LINEAR RECURRENCES THROUGH TILINGS AND MARKOV CHAINS

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ABSTRACT. We present a tiling interpretation for k-th order linear recurrences, which yields new combinatorial proofs for recurrence identities. Moreover, viewing the tiling process as a Markov chain also yields closed form Binet-like expressions for these recurrences.

1. INTRODUCTION

The theme of this paper is the use of tilings and a random tiling process as a general method for understanding and proving identities involving k-th order linear recurrences. Typically, such identities are proved by algebraic means (such as induction or generating functions) which generally give very little insight into their nature; by contrast, our combinatorial approach enables visual interpretations of such identities— facilitating a clearer understanding of them, unifying them, and making them (and their proofs) easy to remember.

A simple example of this approach is the well-known interpretation [4] of Fibonacci numbers (generated by the recurrence $f_n = f_{n-1} + f_{n-2}$, $f_0 = 1$, $f_1 = 1$) as the number of ways to tile an $n \times 1$ board using squares and dominoes. More generally, given non-negative integers G_0 and G_1 and the recurrence $G_n = G_{n-1} + G_{n-2}$ for $n \geq 2$, then G_n counts the number of ways to tile an $n \times 1$ board with squares and dominoes where the initial tile is assigned a *phase*. Specifically, if the initial tile is a domino, we can assign it one of G_0 phases and if the initial tile is a square, we can assign it one of G_1 phases. This interpretation is developed in [2], where it is used as a unifying method for proving a host of identities in [12]. In this paper, we develop a tiling interpretation for higher-order recurrences with non-negative integer coefficients and arbitrary initial conditions. We show how this interpretation can be used to prove associated recurrence identities as well as a new closed form expression for the n-th term of such recurrences, extending formulae of [5, 8, 9, 11].

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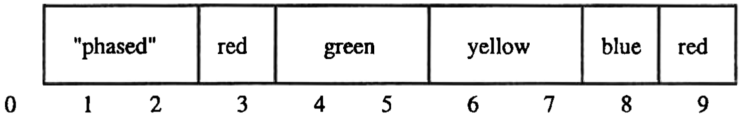

FIGURE 1. A 9-board tiled with colored squares and dominoes. The first tile is given a phase instead of a color.

The paper is organized as follows. In Section 2, we analyze 2nd-order recurrences for motivation, and demonstrate the usefulness of this combinatorial model in deriving several identities. This is followed in Section 3 by a general tiling interpretation for k-th order linear recurrences. Section 4 builds on this interpretation with a Markov chain model to derive a Binet-like formula for "3-bonacci numbers", and this is generalized in Section 5 to handle k-th order linear recurrences with special initial conditions, and further generalized in Section 6 to handle arbitrary initial conditions.

2. 2nd-Order Linear Recurrences

Let s, t, H_0, and H_1 be real numbers, and for $n \geq 2$, define

$$H_n = sH_{n-1} + tH_{n-2}. \tag{1}$$

When s, t, H_0 and H_1 are non-negative integers, then H_n can be given a combinatorial interpretation. We define an *n-board* to be an array of n cells, numbered 1 through n. See Figure 1 for a typical n-board, covered with colored square and domino tiles.

Theorem 1. *For $n \geq 1$, H_n counts the number of ways to cover an n-board with (length one) squares and (length two) dominoes, where all tiles, except for the initial one, are given a color. There are s colors for squares and t colors for dominoes. The initial tile is distinguished in a different way by assigning it a* phase *and there are H_1 phases for an initial square and tH_0 phases for an initial domino.*

Proof. The number of ways to tile a 1-board is H_1. The number of ways to tile a 2-board (with two squares or a single domino) is $sH_1 + tH_0 = H_2$. For $n > 2$, by conditioning on whether the last tile is a square or domino, we have $H_n = sH_{n-1} + tH_{n-2}$. □

This interpretation allows us to combinatorially explain many identities for sequences generated by second-order recurrences. We illustrate with several examples. In the identities that follow we assume that all quantities are non-negative. We shall relax this assumption in the next section.

Identity 1. $t\sum_{k=0}^{n} s^{n-k}H_k = H_{n+2} - s^{n+1}H_1$.

Proof. The right side of this identity counts the number ways to tile an $(n+2)$-board, excluding the tilings consisting of all squares. It remains to show that the left side counts the same quantity.

