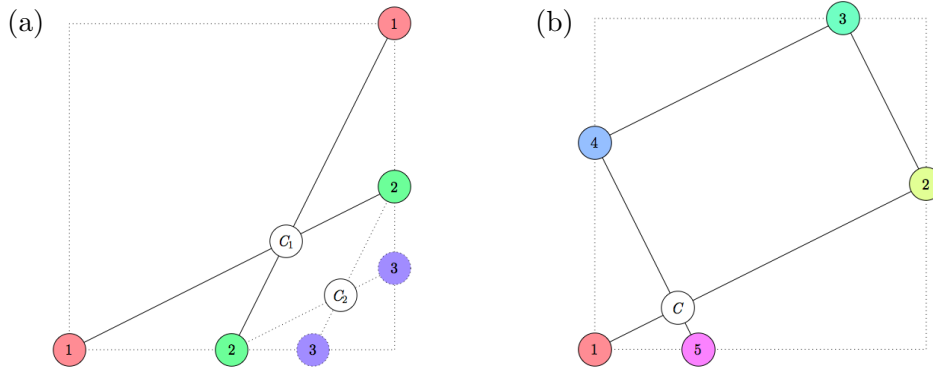


FIGURE 4. (a) For the square board when  $\mathbb{P}$  has moves  $(2, 1)$  and  $(1, 2)$ , the two two-point trajectories starting at  $(0, 0)$  and  $(1, 1)$  have a crossing point at  $C_1 = (2/3, 1/3)$ . The augmentations of these trajectories have a crossing point at  $C_2 = (5/6, 1/6)$ . (b) When  $\mathbb{P}$  has moves  $(2, 1)$  and  $(1, -2)$ , the five-point corner trajectory starting at  $(0, 0)$  has a self-crossing point at  $(1/4, 1/8)$ .

some  $1 \leq i \leq k - 1$  and  $1 \leq j \leq l - 1$ . If  $T_a = T_b$ , we say  $\mathbf{c}$  is a **self-crossing** point of  $T_a$ . See Figure 4.

**Definition 4.4.** Let  $T = [\mathbf{b}_1, \dots, \mathbf{b}_l]$  be a trajectory in  $\mathcal{R}$ . Then  $T$  is a consecutive subsequence of an extended trajectory  $T' = [\dots, \mathbf{b}_1, \dots, \mathbf{b}_l, \dots]$ . We define the *augmentation* of  $T$  to be the sequence of points including  $\mathbf{b}_1$  through  $\mathbf{b}_l$  where we prepend  $\mathbf{b}_0$  from  $T'$  if  $T'$  does not start at  $\mathbf{b}_1$  and we postpend  $\mathbf{b}_{l+1}$  from  $T'$  if  $T'$  does not terminate at  $\mathbf{b}_l$ .

**Remark 4.5.** An augmentation of a cyclical trajectory will no longer be a trajectory because of its repeated vertices. On the other hand, the flow corresponding to the augmentation of a cyclical trajectory  $T$  traces out the entire cycle that the extended trajectory traverses. Furthermore, crossing points of augmentations of trajectories may exist that are not crossing points of the trajectories themselves, as shown in Figure 4(a).

## 5. TRAJECTORIES ON POLYGONAL BOARDS

We apply our discrete dynamical system to the  $q$ -Queens Problem by restricting to general convex polygonal regions  $\mathcal{B}$ . We prove a characterization of the set of vertices  $\mathbf{z} = (\mathbf{z}_1, \dots, \mathbf{z}_q)$  of the inside-out polytope  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  that depends on whether the points  $\mathbf{z}_i$  lie on certain trajectories or are crossing points thereof.

**5.1. Corner trajectories and rigid cycles.** It is natural to extend the notion of rank to a trajectory  $T$  in  $\mathcal{B}$ . We define the *rank* of a trajectory  $T$  to be the rank of the collection of points in  $T$  (recall that the points of the trajectory  $T$  must all be distinct). We characterize the types of trajectories that are of full rank.

**Definition 5.1.** A trajectory  $T$  is called a *corner trajectory* if it contains a corner of  $\mathcal{B}$ .

**Definition 5.2.** Let  $T = [\mathbf{b}_1, \dots, \mathbf{b}_k]$  be a cyclical trajectory. If the point  $(\mathbf{b}_1, \dots, \mathbf{b}_k)$  has full rank,  $T$  is called a *rigid cycle*; otherwise  $T$  is called a *treachery*.

Only for certain choices of  $\mathcal{B}$  and  $\mathbb{P}$  do rigid cycles exist. The characterization of when they exist is open; see Question 7.5.

**Example 5.3.** Let  $\mathcal{B} = [0, 1]^2$  and consider the piece  $\mathbb{P}$  with moves  $\mathbf{m}_1 = (m, 1)$  and  $\mathbf{m}_2 = (-1, m)$  where  $m > 1$ . Choose  $\mathbf{b}_1 = (x_1, y_1)$  along the south edge of  $\mathcal{B}$ , so that  $\mathbf{b}_2 = (x_2, y_2) = s_1 \mathbf{b}_1$  lies



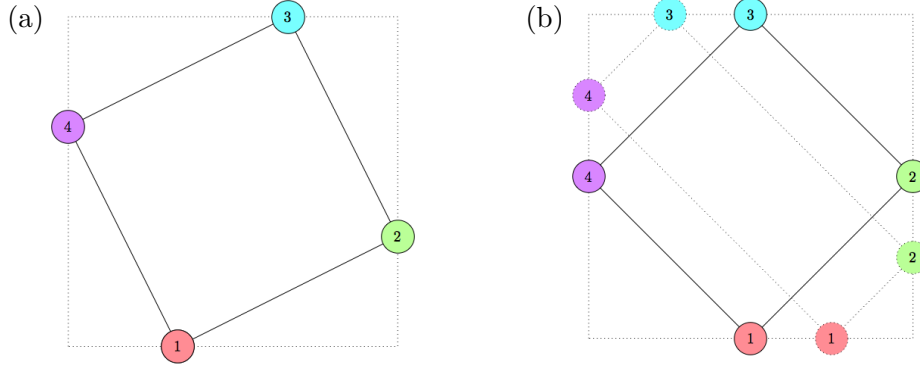


FIGURE 5. (a) For the piece with basic moves  $(2, 1)$  and  $(1, -2)$  the cyclical trajectory starting at  $\mathbf{b}_1 = (\frac{1}{3}, 0)$  is a rigid cycle. See Example 5.3. (b) For the bishop, every cyclical trajectory is a treachery. Note that the solid and dotted trajectories have the same associated hyperplane arrangement  $\mathcal{H}(\mathbf{b})$ . See Example 5.4.

along its east edge,  $\mathbf{b}_3 = (x_3, y_3) = s_2\mathbf{b}_2$  lies along its north edge, and  $\mathbf{b}_4 = (x_4, y_4) = s_1\mathbf{b}_3$  lies along its west edge. If  $\mathbf{b}_1 = s_2\mathbf{b}_4$ , the trajectory  $T = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4]$  is cyclical and the coordinates of the points are given by the system of equations

$$(5.1) \quad \{\mathbf{b}_1 \sim_1 \mathbf{b}_2, \mathbf{b}_2 \sim_2 \mathbf{b}_3, \mathbf{b}_3 \sim_1 \mathbf{b}_4, \mathbf{b}_4 \sim \mathbf{b}_1, y_1 = 0, x_2 = 1, y_3 = 1, x_4 = 0\}.$$

$T$  is a rigid cycle because when  $m > 1$  the unique solution to this system is

$$\mathbf{z} = \left( \frac{1}{1+m}, 0, 1, \frac{1}{1+m}, \frac{m}{1+m}, 1, 0, \frac{m}{1+m} \right).$$

Notice this implies  $\mathbf{z}$  is a vertex of  $([0, 1]^8, \mathcal{A}_{\mathbb{P}}^4)$ . Figure 5(a) shows the special case when  $m = 2$ . This example is generalized in Section 6.2.

**Example 5.4.** When  $\mathcal{B} = [0, 1]^2$  and  $\mathbb{P}$  is the bishop with moves  $(1, 1)$  and  $(1, -1)$ , there are no rigid cycles. Orbits fall into two cases—either they contain two opposite corners of  $\mathcal{B}$  or they form a cyclical trajectory  $T = [(x, 0), (1, 1-x), (1-x, 1), (0, x)]$  which is a rectangle. The points of  $T$  have as their associated hyperplane arrangement the system (5.1) when  $m = 1$ , which is no longer of full rank. We conclude  $T$  is a treachery. Alternatively, we see that all cyclical trajectories satisfy the system (5.1), so by Lemma 3.8, they are not of full rank. See Figure 5(b).

