A q-QUEENS PROBLEM. VII. COMBINATORIAL TYPES OF NONATTACKING CHESS RIDERS

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Abstract. On a convex polygonal chessboard, the number of combinatorial types of nonattacking configuration of three identical chess riders with r moves, such as queens, bishops, or nightriders, equals $r(r^2+3r-1)/3$, as conjectured by Chaiken, Hanusa, and Zaslavsky (2019). Similarly, for any number of identical 3-move riders the number of combinatorial types is independent of the actual moves.

1. Combinatorial Types

Consider a chessboard, say an $n \times n$ square board, and a chess piece $\mathbb{P}$ resembling the queen, bishop, and rook in that its moves along specified lines are unlimited in extent. Such pieces are known as *riders* in fairy chess;[1] an example is the nightrider, which moves any distance in the directions of a knight's move.

Now place several (identical) copies of $\mathbb{P}$ on the board in a nonattacking configuration, i.e., no piece lies on a move line of another piece. The piece $\mathbb{P}_i$ in a particular location on the board divides the board into open regions by its move lines, each region determined by the two move lines of $\mathbb{P}_i$ that bound it. The other pieces must be inside some of these regions, as they cannot be on the move lines. Two nonattacking configurations with the same number of pieces are said to have the same *combinatorial type* if for each pair $\mathbb{P}_i$ and $\mathbb{P}_j$, $\mathbb{P}_j$ lies in the same region of the board with respect to $\mathbb{P}_i$ in both configurations. For example, there are three combinatorial types of configuration for two copies of a 3-move piece, shown in Figure 1.

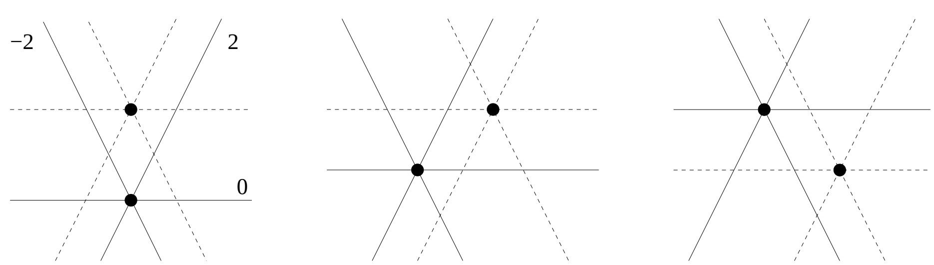

Figure 1. The three combinatorial types of two identical nonattacking riders with three moves along lines of slope 0 and ±2.

How many combinatorial types of nonattacking configuration are there? Call this number $t_{\mathbb{P}}(q)$. *A priori*, the answer could depend on the moves of $\mathbb{P}$, on the number q of pieces, and on the board. Happily, it turns out that the board itself does not matter because every possible combinatorial type can be realized on any sufficiently large board. The set of moves and the value of q remain as relevant variables, which still are relatively complicated information.

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[1]Chess with varied pieces, rules, or boards.

$q\backslash r$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	3	4	5	6
3	1	6	17	36	65	106
4	1	24	151^*	$574_{\mathbb{Q}}{}^*$	?	?
5	1	120	1899^*	$14206_{\mathbb{Q}}{}^*$	?	?
6	1	720	31709^*	$501552_{\mathbb{Q}}{}^*$	?	?

TABLE 1. The number of combinatorial types of nonattacking configuration of q identical riders with r moves.* indicates a number that was computed from Kotěšovec's empirical formulas. The $_{\mathbb{Q}}$ indicates a number that has been computed for queens but may depend on the piece. The numbers marked by ? are unknown and may depend on the piece.