Specifically, we show that the left side counts the number of $(n+2)$-tilings where the last domino occupies cells $k+1$ and $k+2$ for some $0 \leq k \leq n$. For $1 \leq k \leq n$, there are H_k ways to tile cells 1 through k, t ways to color the domino on cells $k+1$ and $k+2$, and s^{n-k} ways to color the squares on cells $k+3$ to $n+2$. Consequently, there are $tH_k s^{n-k}$ such tilings. When $k=0$ there are tH_0 ways to choose the initial domino, and s^n ways to color the subsequent squares, resulting in $tH_0 s^n$ tilings. Altogether, we have $t\sum_{k=0}^{n} s^{n-k}H_k$ tilings of an $(n+2)$-board with at least one domino. □

When tiling a board of even length $2n$, the last square, if it exists, must cover an even cell $2k$ for some $1 \leq k \leq n$. The preceding squares are tiled H_{2k-1} ways and the last square and subsequent $n-k$ dominoes can be colored st^{n-k} ways. Consequently, we have

Identity 2. $H_{2n} = H_0 t^n + s\sum_{k=1}^{n} t^{n-k}H_{2k-1}$,

where the first term on the right side enumerates the all-domino tilings.

Similarly, when tiling a $(2n+1)$-board, a last square must exist at some cell $2k+1$ for some $0 \leq k \leq n$. Separating the $k=0$ case from the rest leads to

Identity 3. $H_{2n+1} = H_1 t^n + s\sum_{k=1}^{n} t^{n-k}H_{2k}$.

The next identity invites a more intricate, but natural interpretation on pairs of tilings.

Identity 4. $H_{2n}^2 = t^{2n}H_0^2 + s\sum_{k=1}^{2n} t^{2n-k}H_{k-1}H_k$.

Proof. The quantity on the left side counts the number of ordered pairs (A, B), where A and B are $(2n)$-tilings. The first term on the right side counts those (A, B) where A and B consist only of dominoes. For any $(2n)$-tiling X, let k_X be the last cell of tiling X covered by a square. If X is all dominoes, set k_X to infinity. For (A, B) to have at least one square, the minimum of k_A and k_B must be finite and even. Let $k = \max\{k_A, k_B - 1\}$. When k is even, A and B have dominoes covering cells $k+1$ through $2n$ and A has a square at cell k. In this way, the number of tilings (A, B) with even k is the number of ways to tile A times the number of ways to tile B, i.e., $H_{k-1}st^{(2n-k)/2} \cdot H_k t^{(2n-k)/2} = st^{2n-k}H_{k-1}H_k$. When k is odd, A has dominoes covering cells k through $2n$ and B has dominoes covering cells $k+2$ through $2n$ and a square at cell $k+1$, so the number of tiling pairs is $H_{k-1}t^{(2n-k+1)/2} \cdot H_k st^{(2n-k-1)/2} = st^{2n-k}H_{k-1}H_k$, the same expression as for even k. Altogether the number of tiling pairs (A, B) with at least one square is $s\sum_{k=1}^{2n} t^{2n-k}H_{k-1}H_k$. □

A similar approach easily leads to

Identity 5. $H_{2n+1}^2 = t^{2n}H_1^2 + s\sum_{k=2}^{2n+1} t^{2n+1-k}H_{i-1}H_k$.

The above formulae are just a few examples of identities for 2nd-order linear recurrences that can be easily assimilated, explained, and remembered by our combinatorial interpretation.

3. k-th Order Linear Recurrences

In this section we present a combinatorial interpretation of sequences generated by k-th order linear recurrences with non-negative integer coefficients. Specifically, by the reasoning and terminology of the last section (i.e., by conditioning on the last tile), we obtain the following theorem.

Theorem 2. *Given non-negative integers $c_1, c_2, \ldots, c_k$, $a_0, a_1, \ldots, a_{k-1}$, consider for $n \geq k$, the linear recurrence*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}. \tag{2}$$

Then for $n \geq 1$, a_n counts the number of ways to tile an n-board using colored tiles of various lengths where each tile, except the initial one, *has a color. Specifically, for $1 \leq i \leq k$, each tile of length i may be assigned any of c_i different colors. The initial tile is assigned a* phase *and if it has length $1 \leq i \leq k$, the number of phases for that tile is*

$$p_i = a_i - \sum_{j=1}^{i-1} c_j a_{i-j}. \tag{3}$$

In particular, an initial square has $p_1 = a_1$ phases, and by (2) and (3), it follows that $p_k = c_k a_0$. For k-th order recurrences where $k > 2$, this combinatorial interpretation is only valid when the initial conditions $a_0, a_1, \ldots, a_{k-1}$ are sufficiently "spread out" so that for $1 \leq i < k$, the number of phases p_i is non-negative. This restriction will be removed later in this section.