**Lemma 5.5.** *Corner trajectories have full rank.*

*Proof.* We show that every corner trajectory  $T = [\mathbf{b}_1, \dots, \mathbf{b}_l]$  has full rank by induction on  $l$ . When  $l = 1$ ,  $\mathbf{b}_1$  is a corner and hence the intersection of two linearly independent fixations; we conclude  $T$  has rank 2.

Now suppose  $l > 1$  is an integer, and all corner trajectories of shorter length  $l' < l$  have full rank. Suppose that  $\mathbf{b}_l$  is not a corner of  $\mathcal{B}$ , so that  $T' = [\mathbf{b}_1, \dots, \mathbf{b}_{l-1}]$  remains a corner trajectory, and therefore has full rank. Then  $\mathbf{z}' = (\mathbf{b}_1, \dots, \mathbf{b}_{l-1})$  is the unique intersection point of a set  $\mathcal{H}'$  of  $2l - 2$  hyperplanes.

The point  $\mathbf{b}_l$  equals  $s_r\mathbf{b}_{l-1}$  for some  $r \in \{1, 2\}$  and also lies along an edge  $\alpha_1x + \alpha_2y = \beta$  of  $\mathcal{B}$ . The set of equations  $\mathcal{E} = \{(\alpha_1, \alpha_2) \cdot \mathbf{z}_l = \beta, \mathbf{b}_{l-1} \sim_r \mathbf{z}_l\}$  is linearly independent because  $\mathbf{b}_l - \mathbf{b}_{l-1}$  is not parallel to the edge  $\alpha_1x + \alpha_2y = \beta$ . (Otherwise, the trajectory would have stopped at  $\mathbf{b}_{l-1}$ .)

Therefore the set of  $2l$  hyperplanes  $\mathcal{H} = \mathcal{H}' \cup \mathcal{E}$  is linearly independent and uniquely defines the vertex  $\mathbf{z} = (\mathbf{b}_1, \dots, \mathbf{b}_l)$ , and we conclude that  $T$  has full rank.

Finally, if  $\mathbf{b}_l$  is a corner of  $\mathcal{B}$ , we can use a similar argument by removing  $\mathbf{b}_1$ . □

**Proposition 5.6.** *The only trajectories of full rank are corner trajectories and rigid cycles.*

*Proof.* We show that non-cyclical trajectories  $T$  that are of full rank must contain a corner. The statement then follows from Lemma 5.6.

Suppose  $T = [\mathbf{b}_1, \dots, \mathbf{b}_l]$  has rank  $2l$  and is not cyclical. Let  $\mathbf{z} = (\mathbf{b}_1, \dots, \mathbf{b}_l) \in (\partial\mathcal{B})^l$ . There exists a set of hyperplanes  $\mathcal{H} \subseteq \mathcal{H}(\mathbf{z})$  with size and rank  $2l$  whose unique intersection point is  $\mathbf{z}$ . Since  $T$  is not cyclical, it either does not contain one of  $s_1\mathbf{b}_l$  and  $s_2\mathbf{b}_l$ , or one of these is equal to  $\mathbf{b}_l$  (this is only when  $T$  is forced to terminate at  $\mathbf{b}_l$ ).

In both cases, there can only be one attack equation between  $\mathbf{b}_l$  and another point  $\mathbf{b}_j$ . Since the points of  $T$  are distinct, for all  $j$  between 2 and  $l-1$ ,  $\mathbf{b}_j$  can only be related to  $\mathbf{b}_{j-1}$  and  $\mathbf{b}_{j+1}$  through attack equations. Finally,  $\mathbf{b}_1$  can only be related to  $\mathbf{b}_2$  through an attack equation. In summary,  $\mathcal{H}$  contains at most  $l-1$  attack hyperplanes. If each  $\mathbf{b}_j$  lies on only one fixation, then  $\mathcal{H}$  contains at most  $2l-1$  hyperplanes, which is impossible because  $\mathcal{H}$  has full rank. Therefore, one of the  $\mathbf{b}_j$  must be a corner of  $\mathcal{B}$ .  $\square$

**5.2. The vertices and denominator of the inside-out polytope.** We will determine the vertices and denominator of the inside-out polytope by understanding points  $(\mathbf{b}_1, \dots, \mathbf{b}_q) \in \mathcal{B}^q$ .

**Lemma 5.7.** *Let  $\mathcal{R}$  be a bounded convex region,  $\mathbb{P}$  be a piece with basic moves  $\mathbf{m}_1$  and  $\mathbf{m}_2$ , and let  $S$  be a finite subset of  $\partial\mathcal{R}$ . Then  $S$  can be partitioned into a set of trajectories  $\mathcal{T}(S)$  that travel along paths parallel to  $\mathbf{m}_1$  and  $\mathbf{m}_2$ .*

*Proof.* Suppose  $S = \{\mathbf{z}_1, \dots, \mathbf{z}_q\}$ . Create a graph with vertices labeled by  $S$  with an edge  $\{\mathbf{z}_i, \mathbf{z}_j\}$  if  $\mathbf{z}_i \neq \mathbf{z}_j$  and  $\mathbf{z}_i = s_1\mathbf{z}_j$  or  $\mathbf{z}_i = s_2\mathbf{z}_j$ . Every vertex in this graph has degree at most 2, so each connected component is either a cycle or a path. Within each connected component the edges will alternate between corresponding to  $s_1$  and  $s_2$ . Writing the vertices of a connected component in the order given by the path or cycle gives a trajectory.  $\square$

**Lemma 5.8.** *Let  $\mathcal{B}$  be a bounded convex polygon,  $\mathbb{P}$  be a piece with basic moves  $\mathbf{m}_1$  and  $\mathbf{m}_2$ , and let  $S$  be a finite subset of  $\partial\mathcal{B}$ . The rank of  $S$  is the sum of the ranks of the trajectories in  $\mathcal{T}(S)$ , and  $S$  has full rank if and only if each of the trajectories in  $\mathcal{T}(S)$  has full rank.*

*Proof.* Let  $S = \{\mathbf{z}_1, \dots, \mathbf{z}_l\}$  and define  $\mathbf{z} = (\mathbf{z}_1, \dots, \mathbf{z}_l)$ . The rank of  $S$  equals the rank of  $\mathcal{H}(\mathbf{z})$ , which includes all hyperplanes from each individual trajectory in  $\mathcal{T}(S)$  and attack equations between distinct trajectories in  $\mathcal{T}(S)$ . These latter equations do not exist unless  $\mathbf{z}_i$  and  $\mathbf{z}_j$  are in different connected components and lie on the same side of  $\mathcal{B}$  that is parallel to a move of  $\mathbb{P}$ . In this case there are two fixations in  $\mathcal{H}(\mathbf{z})$ , with equations involving  $\mathbf{z}_i$  and  $\mathbf{z}_j$  respectively, whose equations imply the attack equation linking  $\mathbf{z}_i$  and  $\mathbf{z}_j$ . This means we can remove the attack equation with this equation from  $\mathcal{H}(\mathbf{z})$  without affecting the rank of  $\mathcal{H}(\mathbf{z})$ . This concludes the proof.  $\square$

This tells us exactly which  $(\mathbf{b}_1, \dots, \mathbf{b}_q) \in (\partial\mathcal{B})^q$  have full rank. We can extend this knowledge to points in  $\mathcal{B}^q$ .

**Theorem 5.9.** *Suppose  $\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_q) \in \mathcal{B}^q$  and partition  $S = \{\mathbf{z}_i\}_{1 \leq i \leq q}$  into  $B \subseteq \partial\mathcal{B}$  and  $C \subseteq \mathcal{B}^\circ$ . Then  $\mathbf{z}$  is a vertex of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  if and only if:*

- (1)  $B$  can be written as the union of corner trajectories and rigid cycles, and
- (2)  $C$  consists of crossing points of augmentations of these corner trajectories and rigid cycles (which may include self-crossing points).

*Proof.* By Lemma 3.4 and Proposition 3.7,  $\mathbf{z}$  is a vertex of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  if and only if  $S$  has full rank. We proceed by induction on the size of  $C$ . When  $|C| = 0$ ,  $S = B$ ; Proposition 5.6 and Lemma 5.8 show that  $S$  has full rank if and only if  $S$  can be decomposed into corner trajectories and rigid cycles.