Let us review the known data (Table 1). The number of combinatorial types is known for very few pieces and pieces with very few moves. Suppose there are r move lines. It is easy to see that $t_{\mathbb{P}}(1) = 1$ and $t_{\mathbb{P}}(2) = r$ [1, Theorem 5.6]. One might expect $t_{\mathbb{P}}(q)$ to depend on the exact set of moves for larger numbers of pieces, but Chaiken and we conjectured in [2, Conjecture 4.4] that for any rider piece with r moves, $t_{\mathbb{P}}(3) = r(r^2 + 3r - 1)/3$. Here we prove that conjecture. We also show that when pieces have only three moves the value of $t_{\mathbb{P}}(q)$ does not depend on the exact moves, no matter how many pieces there are.

On the other hand, $t_{\mathbb{P}}(q)$ could depend on the move set when $q, r \geq 4$. But data is hard to come by. It is hard to compute values by direct counting when q or r is not tiny, i.e., ≤ 2. For $q \geq 3$ or $r \geq 3$ we get $t_{\mathbb{P}}(q)$ from a general theorem. A *quasipolynomial* function $f(n)$ is a function given by a cyclically repeating sequence of polynomials.

Lemma 1 ([1, Theorems 4.1 and 5.3]). *The number $u_{\mathbb{P}}(q; n)$ of nonattacking configurations of q unlabelled copies of $\mathbb{P}$ on an $n \times n$ square board is given by a quasipolynomial function of n of degree $2q$.*

The number of combinatorial types is equal to $u_{\mathbb{P}}(q; -1)$.

Applying Lemma 1 requires knowing the quasipolynomial formula for $u_{\mathbb{P}}(q; n)$. That is hard, and few such formulas are known. We rely on Kotěšovec's heuristic results, especially [3], for most of them. (We have found that, when they can be checked rigorously, as in [2] and its related papers, Kotěšovec's formulas are always correct.) Table 1 shows the results.

2. BACKGROUND

We need precise definitions. The *board* $\mathcal{B}$ is a closed, convex polygonal region in the plane. (In [1, 2] the vertices are assumed rational; here that is unnecessary.) The working board of order n, on which we place pieces, is $(n+1)\mathcal{B}^\circ \cap \mathbb{Z}^2$, $\mathcal{B}^\circ$ denoting the interior of $\mathcal{B}$. For example, take the square board $\mathcal{B} = [0, 1]^2$; then $(n + 1)\mathcal{B}^\circ \cap \mathbb{Z}^2 = [n]^2$ (where $[n]$ means $\{1, 2, \dots, n\}$), which is indeed the ordinary $n \times n$ square chessboard.

The *move set* of a piece $\mathbb{P}$ is the set $\mathbf{M} = \{m_1, \dots, m_r\}$ of integer vectors $m_j = (c_j, d_j)$, the *basic moves*, such that $\mathbb{P}$ can move by any amount λm_j that takes it to an integral point in $\mathcal{B}$. Most important is the *slope*, $\mu_j = d_j/c_j$, a rational number for real pieces; but for counting combinatorial types it need not be rational, as we explain below (Lemma 3).

Given a piece $\mathbb{P}$ with move set $\mathbf{M} = \{m_1, \ldots, m_r\}$, place it at $(x_0, y_0) \in \mathbb{Z}^2$. The points (x, y) that $\mathbb{P}$ attacks are those that satisfy the equation $(x - x_0, y - y_0) = \lambda m_j$ for some $m_j = (c_j, d_j) \in \mathbf{M}$ and real number λ. This equation defines a line l_j through (x_0, y_0) of slope $\mu_j = d_j/c_j$, which we call a *move line* of $\mathbb{P}$. The r move lines form an arrangement of lines that we call the *move-line arrangement* of $\mathbb{P}$, written $\mathscr{M}(\mathbb{P})$. The move-line arrangement forms $2r$ regions, each bounded by two move lines with consecutive slopes.