To demonstrate the utility of our combinatorial interpretation, we give new proofs of the following three identities for "generalized Tribonacci" sequences that were originally proved using matrix methods in [13].

Identity 6. *Consider the sequence generated by integers a_0, a_1, a_2 and for $n \geq 3$, $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3}$. Then for $n \geq 0$,*

$$c_1^n(c_3 a_0 + c_1 a_2) + (c_3 + c_1 c_2)\sum_{i=1}^{n} c_1^{n-i} a_i = c_1 a_{n+2} + c_3 a_n.$$

Proof. In this identity, we tile an $(n+3)$ board with squares, dominoes and triominoes. Each tile of length i, unless it is the first tile of our tiling, has c_i color choices. If the first tile has length i, then it can be phased p_i ways, where $p_1 = a_1$, $p_2 = a_2 - c_1 a_1$, and $p_3 = c_3 a_0$. For $n \geq 1$, (when $n = 0$, the statement is obvious), the right side of our identity counts the number of ways to tile a board of length $n+3$ such that the last tile is either a colored square (which can be preceded a_{n+2} ways) or a colored triomino (which can be preceded a_n ways).

To combinatorially interpret the left side, we first count, for $1 \le i \le n$, tilings whose last domino or triomino begins at cell $i+1$. There are $(c_3 + c_1c_2)c_1^{n-i}a_i$ such tilings since cells $i+1, i+2, i+3$ consists of either a triomino or a domino followed by a square ($c_3 + c_1c_2$ choices), the tiles from cell $i+3$ through $n+3$ must all be squares (c_1^{n-i} choices) and cells 1 through i may be tiled arbitrarily (a_i choices). Note that by our ending condition, a last domino may not begin at cell $(n+2)$. The only uncounted tilings are those with only squares on cells 4 through $n+3$ (c_1^n choices) and cells 1 through 3 contain either a phased triomino ($p_3 = c_3a_0$ choices) or have a square at cell 3 (c_1a_2 choices). Altogether, our tilings can be constructed in $c_1^n(c_3a_0 + c_1a_2) + (c_3 + c_1c_2)\sum_{i=1}^{n} c_1^{n-i}a_i$ ways. □

In similar fashion, we can also prove for sequences $\{a_n\}$ generated by the same recurrence and initial conditions, that

Identity 7. *For* $n \ge 1$,

$$c_2^n(a_2 - c_1a_1) + (c_3 + c_1c_2)\sum_{i=1}^{n} c_2^{n-i}a_{2i-1} = c_2a_{2n} + c_3a_{2n-1}.$$

Proof. Here the right side of the identity counts the $(2n+2)$-tilings that are restricted to end with a domino or triomino. For the left side, we first count, for $1 \le i \le n$, those tilings whose last square or triomino begins at cell $2i$. Such a tiling has cells $2i$, $2i+1$, and $2i+2$ consisting of either a triomino or a square followed by a domino ($c_3 + c_1c_2$ choices) preceded by an arbitrary $(2i-1)$-tiling (a_{2i-1} choices) and followed by all dominoes (c_2^{n-i} choices). The only uncounted tilings are the all-domino tilings ($p_2c_2^n = (a_2 - c_1a_1)c_2^n$ choices). Altogether our tilings can be constructed in $c_2^n(a_2 - c_1a_1) + (c_3 + c_1c_2)\sum_{i=1}^{n} c_2^{n-i}a_{2i-1}$ ways. □

By a similar argument, this time with boards of odd length, we obtain:

Identity 8. *For* $n \ge 1$,

$$c_2^{n-1}(c_3a_0 + c_2a_1) + (c_3 + c_1c_2)\sum_{i=1}^{n-1} c_2^{n-1-i}a_{2i} = c_2a_{2n-1} + c_3a_{2n-2}.$$

Finally, we illustrate the power of the combinatorial approach by establishing the next identity, proved by more sophisticated methods in [7] and [8]:

Identity 9. *Let g_n be the k-th order Fibonacci sequence defined by $g_0 = 1$ and, for $1 \le n < k$, $g_n = g_{n-1} + g_{n-2} + \cdots + g_0$. For $n \ge k$, $g_n = g_{n-1} + g_{n-2} + \cdots + g_{n-k}$. Then for $n \ge 0$,*

$$g_n = \sum_{n_1}\sum_{n_2}\cdots\sum_{n_k}\binom{n_1+n_2+\cdots+n_k}{n_1,n_2,\ldots,n_k},$$

where the summation is over all non-negative integers $n_1, n_2, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n$.