Now let  $|C| = k > 0$ . Suppose  $\mathbf{z}$  has full rank and let  $\mathcal{H} \subseteq \mathcal{H}(\mathbf{z})$  be a set of  $2q$  equations whose unique intersection point is  $\mathbf{z}$ . Up to index reordering, we can choose  $\mathbf{z}_q \in \mathcal{B}^\circ$  and therefore  $\mathbf{z}_q$  is not involved in any fixation equations. Furthermore we can assume  $\mathcal{H}$  contains exactly one attack

equation involving  $\mathbf{z}_q$  of each type, say  $\mathbf{z}_q \sim_1 \mathbf{z}_i$  and  $\mathbf{z}_q \sim_2 \mathbf{z}_j$  for  $1 \leq i, j \leq q-1$ . (If there were more than one, we could replace an equation of the form  $\mathbf{z}_q \sim_1 \mathbf{z}_k$  by  $\mathbf{z}_i \sim_1 \mathbf{z}_k \in \mathcal{H}(\mathbf{z})$ .) The removal of these two equations from  $\mathcal{H}$  gives  $2q-2$  linearly independent equations involving  $\mathbf{z}_1$  through  $\mathbf{z}_{q-1}$ , so  $\mathbf{z}' = (\mathbf{z}_1, \dots, \mathbf{z}_{q-1})$  is a vertex of  $(\mathcal{B}^{q-1}, \mathcal{A}_{\mathbb{P}}^{q-1})$ .

By induction, the set  $S' = \{\mathbf{z}_i\}_{1 \leq i \leq q-1}$  can be partitioned into the sets  $B' = B \subseteq \partial\mathcal{B}$  and  $C' \subseteq \mathcal{B}^\circ$  which satisfy conditions (1) and (2). Every  $\mathbf{z}_k \in C'$  is the crossing point of trajectories involving points of  $B$ , so is related by attacking equations of both types to points of  $B$ . Since  $\mathbf{z}_q \sim_1 \mathbf{z}_i$  and  $\mathbf{z}_q \sim_2 \mathbf{z}_j$ , then by transitivity of  $\sim_r$ ,  $\mathbf{z}_q$  is a crossing point of augmentations of trajectories involving points of  $B$ . (The need for augmentations of trajectories  $T = [\mathbf{b}_1, \dots, \mathbf{b}_l]$  arises because the crossing point may lie along the line segment leaving  $\mathbf{b}_1$  toward  $\mathbf{b}_0$  or along the line segment leaving  $\mathbf{b}_l$  toward  $\mathbf{b}_{l+1}$ .) This completes the proof in the forward direction.

Now, suppose the elements of  $C$  are all crossing points of augmentations of the corner trajectories and rigid cycles of  $\mathcal{T}(B)$ . By the inductive hypothesis,  $\mathbf{z}' = (\mathbf{z}_1, \dots, \mathbf{z}_{q-1})$  has full rank. Let  $\mathcal{H}' \subseteq \mathcal{H}(\mathbf{z})$  be a set of hyperplanes with rank  $2(q-1)$ , whose intersection is  $\mathbf{z}'$ .

Since  $\mathbf{z}_q$  is a crossing point of two of the augmentations of trajectories making up  $B$ , it is linked by two attack equations of different types to points in  $B$ . Since the moves of  $\mathbb{P}$  are linearly independent,  $\mathcal{H}'$  with these two attack equations appended has rank  $2q$ , and  $\mathbf{z}$  is the intersection point of these hyperplanes. Therefore,  $\mathbf{z}$  has full rank.  $\square$

Now that we know the vertices of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$ , we can find its denominator.

**Theorem 5.10.** *The denominator of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  is equal to the least common multiple of the denominators of*

- (1) *Points on rigid cycles of length at most  $q$ ,*
- (2) *Points on corner trajectories of length at most  $q$  that start at corners,*
- (3) *Self-crossing points of augmentations of corner trajectories or rigid cycles of length at most  $q-1$ , and*
- (4) *Crossing points of augmentations of two distinct corner trajectories or rigid cycles whose lengths sum to at most  $q-1$ .*

*Proof.* The denominator of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  is the least common multiple of the denominator of all vertices  $\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_q)$  of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$ . We must determine the set of all points that may occur as a component of some vertex.

Theorem 5.9 says that the set of points  $S = \{\mathbf{z}_i\}$  can be partitioned into corner trajectories, rigid cycles, and crossing points of augmentations of these trajectories. We first consider points on corner trajectories and rigid cycles. Points on rigid cycles  $[\mathbf{b}_1, \dots, \mathbf{b}_l]$  of length  $l \leq q$  will occur as components of the vertex  $(\mathbf{b}_1, \dots, \mathbf{b}_l, \mathbf{b}_l, \dots, \mathbf{b}_l) \in (\partial\mathcal{B})^q$ . Points on corner trajectories  $[\mathbf{b}_1, \dots, \mathbf{b}_l]$  that include the corner  $\mathbf{c}$  will occur as components of a vertex  $(\mathbf{b}'_1, \dots, \mathbf{b}'_q) \in (\partial\mathcal{B})^q$  where  $T = (\mathbf{b}'_1, \dots, \mathbf{b}'_l)$  is a trajectory starting at  $\mathbf{b}'_1 = \mathbf{c}$  and continues until  $l = q$  or until it stops. (If  $l < q$ , we pad our vertex with repeated points  $\mathbf{b}'_{l+1} = \dots = \mathbf{b}'_q = \mathbf{c}$ .)

A point  $\mathbf{c}$  that occurs as a self-crossing point of an augmentation of some trajectory  $T = [\mathbf{b}_1, \dots, \mathbf{b}_l]$  occurs as a vertex  $(\mathbf{b}_1, \dots, \mathbf{b}_l, \mathbf{c}, \dots, \mathbf{c})$  if and only if  $l \leq q-1$ , and a point  $\mathbf{c}$  that occurs as a crossing point of augmentations of trajectories  $T_a = [\mathbf{a}_1, \dots, \mathbf{a}_k]$  and  $T_b = [\mathbf{b}_1, \dots, \mathbf{b}_l]$  occurs as a vertex  $(\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{b}_1, \dots, \mathbf{b}_l, \mathbf{c}, \dots, \mathbf{c})$  if and only if  $k+l \leq q-1$ .  $\square$

**Corollary 5.11.** *If Conjecture 2.1 is true, the period of the counting quasipolynomial  $u_{\mathbb{P}}(q; n)$  on the square board is equal to the least common multiple of the denominators of*

- (1) *Points on rigid cycles of length at most  $q$ ,*
- (2) *Points on corner trajectories of length at most  $q$  that start at corners,*
- (3) *Self-crossing points of augmentations of corner trajectories or rigid cycles of length at most  $q-1$ , and*

- (4) *Crossing points of augmentations of two distinct corner trajectories or rigid cycles whose lengths sum to at most  $q - 1$ .*

Theorem 5.10 allows us to give a new and simpler proof of the main result from [11].

**Corollary 5.12.** *For  $q \geq 3$ , the period of the counting quasipolynomial of the bishop on the square board is 2.*

*Proof.* The only corner trajectories are the diagonals of  $\mathcal{B}$ , and there are no rigid cycles, as shown in Example 5.4. This shows that every vertex  $\mathbf{z}$  of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  has  $z_i$  equal to a corner of  $\mathcal{B}$  or  $(\frac{1}{2}, \frac{1}{2})$ . Therefore the denominator of the IOP is 2, which the period of the counting quasipolynomial must divide. Lemma 3.3(III) from [8] shows that the coefficient of  $n^{2q-6}$  has period 2 for  $q \geq 3$ , which completes the proof.  $\square$

## 6. TWO-MOVE RIDERS ON SQUARE BOARDS

We now restrict to the square board  $\mathcal{B} = [0, 1]^2$  and investigate the denominator  $D([0, 1]^{2q}, \mathcal{A}_{\mathbb{P}}^q)$  of the inside-out polytope for some two-move riders. Our analysis is broken into cases depending on the signs and magnitudes of the slopes  $d_1/c_1$  and  $d_2/c_2$ . We will notate the open edges of  $\mathcal{B}$  counterclockwise by

$$E_1 = (0, 1) \times \{0\}, E_2 = \{1\} \times (0, 1), E_3 = (0, 1) \times \{1\}, \text{ and } E_4 = \{0\} \times (0, 1).$$

**6.1. Slopes of the same sign.** First consider a piece whose moves have slopes of the same sign. The non-trivial extended trajectories converge to the fixed points of the dynamical system. This proposition does not require the slopes to be rational.