Suppose our q pieces are $\mathbb{P}_1, \ldots, \mathbb{P}_q$. We write $\mathscr{M}_i = \mathscr{M}(\mathbb{P}_i)$ for the arrangement and l^i_j for the move line on piece $\mathbb{P}_i$. Formally, the *combinatorial type* of a nonattacking configuration of q pieces is the list of the sides of move lines l^i_j occupied by each piece $\mathbb{P}_k$, for all (i, j, k) such that $i \neq k$.

In [1] Chaiken and we defined two kinds of isotopy of nonattacking configurations, each of which produces an equivalence relation on them. The simpler one is continuous isotopy. Allow pieces to occupy any real point in the board. Continuous isotopy means moving the pieces around in any way that keeps the configuration nonattacking throughout. For discrete isotopy the pieces stay on integral points but we are allowed to refine the grid at will, by increasing n by a large multiplier. The following lemma is partly explicit and partly implicit in [1, Section 5].

Lemma 2. *Isotopy, discrete isotopy, and having the same combinatorial type produce the same equivalence relation on nonattacking configurations.*

Consequently, to count combinatorial types it is not necessary to place pieces on integral points of a dilated board; they can be placed anywhere in the polygon $\mathcal{B}$.

Proof. The equivalence of isotopy and discrete isotopy is [1, Theorem 5.4]. It was stated in [1] for boards with rational vertices but that restriction is unnecessary.

Obviously, isotopy preserves combinatorial type. The converse is also true: two nonattacking configurations with the same combinatorial type are isotopic. This follows from the fact that a labelled combinatorial type corresponds to a region of a hyperplane arrangement in $\mathbb{R}^{2q}$, as shown in [1, Lemma 5.2]. Thus, isotopy is equivalent to having the same combinatorial type. □

In the previous work with Chaiken we assumed rational slopes. For combinatorial types that is unnecessary, just as it is not necessary to place pieces on board squares, because by the following lemma, allowing irrational slopes produces no new combinatorial types. This gives us helpful freedom in proofs.

Lemma 3. *The number of combinatorial types of q copies of a piece $\mathbb{P}$ with slopes in $\mathbb{R} \cup \{\infty\}$ is also the number for some piece $\mathbb{P}'$ with rational slopes.*

Proof. The definition of combinatorial type does not depend on having rational slopes.

Suppose the slope set is not entirely rational, and for each nonattacking combinatorial type consider a configuration that has that type. Varying the slopes very slightly (for all the configurations) while holding the pieces in each configuration fixed in place does not change the combinatorial type of any configuration. Thus, the irrational or infinite slopes can be shifted to nearby rational values without changing any of the combinatorial types. As there are only finitely many combinatorial types, all combinatorial types that can be realized with the original slopes can be realized with rational slopes. □

3. Three riders and any number of moves

The formula for the number of combinatorial types of configuration of three riders is based on counting regions of overlapping move-line arrangements. We use a classic result of Steiner.

Lemma 4 (Steiner [5]). *Suppose we have k lines in the plane $\mathbb{R}^2$, whose intersection points consist, for each $p \geq 2$, of n_p points at which p lines intersect. Then the number of regions formed by the lines is $1 + k + \sum_{p\geq 2}(p-1)n_p$.*

Suppose we have such an arrangement of lines and a board $\mathcal{B}$. By taking a sufficiently large dilation of $\mathcal{B}$ we can ensure that all intersection points are inside the dilation and all regions in the plane do intersect the dilation. Thus, for counting regions in the dilated board we can ignore the board and pretend it is the whole plane. That is why the number of combinatorial types of nonattacking configuration is independent of the board.

Theorem 5. *On any board, let $\mathbb{P}$ be a rider with r moves. The number of combinatorial types of 3 nonattacking unlabelled copies of $\mathbb{P}$ on dilations of the board equals $\frac{1}{3}r(r^2+3r-1)$.*

Proof. We suppose the three pieces are labelled $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3$. To make a nonattacking configuration we place $\mathbb{P}_1$ anywhere in the (open) board, then $\mathbb{P}_2$ inside a region of the move-line arrangement centered at $\mathbb{P}_1$, and finally $\mathbb{P}_3$ inside any region of the combined move-line arrangements of $\mathbb{P}_1$ and $\mathbb{P}_2$; that is, of $\mathscr{M}_{12} = \mathscr{M}_1 \cup \mathscr{M}_2$.