Proof. Here g_n counts the number of ways to tile an n-board with (colorless and phaseless) tiles of length at most k. (This may be seen directly by conditioning on the last tile or one can derive from equation (3) that for $1 \le j \le k$, $c_j = 1$ and $p_j = 2^{j-1} - (2^{j-1} - 1) = 1$.) The right side of this identity conditions on how many such tilings use exactly n_i tiles of length i for $1 \le i \le k$. To be non-zero, the sum of the lengths of the tiles must be n. The number of ways to arrange these $n_1 + n_2 + \cdots + n_k$ tiles is given by the multinomial coefficient. □

We proved Identities 1 through 5 under the assumption that the initial conditions were non-negative, and Identities 6 through 8 used the stronger assumption that the initial conditions were sufficiently spread out so that $p_i \ge 0$ for $1 \le i \le k-1$. We conclude this section by demonstrating that, by exploiting linearity, these identities remain true for arbitrary real (or complex) initial conditions.

For any given numbers a and b, let $\mathbf{H}$ denote the set of all sequences $(H_0, H_1, \ldots)$ that satisfy the recurrence of equation (1), where the initial conditions H_0 and H_1 are arbitrary real (or complex) numbers. Then $\mathbf{H}$ is a two-dimensional real vector space, with basis sequences $H(1,0)$ and $H(0,1)$ where $H(x,y)$ is the sequence in $\mathbf{H}$ with initial conditions $H_0 = x$ and $H_1 = y$. The function $L : \mathbf{R}^2 \to \mathbf{H}$ is linear, where $L(x,y) = H(x,y)$. Many identities can be viewed as a linear function $I : \mathbf{H} \to R$. For example, Identity 1 can be viewed as $I : \mathbf{H} \to \mathbf{R}$ defined for $H \in \mathbf{H}$ by $I(H) = t\sum_{k=0}^{n} s^{n-k}H_k - H_{n+2} - s^{n+1}H_1$. Identity 1 asserts that $I(H) = 0$ for all $H \in \mathbf{H}$ where H is of the form $H(x,y)$ with x and y non-negative integers. Since the composed linear function $I \circ L : \mathbf{R}^2 \to \mathbf{R}$ is equal to 0 for basis vectors $(1,0)$ and $(0,1)$, then the identity is true for all initial conditions. The same argument applies to Identities 2 and 3.

To extend this reasoning to linear k-th order recurrences like Identites 6, 7, and 8, we simply need to find a basis of non-negative integer vectors that are sufficiently "spread out". This can always be done. For instance, if $k = 3$ in recurrence (2), a suitable basis would be $\{(0,0,1), (0,1,c_1), (1,c_1,c_1^2 + c_2)\}$. Thus Identities 6, 7, and 8 are valid for any initial conditions.

Finally, identities such as 4 and 5 can be viewed as quadratic functions on $\mathbf{H} \times \mathbf{H}$, that is they are of the form $Q(H', H'') = 0$ where $H', H'' \in \mathbf{H}$, and are linear in both H' and H''. Thus if the identity holds for any pair of basis vectors in $\mathbf{H}$, (e.g., when H' and H'' begin with $(0,1)$ or $(1,0)$) then the identity holds for all initial conditions. A similar argument can be made for quadratic identities for linear k-th order recurrences when $k > 2$.

4. Binet's Formula for 3-bonacci Tilings

So far the tiling interpretation of k-th order recurrences has yielded natural combinatorial proofs of several identities for such recurrences. We now show how our combinatorial interpretation, together with a stochastic element, even allows us to prove recurrence identities involving irrational numbers.

A closed-form expression for the n-th Fibonacci number (where $f_0 = f_1 = 1$) is given by Binet's formula:

$$f_n = \frac{1}{\sqrt{5}}\left[\phi^{n+1} - \left(\frac{-1}{\phi}\right)^{n+1}\right], \tag{4}$$

where ϕ is the golden mean $(1+\sqrt{5})/2$. A novel proof can be obtained through a combinatorial interpretation of f_n as the number of ways to tile an n-board using squares and dominoes. See [1]. The above formula, which can be generalized to handle Fibonacci recurrences with arbitrary initial conditions, then arises by interpreting the tiling as a process in which a square or domino is laid sequentially on a board that is infinitely long.

We define $F_{k,n}$ to be the number of ways of tiling an n-board with two types of tiles: squares and k-ominoes (a tile that covers k cells). Naturally, for $0 \le n \le k-1$, $F_{k,n} = 1$ and for $n \ge k$ $F_{k,n} = F_{k,n-1} + F_{k,n-k}$. Now we show how a random tiling process yields a formula analogous to Equation (4) for $F_{3,n}$, the "3-bonacci" numbers (as opposed to the "tribonacci" numbers that satisfy $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ [10]). This will motivate the analysis in the following sections, in which more general recurrences are handled.