**Proposition 6.1.** *Let  $\mathcal{B}$  be the square board and  $\mathbb{P}$  have moves with real-valued slopes  $m_1$  and  $m_2$  of the same sign. Every extended trajectory  $T = [\mathbf{b}_n]_{n \in \mathbb{Z}}$  with more than one point is doubly infinite, with its points converging to  $(0, 1)$  as  $n$  approaches  $+\infty$  and  $(1, 0)$  as  $n$  approaches  $-\infty$  or vice versa.*

*Proof.* Assume  $0 < m_1 < m_2$ . The points  $(0, 1)$  and  $(1, 0)$  are fixed points of the system; no trajectories enter or leave. We show the orbit of every other point of  $\partial\mathcal{B}$  is infinite. Define the sets

$$Z_1 = E_1 \cup E_4 \cup \{(0, 0)\} \text{ and } Z_2 = E_2 \cup E_3 \cup \{(1, 1)\}.$$

When  $\mathbf{b} \in Z_1$ , both  $s_1\mathbf{b}$  and  $s_2\mathbf{b}$  are in  $Z_2$  and  $s_1\mathbf{b}$  is to the southeast of  $s_2\mathbf{b}$ ; when  $\mathbf{b} \in Z_2$ , both  $s_1\mathbf{b}$  and  $s_2\mathbf{b}$  are in  $Z_1$  and  $s_1\mathbf{b}$  is to the northwest of  $s_2\mathbf{b}$ .

Therefore, when  $\mathbf{b} \in Z_1$ , then  $s_2\mathbf{b} \in Z_2$ , so  $s_1s_2\mathbf{b}$  is to the northwest of  $s_2s_2\mathbf{b} = \mathbf{b}$  in  $Z_1$ , and hence  $s_1s_2\mathbf{b}$  is closer to  $(0, 1)$  than  $\mathbf{b}$  is. This is the first of the following statements, all of which follow similarly.

$$\text{When } \mathbf{b} \in Z_1, 0 < |(0, 1) - s_1s_2\mathbf{b}| < |(0, 1) - \mathbf{b}| \text{ and } 0 < |(1, 0) - s_2s_1\mathbf{b}| < |(1, 0) - \mathbf{b}|.$$

$$\text{When } \mathbf{b} \in Z_2, 0 < |(1, 0) - s_1s_2\mathbf{b}| < |(1, 0) - \mathbf{b}| \text{ and } 0 < |(0, 1) - s_2s_1\mathbf{b}| < |(0, 1) - \mathbf{b}|.$$

We conclude that the extended trajectory  $T$  is doubly infinite with one tail going northwest and one tail going southeast; we now show its points converge to  $(1, 0)$  or  $(0, 1)$ . When successive points alternate between neighboring sides, the distance to  $(1, 0)$  or  $(0, 1)$  along the same edge decreases geometrically. The trajectory may first alternate between diametrically opposite sides, but in that case, the distance between consecutive points along the same edge is a positive constant, so the trajectory eventually begins to alternate between neighboring sides.

The negative slope case follows by symmetry.  $\square$

We now apply Theorem 5.10 to find  $D([0, 1]^{2q}, \mathcal{A}_{\mathbb{P}}^q)$  when  $0 < m_1 < 1 < m_2$ . This restriction avoids a much more complicated formula that arises from the behavior of the crossing points in the general case.

**Theorem 6.2.** Suppose  $\mathbb{P}$  has moves  $\mathbf{m}_1 = (c_1, d_1)$  and  $\mathbf{m}_2 = (c_2, d_2)$ , satisfying  $0 < \frac{d_1}{c_1} < 1 < \frac{d_2}{c_2}$ . The denominator of  $([0, 1]^{2q}, \mathcal{A}_{\mathbb{P}}^q)$  is the least common multiple of the denominators of the first  $q$  terms of the following sequence defined for  $i \geq 1$

$$(6.1) \quad \begin{cases} \left(1, \left(\frac{d_1 c_2}{c_1 d_2}\right)^{\frac{i-1}{2}}\right) & \text{for } i \text{ odd} \\ \left(\frac{d_1}{c_1} \left(\frac{d_1 c_2}{c_1 d_2}\right)^{\frac{i}{2}-1}, \frac{c_2}{d_2} \left(\frac{d_1 c_2}{c_1 d_2}\right)^{\frac{i}{2}-1}\right) & \text{for } i \text{ even} \end{cases}$$

and the denominators of the first  $\lfloor (q-1)/2 \rfloor$  terms of the following sequence defined for  $i \geq 1$

$$(6.2) \quad \begin{cases} \left(\frac{d_1 c_2}{c_1 d_2}\right)^{\frac{i-1}{2}} \left(\frac{c_2(d_1-c_1)}{c_1 d_2 - c_2 d_1}, \frac{d_1(d_2-c_2)}{c_1 d_2 - c_2 d_1}\right) & \text{for } i \text{ odd} \\ \left(\frac{d_1 c_2}{c_1 d_2}\right)^{\frac{i}{2}} \left(\frac{c_1(c_2-d_2)}{c_1 d_2 - c_2 d_1}, \frac{d_2(c_1-d_1)}{c_1 d_2 - c_2 d_1}\right) & \text{for } i \text{ even} \end{cases}$$

*Proof.* By Proposition 6.1, all orbits of points other than  $(1, 0)$  and  $(0, 1)$  are infinite, so there are no rigid cycles. These extended trajectories also have no self-crossing points. Therefore the denominator  $D(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  can be found by calculating the coordinates of all points on corner trajectories of length at most  $q$  starting at  $(0, 0)$  or  $(1, 1)$ , and crossing points of augmentations of the same whose lengths sum to at most  $q-1$ .

The trajectories  $T = [\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_q]$  starting at  $\mathbf{b}_1 = (0, 0)$  with initial velocities  $\mathbf{m}_1$  and  $\mathbf{m}_2$  respectively have coordinates

$$\mathbf{b}_i = \begin{cases} \left(1 - \left(\frac{d_1 c_2}{c_1 d_2}\right)^{\frac{i-1}{2}}, 0\right) & \text{for } i \text{ odd} \\ \left(1, \frac{d_1}{c_1} \left(\frac{d_1 c_2}{c_1 d_2}\right)^{\frac{i}{2}-1}\right) & \text{for } i \text{ even} \end{cases} \quad \text{and} \quad \mathbf{b}_i = \begin{cases} \left(0, 1 - \left(\frac{d_1 c_2}{c_1 d_2}\right)^{\frac{i-1}{2}}\right) & \text{for } i \text{ odd} \\ \left(\frac{c_2}{d_2} \left(\frac{d_1 c_2}{c_1 d_2}\right)^{\frac{i}{2}-1}, 1\right) & \text{for } i \text{ even} \end{cases}$$

If instead  $T$  starts at  $\mathbf{b}_1 = (1, 1)$  with initial velocities  $\mathbf{m}_1$  and  $\mathbf{m}_2$ , the coordinates are respectively

$$\mathbf{b}_i = \begin{cases} \left(\left(\frac{d_1 c_2}{c_1 d_2}\right)^{\frac{i-1}{2}}, 1\right) & \text{for } i \text{ odd} \\ \left(0, 1 - \frac{d_1}{c_1} \left(\frac{d_1 c_2}{c_1 d_2}\right)^{\frac{i}{2}-1}\right) & \text{for } i \text{ even} \end{cases} \quad \text{and} \quad \mathbf{b}_i = \begin{cases} \left(1, \left(\frac{d_1 c_2}{c_1 d_2}\right)^{\frac{i-1}{2}}\right) & \text{for } i \text{ odd} \\ \left(1 - \frac{c_2}{d_2} \left(\frac{d_1 c_2}{c_1 d_2}\right)^{\frac{i}{2}-1}, 0\right) & \text{for } i \text{ even} \end{cases}$$

An example of these trajectories is shown in Figure 6.

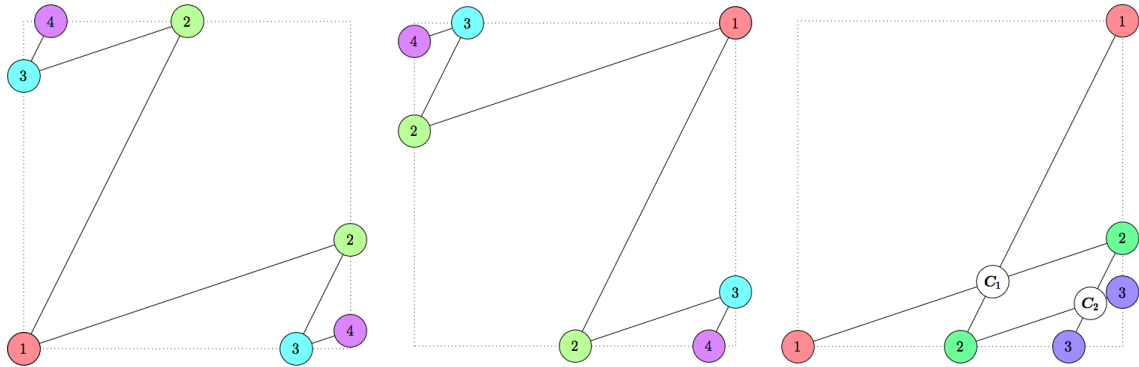


FIGURE 6. For the piece with moves  $(1, 2)$  and  $(3, 1)$  we illustrate the four corner trajectories starting at  $(0, 0)$  or  $(1, 1)$ . The right image shows two crossing points of these trajectories.