According to Lemma 2, the number of combinatorial types is the number of ways to choose the region for $\mathbb{P}_2$ and then for $\mathbb{P}_3$.

When we place $\mathbb{P}_2$, we put it inside a region of $\mathscr{M}_1$, which we choose out of $2r$ possible regions. Due to Lemma 2, any point in that region is equivalent to any other.

When we place $\mathbb{P}_3$, we put it in a region of $\mathscr{M}_{12}$, so we need to calculate the number of regions of this arrangement. For that, we need to find the intersection points. First, $\mathbb{P}_i$ is located on an r-fold intersection, as it is on all lines of $\mathscr{M}_i$ but no other lines (since $\mathbb{P}_1$ and $\mathbb{P}_2$ are nonattacking). Thus, $n_r = 2$. There are no multiple intersections away from the pieces because two lines of the same $\mathscr{M}_i$ already meet at $\mathbb{P}_i$. Line l_j^1 intersects every line of $\mathscr{M}_2$ at a separate point, except for l_j^2, to which it is parallel. Thus, $n_2 = r(r-1)$.

It follows from Lemma 4 that $\mathscr{M}_{12}$ has $1+2r+[r(r-1)+2(r-1)] = r^2+3r-1$ regions.

Since $\mathbb{P}_2$ could have been in any of the $2r$ regions of $\mathscr{M}_1$, we multiply by $2r$.

Finally, we divide by 3! because the three pieces are actually unlabelled.

To complete the proof we show that the combinatorial types of regions of $\mathscr{M}_{12}$ are independent of where we place $\mathbb{P}_2$ within a fixed region of $\mathscr{M}_1$. We do this by describing all the regions of $\mathscr{M}_{12}$ and observing that the descriptions depend only on the region of $\mathscr{M}_1$ occupied by $\mathbb{P}_2$.

By rotation we can ensure that the region C of $\mathscr{M}_1$ in which we will place $\mathbb{P}_2$ contains the positive horizontal ray out of $\mathbb{P}_1$. By another affine transformation we can arrange the slopes to be $0 = \mu_1 < \cdots < \mu_{r-1} < \mu_r = \infty$ and C to be the fourth quadrant. Thus the argument for any choice of C is the same and we can assume a simple form for the move-line arrangements.

The irrelevance of where $\mathbb{P}_2$ lies in C is obvious from Figure 2. The regions of $\mathscr{M}_i$ are cones with apex at $\mathbb{P}_i$ and a region of $\mathscr{M}_{12}$ is the intersection of two of these cones. Each cone is either C_j^i, bounded by rays with directions m_{j-1} and m_j for $j = 2, \ldots, r$ or $-m_r$ and m_1 for C_1^i, or one of the negatives $C_{-j}^i = -C_j^i$. The four cones with $j = 1$ are special since they are quadrants and the pieces are in two of them: $\mathbb{P}_2$ in C_1^1 with apex $\mathbb{P}_1$ and $\mathbb{P}_1$ in $-C_1^2$ with apex $\mathbb{P}_2$. That explains why the exact location of $\mathbb{P}_2$ is unimportant. □