Suppose we are given an infinitely long board with cells numbered $1, 2, 3, \ldots$, which we shall cover with squares and triominoes in a random manner. Specifically, starting at cell 1, we place a square with probability $1/\tau_1$ and place a triomino with probability $1/\tau_1^3$, where τ_1 is the (unique) real root of

$$\frac{1}{\tau} + \frac{1}{\tau^3} = 1.$$

This ensures that the probability that our tiling begins with a specific length n tiling is τ_1^{-n}, regardless of how many squares or triominoes are used. We see that τ_1 satisfies the characteristic equation

$$\tau^3 - \tau^2 - 1 = 0. \tag{5}$$

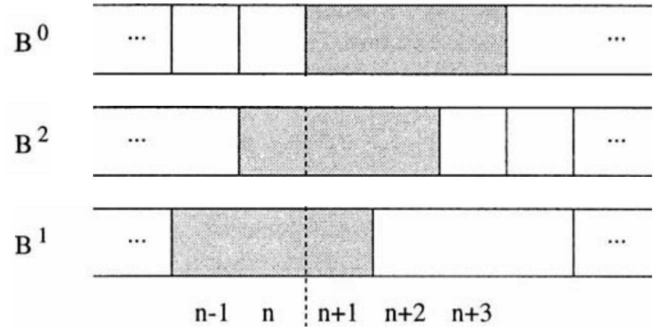

FIGURE 2. Examples of the three Markov Chain states at cell n.

Descartes's rule of signs shows that a positive real root τ_1 of this equation exists and is unique (the number of positive real roots is bounded by, and of the same parity as, the number of sign changes in the coefficients. We denote the other two (complex) roots of this equation by τ_2 and τ_3.

We say that a tiling is *breakable at cell n* if a new tile begins at cell $(n+1)$. For example, the tiling in Figure 1 is breakable at cells $0, 2, 3, 5, 7, 8$. Let q_n denote the probability that the tiling is breakable at cell n. Since there are $F_{3,n}$ ways to tile the first n cells, and each such tiling has probability $1/\tau_1^n$ of occurring, we have

$$q_n = \frac{F_{3,n}}{\tau_1^n}. \tag{6}$$

We determine q_n (and hence $F_{3,n}$) using a stochastic model. The process of randomly placing tiles as we advance one unit along the board can be described by a Markov chain that moves between three states: B^0 (breakable at the current cell), B^1 (a triomino ends one cell later), and B^2 (a triomino ends two cells later). (See Figure 2.)

The matrix of transition probabilities is:

$$P = \begin{array}{c} \\ B^2 \\ B^1 \\ B^0 \end{array} \begin{array}{c} \begin{array}{ccc} B^2 & B^1 & B^0 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{\tau_1^3} & 0 & \frac{1}{\tau_1} \end{pmatrix} \end{array}$$

where p_{ij} is the probability of going from state i to state j. The 1s in the first two rows occur because once a triomino is placed, the next two states in the process are determined. If the current state is B^2, then the next state must be B^1. If the current state is B^1, then the next state must be

B^0. If the current state is B^0, then the next state is B^2 if a triomino is placed (as in Figure 2), or B^0 if a square is placed after the break. At time (cell) 0, the chain begins in the breakable state. So q_n, the probability that this tiling is breakable at cell n, is the $(3,3)$ entry of P^n. By diagonalizing P, we obtain:

$$P^n = \begin{bmatrix} 1 & 1 & 1 \\ \frac{\tau_2}{\tau_1} & \frac{\tau_3}{\tau_1} & 1 \\ \frac{\tau_2^2}{\tau_1^2} & \frac{\tau_3^2}{\tau_1^2} & 1 \end{bmatrix} D^n \begin{bmatrix} \frac{\tau_1\tau_3}{(\tau_2-\tau_1)(\tau_2-\tau_3)} & \frac{\tau_1(\tau_1+\tau_3)}{(\tau_2-\tau_1)(\tau_2-\tau_3)} & \frac{\tau_1^2}{(\tau_2-\tau_1)(\tau_2-\tau_3)} \\ \frac{\tau_1\tau_2}{(\tau_3-\tau_1)(\tau_3-\tau_2)} & \frac{\tau_1(\tau_1+\tau_2)}{(\tau_3-\tau_1)(\tau_3-\tau_2)} & \frac{\tau_1^2}{(\tau_3-\tau_1)(\tau_3-\tau_2)} \\ \frac{\tau_2\tau_3}{(\tau_1-\tau_2)(\tau_1-\tau_3)} & \frac{\tau_1(\tau_2+\tau_3)}{(\tau_1-\tau_2)(\tau_1-\tau_3)} & \frac{\tau_1^2}{(\tau_1-\tau_2)(\tau_1-\tau_3)} \end{bmatrix},$$

where

$$D = \begin{bmatrix} \left(\frac{\tau_2}{\tau_1}\right) & 0 & 0 \\ 0 & \left(\frac{\tau_3}{\tau_1}\right) & 0 \\ 0 & 0 & \left(\frac{\tau_1}{\tau_1}\right) \end{bmatrix}.$$