We must now find all crossing points  $\mathbf{p} \in \mathcal{B}^\circ$ . We consider crossing points of the first and fourth trajectories—the other crossing points arise from a 180-degree rotation around  $(\frac{1}{2}, \frac{1}{2})$  and have the same denominators.

Let  $T_a = [\mathbf{a}_1, \dots, \mathbf{a}_k]$  be the first trajectory and let  $T_b = [\mathbf{b}_1, \dots, \mathbf{b}_l]$  be fourth trajectory. The points lying along  $E_1$  starting at  $(0, 0)$  and moving eastward are  $\mathbf{a}_1, \mathbf{b}_2, \mathbf{a}_3, \mathbf{b}_4, \dots$  and the

points lying along  $E_2$  starting at  $(1, 1)$  and moving southward are  $\mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_3, \mathbf{a}_4, \dots$ . Because line segments only have one of two slopes and because the points are connected in increasing order in the trajectory, the only crossing points of line segments from  $\mathbf{a}_i$  and  $\mathbf{a}_{i+1}$  and from  $\mathbf{b}_j$  and  $\mathbf{b}_{j+1}$  occur when  $i = j$ .

Solving  $\mathbf{p} \sim_1 \mathbf{a}_i$  and  $\mathbf{p} \sim_2 \mathbf{b}_i$  for  $\mathbf{p}$  gives

$$\mathbf{p} = \begin{cases} (1, 0) + \left(\frac{d_1 c_2}{c_1 d_2}\right)^{\frac{i-1}{2}} \left(\frac{c_2(d_1 - c_1)}{c_1 d_2 - c_2 d_1}, \frac{d_1(d_2 - c_2)}{c_1 d_2 - c_2 d_1}\right) & \text{for } i \text{ odd} \\ (1, 0) + \left(\frac{d_1 c_2}{c_1 d_2}\right)^{\frac{i}{2}} \left(\frac{c_1(c_2 - d_2)}{c_1 d_2 - c_2 d_1}, \frac{d_2(c_1 - d_1)}{c_1 d_2 - c_2 d_1}\right) & \text{for } i \text{ even} \end{cases},$$

which will be a crossing point when  $i \leq [(q-1)/2]$ . The result follows from Theorem 5.10.  $\square$

**Corollary 6.3.** *Let  $\mathcal{B} = [0, 1]^2$  be the square board, and  $\mathbb{P}$  be the inclined nightrider. Then the denominator of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  is:*

$$\begin{cases} 1 & q = 1 \\ 2 & q = 2 \\ 3 \cdot 2^{q-1} & q \geq 3 \end{cases}$$

*Proof.* For the inclined nightrider with moves  $(1, 2)$  and  $(2, 1)$ , the denominators in Sequence (6.1) are  $2^{i-1}$  and the denominators in Sequence (6.2) are  $3 \cdot 2^{i-1}$ , so a factor of 3 will appear in the denominator for all  $q \geq 3$ .  $\square$

**6.2. Slopes of opposite signs.** We now investigate the dynamics of trajectories for a piece  $\mathbb{P}$  with moves  $(c_1, d_1)$  and  $(c_2, d_2)$ , where  $0 < d_1/c_1 < 1$ , and  $d_2/c_2 < -1$ . (This is a generalization of the orthogonal nightrider.) We let  $c_1, d_1, d_2 > 0$  and  $c_2 < 0$ . In this dynamical system, extended trajectories converge to a single rigid cycle. The general case when the moves are of opposite signs is presented as an open question in Section 7.

We first consider real-valued slopes  $m_1$  and  $m_2$  satisfying  $0 < m_1 < 1$  and  $m_2 < -1$ . The point  $\mathbf{b} = \left(\frac{m_1 - 1}{m_1 + m_2}, 0\right) \in \partial\mathcal{B}$  has orbit

$$(6.3) \quad \mathcal{O} = \left\{ \left(\frac{m_1 - 1}{m_1 + m_2}, 0\right), \left(1, \frac{m_1(1 + m_2)}{m_1 + m_2}\right), \left(\frac{1 + m_2}{m_1 + m_2}, 1\right), \left(0, \frac{m_2(1 - m_1)}{m_1 + m_2}\right) \right\}.$$

An example is shown in Figure 7.

The four points of  $\mathcal{O}$  form a rigid cycle because they are the solution to the system of equations

$$\{\mathbf{z}_1 \sim_1 \mathbf{z}_2, \mathbf{z}_2 \sim_2 \mathbf{z}_3, \mathbf{z}_3 \sim_1 \mathbf{z}_4, \mathbf{z}_4 \sim_2 \mathbf{z}_1, y_1 = 0, x_2 = 1, y_3 = 1, x_4 = 0\},$$

which has full rank. In fact,  $\mathcal{O}$  is the only rigid cycle in the system and is an attractor for all other trajectories.

**Theorem 6.4.** *Let  $\mathcal{B}$  be the square board and  $\mathbb{P}$  have moves with real-valued slopes  $m_1$  and  $m_2$  satisfying  $0 < m_1 < 1$  and  $m_2 < -1$ . The orbit  $\mathcal{O}$  in Equation (6.3) is the only finite orbit in  $\partial\mathcal{B}$ . Further, suppose  $T = [\mathbf{b}_n]_{n \in \mathbb{Z}}$  is an extended trajectory disjoint from  $\mathcal{O}$ . Then as  $n$  both increases and decreases,  $T$  either stops at a corner or converges to  $\mathcal{O}$ . (In other words,  $\mathcal{O}$  is the  $\omega$ -limit set of  $T$ .)*

*Proof.* Restricting the antipode map  $s_1$  to the domain  $\overline{E_1}$  is a linear contraction  $s_1 \overline{E_1} \rightarrow \overline{E_2}$  with a factor of  $m_1$  because

$$|s_1(x_1, 0) - s_1(x_2, 0)| = |(1, m_1(1 - x_1)) - (1, m_1(1 - x_2))| = m_1|x_1 - x_2|.$$

Similarly,  $s_2 : \overline{E_2} \rightarrow \overline{E_3}$  is a linear contraction with a factor of  $|\frac{1}{m_2}|$ ,  $s_1 : \overline{E_3} \rightarrow \overline{E_4}$  is a linear contraction with a factor of  $m_1$ , and  $s_2 : \overline{E_4} \rightarrow \overline{E_1}$  is a linear contraction with a factor of  $|\frac{1}{m_2}|$ .

For any point  $\mathbf{b}_0 \in \partial\mathcal{B} \setminus \mathcal{O}$ , we investigate the extended trajectory  $T = [\mathbf{b}_n]_{n \in \mathbb{Z}}$  where we choose  $\mathbf{b}_1 = \varphi(\mathbf{b}_0)$  to be on the next side counterclockwise from  $\mathbf{b}_0$ . (This is well defined because of the restrictions on  $m_1$  and  $m_2$ .) By the above reasoning, this sequence continues along sides of  $\mathcal{B}$  in a

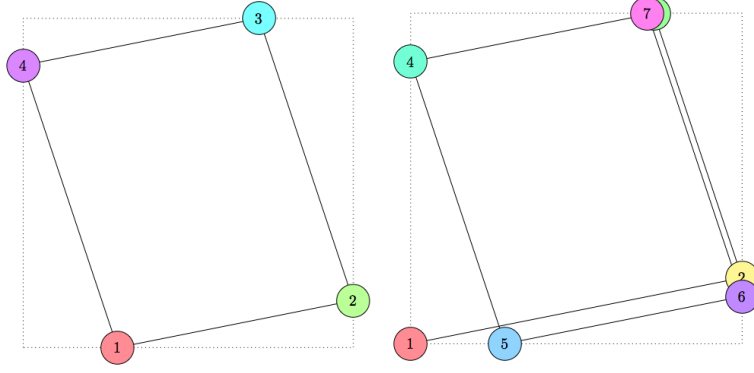


FIGURE 7. The left image shows the rigid cycle  $\mathcal{O}$  in Equation (6.3) for the piece with moves  $(5, 1)$  and  $(-1, 3)$ . The right image is a corner trajectory in the same system, whose points converge to  $\mathcal{O}$ .

counterclockwise manner as  $n \rightarrow +\infty$ . Suppose  $o$  is the element of  $\mathcal{O}$  on the same side of  $\partial\mathcal{B}$  as  $\mathbf{b}_0$ . Then we know that  $\varphi^4(o) = o$  and

$$|\varphi^{4k}(\mathbf{b}_0) - o| = \frac{m_1^{2k}}{m_2^{2k}} |\mathbf{b}_0 - o|.$$

We conclude that  $\mathbf{b}_n$  is defined for all  $n \geq 0$  and  $\mathcal{O}$  is the  $\omega$ -limit set of  $T$  as  $n \rightarrow \infty$ . This also ensures that  $\mathcal{O}$  is the only finite orbit.

On the other hand, if we apply  $\varphi^{-1}$  repeatedly to  $\mathbf{b}_0$ , the points visited can not indefinitely cycle among the sides of  $\mathcal{B}$  in a clockwise manner because each application of  $\varphi^{-1}$  is an expansion. Therefore this sequence either stops at a corner, or two successive points  $\mathbf{b}_{-N+1}$  and  $\mathbf{b}_{-N}$  are on opposite edges of  $\mathcal{B}$ . When this occurs,  $\mathbf{b}_{-N-1}$  is on the edge counterclockwise from  $\mathbf{b}_{-N}$  and the sequence  $[\mathbf{b}_{-n}]_{n \geq N}$  continues in a counterclockwise manner, which means that it is defined for all  $n \geq N$  and  $\mathcal{O}$  is the  $\omega$ -limit set of  $T$  as  $n \rightarrow \infty$ .  $\square$

We now compute  $D([0, 1]^{2q}, \mathcal{A}_{\mathbb{P}}^q)$  when  $\mathbb{P}$  has orthogonal slopes of the form  $(m, 1)$  and  $(1, -m)$ .

**Theorem 6.5.** *Let  $\mathcal{B} = [0, 1]^2$  be the square board, and  $\mathbb{P}$  be the piece with moves  $(m, 1)$  and  $(1, -m)$ . Then the denominator of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  is:*

$$\begin{cases} 1 & q = 1 \\ m & q = 2 \\ m^4 + m^2 & q = 3 \\ \text{lcm}(m^2 + 1, m + 1) \cdot m^{q-1} & q \geq 4 \end{cases}.$$

*Proof.* For this piece  $\mathbb{P}$ , the rigid cycle

$$\mathcal{O} = \{(1/(m+1), 0), (1, 1/(m+1)), (m/(m+1), 1), (0, m/(m+1))\}$$

contributes a denominator of  $m+1$  when  $q \geq 4$ .

Each corner is the start of one corner trajectory; by symmetry about  $(\frac{1}{2}, \frac{1}{2})$  the  $k$ -th point along every trajectory has the same denominator. The trajectory  $T = [\mathbf{b}_1, \mathbf{b}_2, \dots]$  starting at  $\mathbf{b}_1 = (0, 0)$  has coordinates

$$\mathbf{b}_k = \begin{cases} (0, \frac{m}{m+1}) - \frac{1}{m^{k-1}} (0, \frac{1}{m+1}) & k \equiv 0 \pmod{4} \\ (\frac{1}{m+1}, 0) - \frac{1}{m^{k-1}} (\frac{1}{m+1}, 0) & k \equiv 1 \pmod{4} \\ (1, \frac{1}{m+1}) + \frac{1}{m^{k-1}} (0, \frac{1}{m+1}) & k \equiv 2 \pmod{4} \\ (\frac{m}{m+1}, 1) + \frac{1}{m^{k-1}} (\frac{1}{m+1}, 0) & k \equiv 3 \pmod{4} \end{cases},$$

whose denominator is  $m^{k-1}$  for all  $k$ . (Notice, for example, that  $m^{k-1} - 1$  is divisible by  $m + 1$  for  $k$  odd.)

We must also determine the denominators of crossing points of augmentations of trajectories and rigid cycles. The key insight is that every crossing point  $\mathbf{c} = (x, y)$  lies on the lines  $x - my = r$  and  $mx + y = s$  for some rational numbers  $r$  and  $s$  whose denominators divide the smaller of the denominators of the two points on  $\partial\mathcal{B}$  that the lines intersect. Solving these equations for  $x$  and  $y$  we see  $x = (r + ms)/(m^2 + 1)$  and  $y = (s - mr)/(m^2 + 1)$ . In essence, a crossing point of the augmentation of trajectories and rigid cycles can not contribute anything new to  $([0, 1]^{2q}, \mathcal{A}_{\mathbb{P}}^q)$  other than  $(m^2 + 1)$ . This contribution of  $(m^2 + 1)$  will indeed occur when  $q \geq 3$  because, for example, the augmentations of the one-point corner trajectories  $T_a = [(0, 0)]$  and  $T_b = [(1, 0)]$  have the crossing point  $\mathbf{c} = (\frac{m^2}{m^2+1}, \frac{m}{m^2+1})$ .  $\square$

**Remark 6.6.** In the above formula the reader may find it useful to note that

$$\text{lcm}(m^2 + 1, m + 1) = \begin{cases} (m^2 + 1)(m + 1) & \text{if } m \text{ is even} \\ (m^2 + 1)(m + 1)/2 & \text{if } m \text{ is odd} \end{cases}.$$

This is because  $\text{lcm}(m^2 + 1, m + 1) = \text{lcm}(m^2 - m, m + 1)$ , and  $(m - 1)$ ,  $m$ , and  $(m + 1)$  only share a factor if  $m$  is odd, for which the common factor is 2.

The proof for the general case of pieces with orthogonal slopes  $(c, d)$  and  $(d, -c)$  can be approached similarly but the formula is not nearly as clean. Theorem 6.5 applies to the orthogonal nightrider with moves  $(2, 1)$  and  $(1, -2)$ .

**Corollary 6.7.** *Let  $\mathcal{B} = [0, 1]^2$  be the square board, and  $\mathbb{P}$  be the orthogonal nightrider. Then the denominator of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  is:*

$$\begin{cases} 1 & q = 1 \\ 2 & q = 2 \\ 20 & q = 3 \\ 15 \cdot 2^{q-1} & q \geq 4 \end{cases}.$$

**6.3. Slopes that sum to zero.** We analyze one more case—when the pieces  $\mathbb{P}$  have moves  $(c, d)$  and  $(-c, d)$ . In this case, the dynamical system is identical to billiards on a square board.

A key technique from polygonal billiards is the *unfolding* of a trajectory, where the polygon is reflected along edges that the trajectory encounters. (See, for example, Chapter 3 of [19].) Because the angle of incidence equals the angle of reflection, the trajectory lies along a single line in this unfolded path. (A visualization is given in Figure 8.)

**Proposition 6.8.** *Let  $\mathcal{B}$  be the square board and  $\mathbb{P}$  have moves with with rational slopes  $m_1$  and  $m_2$  satisfying  $m_2 = -m_1$ . There are no rigid cycles.*

*Proof.* Section 3.1 of [19] shows that on the square board, the orbit of every point  $\mathbf{b} \in \partial\mathcal{B}$  is finite. Therefore, trajectories that start at a corner end at a corner and every other trajectory is cyclical.