The $(3,3)$ entry of P^n simplifies to

$$q_n = \frac{1}{\tau_1{}^n}\left[\frac{\tau_1^{n+1}}{3\tau_1 - 2} + \frac{\tau_2^{n+1}}{3\tau_2 - 2} + \frac{\tau_3^{n+1}}{3\tau_3 - 2}\right].$$

It follows directly from (6) that

$$F_{3,n} = \frac{\tau_1^{n+1}}{3\tau_1 - 2} + \frac{\tau_2^{n+1}}{3\tau_2 - 2} + \frac{\tau_3^{n+1}}{3\tau_3 - 2}, \tag{7}$$

giving a closed form expression for $F_{3,n}$ in terms of the roots of Equation (5).

5. A Binet-like Formula for k-th Order Linear Recurrences

We next observe how our Markov chain model changes when we consider the generalized tiling interpretation of k-th order linear recurrences described in Section 3. We first describe these for "ideal" initial conditions, then extend our results to arbitrary conditions. The ideal initial conditions arise from setting $p_i = c_i$ for $1 \leq i \leq k$ in the tiling interpretation for such recurrences. Equivalently, for a given k-th order linear recurrence, we define $a_j = 0$ for $j < 0$, $a_0 = 1$, and for $n \geq 1$

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}. \tag{8}$$

This recurrence has characteristic polynomial

$$f(x) = x^k - c_1 x^{k-1} - c_2 x^{n-2} - \cdots - c_{k-1} x - c_k. \tag{9}$$

Let $\mu = \mu_1$ denote the unique positive real root of Equation (9) (which exists by Descartes's rule of signs), and let $\mu_2, \mu_3, \ldots, \mu_k$ denote the other roots. We consider only the case where the roots are all distinct.

As in the last section, we now create a random tiling of an infinitely long board. We begin by placing a random colored tile beginning at cell 1. For $1 \leq i \leq k$, such a tile will have length i with probability c_i/μ^i, and the color will be chosen at random (uniformly) from the c_i available colors. (Thus any colored tile of length i has probability $1/\mu^i$ of being selected.) All subsequent tiles will be chosen randomly and independently with these probabilities. Notice that these probabilities sum to 1 since $\sum_{i=1}^{k} \frac{c_i}{\mu^i} = 1$ follows from Equation (9).

As before, let q_n denote the probability that the tiling is breakable at cell n. Since there are a_n ways to tile cells 1 through n, each with probability $1/\mu^n$, we have

$$q_n = \frac{a_n}{\mu^n}. \tag{10}$$

Mouline and Rachidi [6] study the asymptotic behavior of this expression; in contrast, we are concerned with obtaining an exact Binet-like expression for q_n.

The Markov chain represented by our tiling consists of k states: B^0, B^1, $B^2, \ldots, B^{k-1}$, where B^0 is the state in which the tiling is breakable at the current cell, and B^i is the state in which the current tile ends after i more cells. The matrix P of transition probabilities is:

$$P = \begin{array}{c} \\ B^{k-1} \\ B^{k-2} \\ B^{k-3} \\ \vdots \\ B^1 \\ B^0 \end{array} \begin{array}{c} \begin{array}{cccccc} B^{k-1} & B^{k-2} & B^{k-3} & B^{k-4} & \cdots & B^0 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ 0 & 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & & 0 & 1 \\ \frac{c_k}{\mu^k} & \frac{c_{k-1}}{\mu^{k-1}} & \frac{c_{k-2}}{\mu^{k-2}} & \cdots & \frac{c_2}{\mu^2} & \frac{c_1}{\mu} \end{pmatrix} \end{array}$$

where p_{ij} is the probability of going from state i to state j.

A similar matrix appears in Kalman [5], where it is derived by the matrix representation of a k-th order linear recurrence. We follow a similar analysis to derive an expression for q_n. At time (cell) 0, the chain begins in the breakable state. Hence q_n, the probability that the tiling is breakable at cell n, is the (k, k) entry of P^n:

$$q_n = [0, 0, 0, \ldots, 1] P^n [0, 0, 0, \ldots, 1]^{\mathsf{T}}. \tag{11}$$

The eigenvalues λ_i of the matrix P are determined by taking the determinant of $\lambda I - P$, which yields:

$$(\lambda\mu)^k - c_1(\lambda\mu)^{k-1} - c_2(\lambda\mu)^{k-2} - \cdots - c_{k-1}(\lambda\mu) - c_k = 0. \tag{12}$$

This expression for λ shows that $(\lambda\mu)$ satisfies the characteristic equation (9), so the k eigenvalues of P are related to the k roots of (9) by:

$$\lambda_i = \mu_i/\mu,$$

for $1 \leq i \leq k$. By equation (12),the vector $[1, \lambda_i, \lambda_i^2, ..., \lambda_i^{k-1}]^\mathsf{T}$ is an eigenvector corresponding to λ_i. Using the Vandermonde array

$$S = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_k \\ {\lambda_1}^2 & {\lambda_2}^2 & {\lambda_3}^2 & \cdots & {\lambda_k}^2 \\ \vdots & & & & \vdots \\ {\lambda_1}^{k-1} & {\lambda_2}^{k-1} & {\lambda_3}^{k-1} & \cdots & {\lambda_k}^{k-1} \end{bmatrix},$$

and the diagonal matrix D (with $d_{ii} = \lambda_i$), we can diagonalize $P = SDS^{-1}$. Using this diagonalization and (11) we have

$$q_n = [0, 0, 0, \ldots, 1]SD^nS^{-1}[0, 0, 0, \ldots, 1]^\mathsf{T}.$$

The product of the first three matrices is $[\lambda_1^{n+k-1}, \lambda_2^{n+k-1}, \ldots, \lambda_k^{n+k-1}]$. Letting the remaining two matrices be represented by $[y_1, y_2, \ldots, y_k]^\mathsf{T}$, we have

$$q_n = \sum_{i=1}^{k} y_i \lambda_i^{n+k-1}. \tag{13}$$

We can find the y_i by solving $S[y_1, y_2, y_3, \ldots, y_k]^\mathsf{T} = [0, 0, 0, \ldots, 1]^\mathsf{T}$. Cramer's Rule gives

$$y_i = \frac{\mu^{k-1}}{\prod_{j \neq i}(\mu_i - \mu_j)}. \tag{14}$$

Note that the denominator can be expressed as $f'(\mu_i)$, where f is the characteristic polynomial in equation (9). Combining (10), (13), and (14),

$$a_n = \sum_{i=1}^{k} \frac{\mu_i^{n+k-1}}{f'(\mu_i)} \tag{15}$$

Our Markov chain method has thus yielded a closed form Binet-like expression for the recurrence a_n in terms of the roots of the characteristic equation. Since we have assumed that the roots are distinct, $f'(\mu_i) \neq 0$. This equation was derived in [5, 9] using a recurrence matrix rather than a Markov chain approach.

As a specific case of this formula, consider a tiling of a board using (colorless and phaseless) squares and k-ominoes. This yields the characteristic equation

$$x^k - x^{k-1} - 1 = 0$$

whose roots μ_i are manifested in the Binet-like formula

$$F_{k,n} = \sum_{i=1}^{k} \frac{\mu_i^{n+1}}{k\mu_i - k + 1}.$$

This formula recovers (7) in the case $k = 3$ and the original Binet formula (4) when $k = 2$.

6. Extension to Arbitrary Initial Conditions

Let c_i, $1 \le i \le k$, be nonnegative integers, and let $A_0, A_1, \ldots, A_{k-1}$ be integers. Consider the sequence $\alpha_i = A_i$ for $0 \le i \le k$ and, for $n \ge k$, let

$$\alpha_n = c_1\alpha_{n-1} + c_2\alpha_{n-2} + \cdots + c_{k-1}\alpha_{n-k+1} + c_k\alpha_{n-k}. \tag{16}$$

We now prove a Binet-like formula for this very general recurrence. Theorem 3 generalizes Equation (15) and extends the formulae found in [5, 8, 9].

Theorem 3. *Given a recurrence of the form (16), with initial conditions* A_m, $0 \le m \le k-1$,

$$\alpha_n = \sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{m=1}^{j} \frac{c_j\ A_{k-m}\ \mu_i^{n+m-j-1}}{f'(\mu_i)} \tag{17}$$

holds whenever the characteristic polynomial $f(x) = x^k - c_1x^{k-1} - c_2x^{n-2} - \cdots - c_{k-1}x - c_k$ *has distinct roots* μ_i, $1 \le i \le k$.