Suppose  $T = [\mathbf{b}_1, \dots, \mathbf{b}_l]$  is a cyclical trajectory, with associated hyperplane arrangement  $\mathcal{H} = \mathcal{H}(\mathbf{b}_1, \dots, \mathbf{b}_l)$ . Unfold  $T$  starting at  $\mathbf{b}_1$  along the line  $\ell$  defined by  $y = m_1x + b$  for some  $b \in \mathbb{R}$ . The integral horizontal and vertical lines ( $x = r$  and  $y = s$  for integers  $r$  and  $s$ ) that the line passes through correspond to the fixations in  $\mathcal{H}$ . Because  $T$  contains no corners of  $\mathcal{B}$ ,  $\ell$  does not pass through any points in the integer lattice, and therefore there is some  $\varepsilon > 0$  such that the line  $\ell'$  defined by  $y = m_1x + b + \varepsilon$  passes through the integral horizontal and vertical lines in the same order and correspond to the same fixations from  $\mathcal{H}$ . We conclude that the trajectory  $T'$  created by refolding  $\ell'$  has the same defining associated hyperplane arrangement as  $T$ , so  $T$  is not a rigid cycle by Lemma 3.8.  $\square$



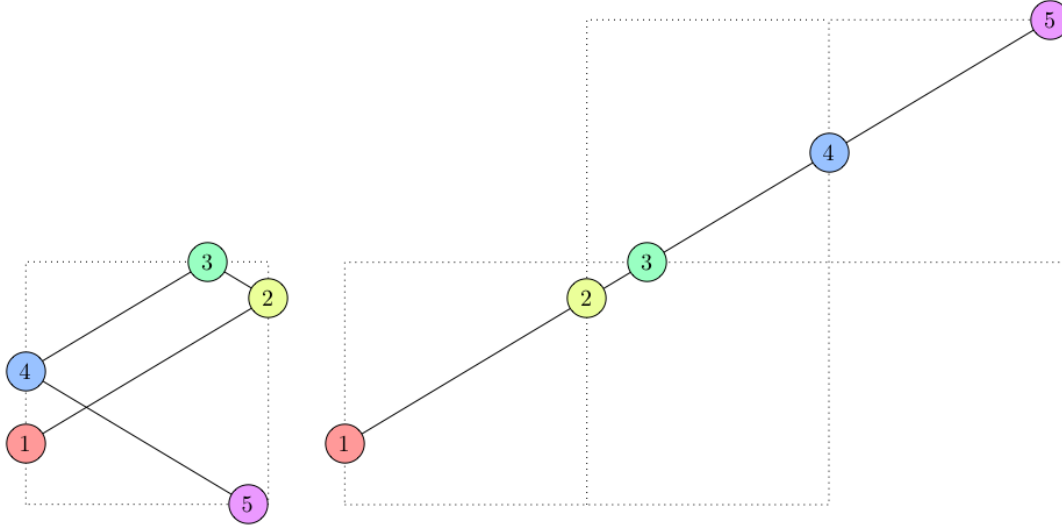


FIGURE 8. The unfolding of a trajectory for the piece with moves  $(5, 3)$  and  $(5, -3)$  starting at  $\mathbf{b}_1 = (0, \frac{1}{4})$ . The trajectory on the left becomes the single line on the right when the unit square is reflected along the edges that are encountered.

**Theorem 6.9.** *Let  $\mathcal{B}$  be the square board and  $\mathbb{P}$  be the piece with moves  $(c, d)$  and  $(c, -d)$ . Then  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  has denominator*

$$\begin{cases} 1 & q = 1 \\ \hat{d} & q = 2 \\ 2\hat{d} & 3 \leq q \leq \lceil \hat{d}/\hat{c} \rceil \\ 2\hat{c}\hat{d} & q \geq \lceil \hat{d}/\hat{c} \rceil + 1 \end{cases},$$

where  $\hat{c} = \min(|c|, |d|)$  and  $\hat{d} = \max(|c|, |d|)$ .

*Proof.* By symmetry, we only need to consider the case  $0 < c < d$ .

Without rigid cycles, the denominator of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  only depends on corner trajectories and the crossing points of their augmentations. Let  $T = [\mathbf{b}_1, \dots, \mathbf{b}_k]$  be the corner trajectory starting at  $\mathbf{b}_1 = (0, 0)$ . Unfold  $T$  to lie on the line  $\ell$  of slope  $d/c$  through  $(0, 0)$ . For  $1 \leq i \leq k$ , notate the image of  $\mathbf{b}_i$  under this unfolding to be  $\mathbf{b}'_i$ ; observe that  $\mathbf{b}_i$  and  $\mathbf{b}'_i$  have the same denominator. This denominator will either be  $c$  or  $d$  depending on whether  $\ell$  is intersecting a line of the form  $x = r$  (for which  $\mathbf{b}'_i = (r, \frac{dr}{c})$ ) or a line of the form  $y = s$  (for which  $\mathbf{b}'_i = (\frac{cs}{d}, s)$ ). The denominators of  $\mathbf{b}'_i$  will all be  $d$  until  $\ell$  meets the line  $x = 1$ . Therefore the contribution to the denominator from corner trajectories is 1 if  $q = 1$ ,  $d$  if  $1 < q \leq \lceil d/c \rceil$  and  $cd$  when  $q > \lceil d/c \rceil$ .

We must also determine the relevant crossing points of augmentations of (possibly concurrent) trajectories  $T_a = [\mathbf{a}_1, \dots, \mathbf{a}_k]$  and  $T_b = [\mathbf{b}_1, \dots, \mathbf{b}_l]$ . By the above reasoning, every point  $\mathbf{a}_i$  and  $\mathbf{b}_i$  is either of the form  $(\frac{v_i}{d}, u)$  or  $(u, \frac{w_i}{c})$  for  $u \in \{0, 1\}$  and integers  $v_i$  and  $w_i$ , and furthermore because the slopes have magnitude greater than one, at least one endpoint of the line segment between  $\mathbf{b}_i$  and  $\mathbf{b}_{i+1}$  (and  $\mathbf{b}_i$  and  $\mathbf{b}_{i+1}$ ) is of the latter form. This means that any crossing point  $\mathbf{c} = (x, y)$  can be found by solving two equations of the form

$$y - \frac{w_1}{c} = \frac{d}{c}(x - u_1) \text{ and } y - \frac{w_2}{c} = -\frac{d}{c}(x - u_2),$$

from which

$$x = \frac{du_1 + du_2 + w_2 - w_1}{2d} \text{ and } y = \frac{du_2 - du_1 + w_1 + w_2}{2c}.$$

Therefore, a crossing point of the augmentation of trajectories can not contribute anything to  $([0, 1]^{2q}, \mathcal{A}_{\mathbb{P}}^q)$  other than  $2cd$ .

A contribution of 2 will definitely occur when  $q \geq 3$  because we can see that the augmentations of the one-point corner trajectories  $T_a = [(0, 0)]$  and  $T_b = [(0, 1)]$  have the crossing point  $\mathbf{c} = (\frac{c}{2d}, \frac{1}{2})$ .

It remains to show that a contribution of  $c$  does not occur when  $c > 1$  and  $q \leq [d/c]$ . By symmetry we choose  $T_a$  to start at  $\mathbf{a}_1 = (0, 0)$  and consider the options for trajectories  $T_b$  where the lengths of  $T_a$  and  $T_b$  sum to at most  $[d/c] - 1$ . If  $T_b$  also starts at  $(0, 0)$ , then neither augmented flow reaches  $x = 1$  and no crossing points exist. If  $T_b$  starts at  $(0, 1)$ , again neither augmented flow reaches  $x = 1$  and the only crossing points are of the form  $\mathbf{c} = (r\frac{c}{2d}, \frac{1}{2})$  for odd integers  $r$ . If  $T_b$  starts at  $(0, 1)$  or  $(1, 1)$ , the augmentations of  $T_a$  does not reach far enough to the right to reach the augmentation of  $T_b$ . This concludes the proof.  $\square$

We now apply Theorem 6.9 to the lateral nightrider with basic moves  $(2, 1)$  and  $(2, -1)$ .

**Corollary 6.10.** *Let  $\mathcal{B}$  be the square board and  $\mathbb{P}$  be the lateral nightrider. Then the denominator of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  is*

$$\begin{cases} 1 & q = 1 \\ 2 & q = 2 \\ 4 & q \geq 3 \end{cases}.$$

## 7. OPEN QUESTIONS

The variables that determine the behavior of a particle's flow in mathematical billiards are the shape of the region as well as the initial position and initial direction of the particle. In our dynamical system, the key variables are the shape of the board, the slopes of the moves, and the initial position of the particle. The similarity between the behavior of the flows in the two dynamical systems leads to many open questions.

**7.1. Fruitful regions and moves.** In the study of convex billiards, circles, ellipses, and curves of constant width have produced beautiful results; as have rational polygons, where internal angles are rational multiples of  $\pi$  [19]. This leads to questions about which choices of board and moves are fruitful in our dynamical system.

**Question 7.1.** What properties of polygonal or general convex boards imply predictable behavior for some choices of moves?

**Question 7.2.** What restrictions on moves are more likely to produce predictable behavior on a wide variety of boards?

**7.2. Properties of trajectories.** In convex billiards, a classic unsolved question is whether every polygon has a periodic orbit, which has applications to the physics of point masses [15]. It is known that every rational polygon and every acute triangle has a periodic orbit. For square regions, it is further known that a billiard trajectory is periodic if the slope of the particle's initial direction is rational, and ergodic otherwise. We ask similar questions about our dynamical system and share our initial findings.

**Question 7.3.** Given a polygonal board  $\mathcal{B}$  (or an arbitrary convex board  $\mathcal{B}$ ), what conditions on the slopes  $m_1$  and  $m_2$  will ensure that there is a periodic orbit in  $\mathcal{B}$ ?

**Question 7.4.** For which choice of board  $\mathcal{B}$ , slopes  $m_1$  and  $m_2$ , and initial point  $\mathbf{b}$  is the extended trajectory through  $\mathbf{b}$  ergodic?

To apply Theorems 5.9 and 5.10, we must understand the periodic orbits and also be able to determine the rank of their corresponding cyclical trajectories. This leads to the following refinement of the Question 7.3.

**Question 7.5.** For which choice of board  $\mathcal{B}$  and slopes  $m_1$  and  $m_2$  does there exist a rigid cycle? And under which conditions is there a unique rigid cycle?

In Section 6 we provided information about these questions for the square board in several cases. However, the case when  $m_1$  and  $m_2$  have opposite signs is not fully understood.

Several types of dynamics have emerged in this case. The simplest situation is when all trajectories are cyclical. This occurs when  $m_2 = -m_1$  (see Section 6.3) and this also appears to occur when  $m_1 = \frac{1}{3}$  and  $m_2 = -\frac{2}{3}$ . (See Figure 9.)

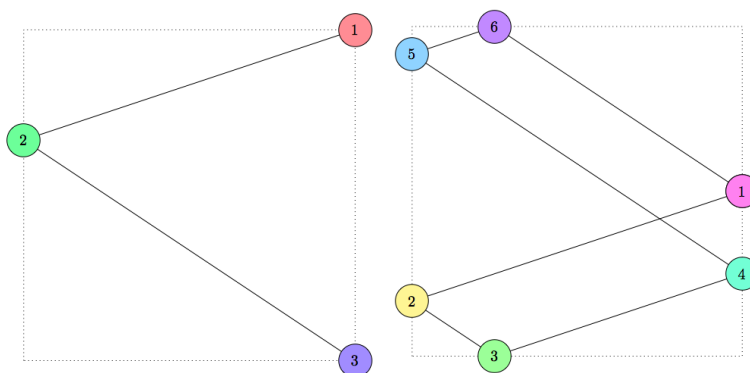


FIGURE 9.  $m_1 = \frac{1}{3}$  and  $m_2 = -\frac{2}{3}$ . The first trajectory begins at  $(1, 0)$ , while the second begins at  $(1, \frac{1}{2})$ . The other points we tested on  $\partial\mathcal{B}$  also have periodic orbits.

Convergent behavior also occurs, similar to what we saw in Figure 7 from Section 6.2 in which all trajectories converge to the same rigid cycle. When  $m_1 = \frac{3}{10}$  and  $m_2 = -\frac{4}{10}$ , trajectories converge to a single finite orbit, as shown in Figure 10.

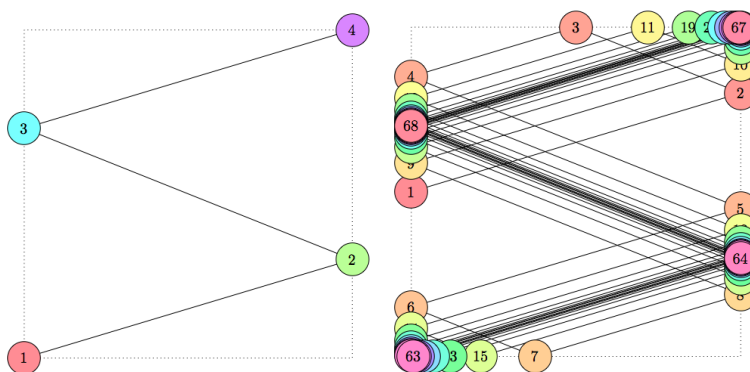


FIGURE 10.  $m_1 = \frac{3}{10}$  and  $m_2 = -\frac{4}{10}$ . The first trajectory begins at  $(0, 0)$ , and the second begins at  $(0, \frac{1}{2})$ . The points of the first trajectory seem to form the  $\omega$ -limit set of the second.

Ergodic behavior also arises when  $|m_1|$  and  $|m_2|$  are both less than 1. For example, when  $m_1 = \frac{1}{3}$  and  $m_2 = -\frac{1}{4}$ , it appears that the orbit of  $(0, 0)$  is dense in  $\partial\mathcal{B}$ —see Figure 11.

The variety of behaviors for pieces with slopes of opposite signs leads us to ask for a classification for these behaviors on the square board.

**Question 7.6.** Classify the behavior of trajectories on the square board for every choice of pieces with moves along slopes  $m_1$  and  $m_2$ . Under what conditions will there be a periodic orbit and what is it? Under what conditions will the behavior of the system be ergodic?

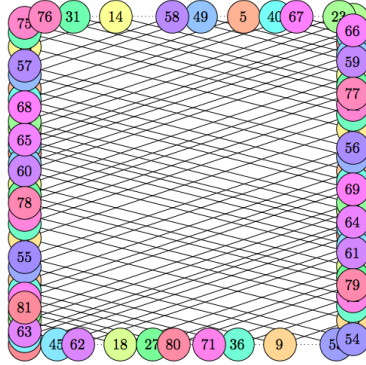


FIGURE 11.  $m_1 = \frac{1}{3}$  and  $m_2 = -\frac{1}{4}$ . These are the first 80 points in the orbit of  $(0, 0)$ , which appears to be dense in  $\partial\mathcal{B}$ .

We remark that in Sections 6.1 and 6.2 the dynamics do not depend on the rationality of  $m_1$  and  $m_2$ , but in Section 6.3 they do. We are not sure why this is the case.

**Question 7.7.** Which results hold for irrational slopes in addition to rational slopes?

**7.3. Generalizations of our dynamical system.** There are many ways that the discrete dynamical system for billiards generalizes; we wonder if our model can also be generalized further. First, we ask if it is possible to generalize the board  $\mathcal{B}$  to regions that are fruitful in the study of billiards.

**Question 7.8.** Can our dynamical system be generalized to non-convex regions? To hyperbolic models? To a system similar to outer billiards?

We also wonder if we can remove the restriction that there are only two moves.

**Question 7.9.** Is there a way to make sense of such a dynamical system involving more than two moves?

Could studying such a dynamical system be useful in the study of three-move riders, or riders with more moves? One possible way to allow for more moves is to require that the moves be applied in a cyclical fashion. When there are only two moves, the trajectory must always lie in the plane spanned by those two vectors. If one is able to find a way to involve more than two moves, the dynamical system may be able to generalize to higher dimensions.

**Question 7.10.** Is there a higher-dimensional analog of this dynamical system, similar to billiards in a polytope?

**7.4. Dynamical System Theory.** Inspired by dynamical systems theory we can ask about the stability of our dynamical system by perturbing the board, perturbing the set of moves, and perturbing the particle's initial position.

**Question 7.11.** How does a slight perturbation of the board impact the behavior of the trajectories? Of the existence or uniqueness of the rigid cycles? How do the changes depend on the piece's moves?

**Question 7.12.** How does a slight perturbation of the piece's move vectors impact the behavior of the trajectories? Of the existence or uniqueness of the rigid cycles? How do the changes depend on the board?

**Question 7.13.** Do two trajectories that start from sufficiently close points  $\mathbf{b}$  and  $\mathbf{b}'$  have the same behavior? If  $\mathbf{b}$  is periodic, must  $\mathbf{b}'$  be periodic? Must they have the same period?

A positive answer to the last question, for a specific board and set of moves, would prove that the corresponding cyclical trajectories are not rigid cycles, similar to the argument given in Proposition 6.8.

Crossing points of trajectories are central to the study of our dynamical system, but there does not appear to be much focus on them in the discrete dynamical system literature. Perhaps such a question can inspire new directions of research in existing dynamical systems.

**Question 7.14.** What are the coordinates of crossing points of trajectories in existing discrete dynamical systems, including billiards? For which discrete dynamical systems are the formulas of the coordinates of these crossing points easy to calculate? Do the denominators of these coordinates behave predictably?

**7.5. Periods and Denominators.** An important question in Ehrhart Theory is the relationship between the period of an Ehrhart quasipolynomial and the denominator of its corresponding polytope (or inside-out polytope).

We have found the denominator of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  for several classes of two-move riders  $\mathbb{P}$  when  $\mathcal{B}$  is the square board. This gives us provable bounds on the period of the Ehrhart quasipolynomial of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$ , and we can use this to explicitly compute  $u_{\mathbb{P}}(q; n)$  through brute force. This may give insight on the period of  $u_{\mathbb{P}}(q; n)$ .

**Question 7.15.** Is the period always equal to the denominator of  $(\mathcal{B}^q, \mathcal{A}_{\mathbb{P}}^q)$  when  $\mathbb{P}$  is a two-move rider?

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