The denominator $f'(\mu_i)$ vanishes if and only if μ_i is a repeated root, so the expression (17) is always valid when it is defined.

Note that for the Fibonacci sequence where $k = 2$, $c_1 = c_2 = 1$, $A_0 = A_1 = 1$, the inner two sums in (17) become

$$\frac{1}{2\mu_i - 1}\left[\mu_i^{n-1} + (\mu_i^{n-2} + \mu_i^{n-1})\right] = \frac{1}{2\mu_i - 1}\left[\mu_i^{n-1} + \mu_i^{n}\right] = \frac{\mu_i^{n+1}}{2\mu_i - 1}.$$

Noting for $i = 1, 2$ that $2\mu_i - 1 = \pm\sqrt{5}$, and summing over i, then recovers Equation (4).

This theorem will be proved using a set of "basis sequences" $e_n^0, e_n^1, \ldots, e_n^{k-1}$ satisfying (16). Then every series satisfying (16) can be represented as a linear combination of these basis series.

Proof. The set of all sequences that satisfy the recurrence (16) forms a k-dimensional vector space. Each sequence is completely determined by its first k terms.

Let e_n^i be the n-th term of the sequence determined by the initial conditions $e_i^i = 1$ and $e_j^i = 0$ for all other j in $0, 1, \ldots, k-1$. (Thus the vector of initial conditions has a 1 in the i-th position and zeroes elsewhere.) Any

k-th-order linear recurrence with initial conditions $A_0, A_1, \ldots, A_{k-1}$ can then be represented as

$$a_n = A_0 e_n^0 + A_1 e_n^1 + A_2 e_n^2 + \cdots + A_{k-1} e_n^{k-1} \tag{18}$$

The basis series e_n^i can be expressed as a linear combination of "shifts" of the specific recurrence a_n studied in the last section, arising from setting $p_i = c_i$ in the tiling interpretation. We list the first few terms below:

Term	a_n
0	1
1	c_1
2	$c_1 a_1 + c_2$
3	$c_1 a_2 + c_2 a_1 + c_3$
$k-2$	$c_1 a_{k-3} + c_2 a_{k-4} + \cdots + c_{k-2}$
$k-1$	$c_1 a_{k-2} + c_2 a_{k-3} + \cdots + c_{k-2} a_1 + c_{k-1}$

Note that this series can also be obtained by setting $a_0 = 1$ and $a_{-1} = a_{-2} = \cdots = a_{1-k} = 0$ and using the recurrence (8) to generate the later terms. Using these negative-indexed terms and the table above, observe that for all $n \geq 1$,

$$e_n^0 = a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_{k-1} a_{n-k+1} = c_k a_{n-k}.$$

In general, we can express the basis series e_n^i, for $0 \leq i \leq k-1$, in terms of $(k-i)$ "shifts" of the sequence a_n:

$$\begin{aligned} e_n^i &= a_{n-i} - c_1 a_{n-i-1} - \cdots - c_{k-i-1} a_{n-k+1} \\ &= c_{k-i} a_{n-k} + \cdots + c_k a_{n-k-i}, \end{aligned} \tag{19}$$

where the last equality follows from Equation (8) applied to a_{n-i}.

Hence by equations (16), (19), and (15),

$$\begin{aligned}
\alpha_n &= \sum_{\ell=0}^{k-1} A_\ell \, e_n^\ell \\
&= \sum_{\ell=0}^{k-1} \sum_{j=k-\ell}^{k} A_\ell \, c_j \, a_{n-\ell-j} \\
&= \sum_{\ell=0}^{k-1} \sum_{j=k-\ell}^{k} \sum_{i=1}^{k} A_\ell \, c_j \, \frac{\mu_i^{n-\ell-j+k-1}}{f'(\mu_i)} \\
&= \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{\ell=k-j}^{k-1} \frac{A_\ell \, c_j \, \mu_i^{n-\ell-j+k-1}}{f'(\mu_i)} \\
&= \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{m=1}^{j} \frac{A_{k-m} \, c_j \, \mu_i^{n+m-j-1}}{f'(\mu_i)}
\end{aligned}$$

as desired. □

7. Discussion

Our combinatorial interpretation of linear recurrences as solutions to tiling problems gives a powerful method for understanding recurrence identities. This approach allows one to quickly assimilate and visually interpret recurrence identities as well as their proofs. Moreover, an associated Markov chain on tilings even allows one to recover identities that at first glance to not appear to be combinatorial, such as the "Binet-like" formula of Theorem 3. Our tiling and random tiling intepretations are a unifying approach to understanding k-th order linear recurrences.

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