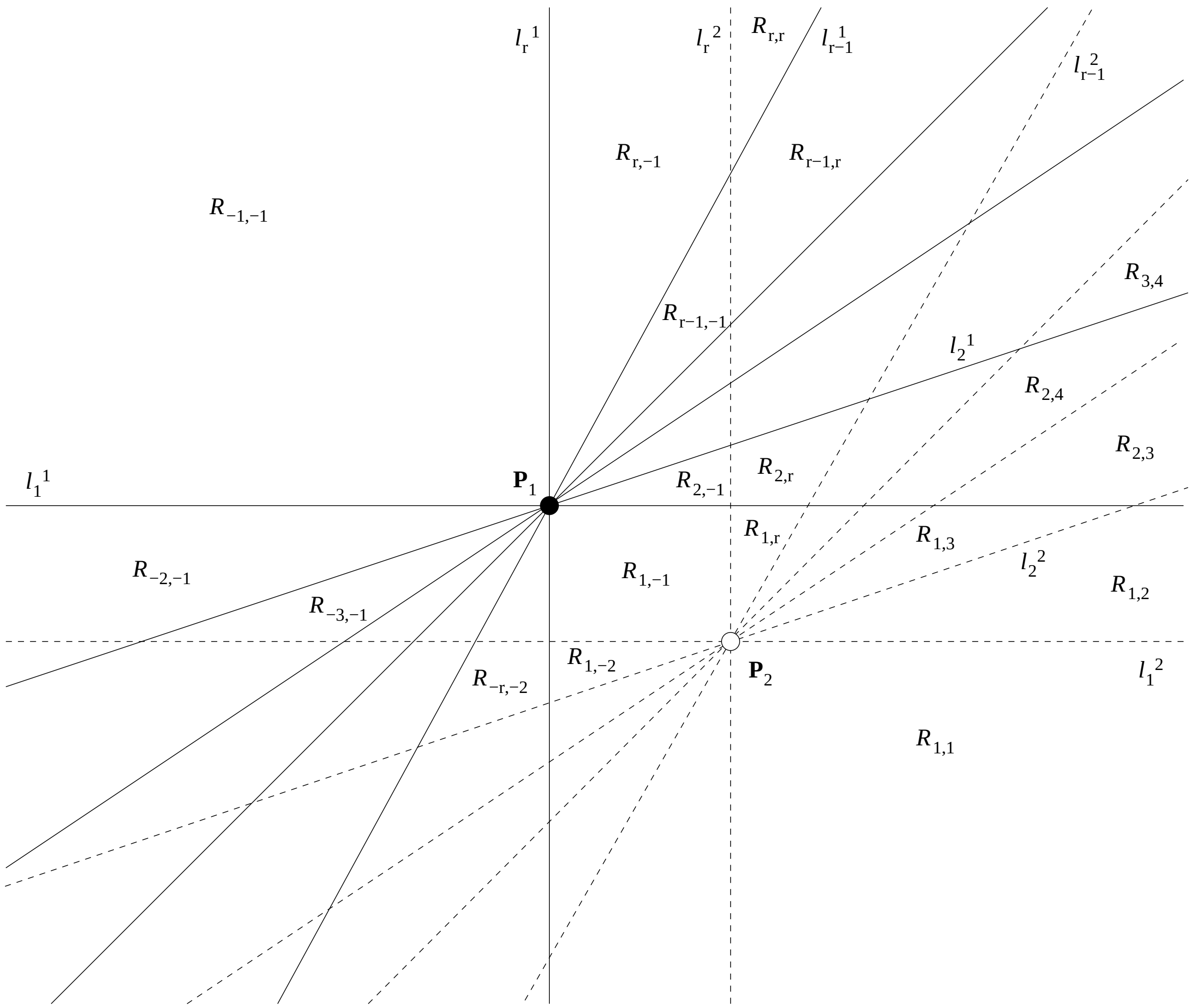

Figure 2. The regions of $\mathscr{M}_{12}$. It is clear that moving $\mathbb{P}_2$ left, right, up, or down within C_1^1 will not change either the regions formed by intersecting pairs of cones, $R_{jk} = C_j^1 \cap C_k^2$, or the boundary lines of each region.

4. Any number of riders having three moves

We have another theorem involving the number three. We can show that every rider with three moves has the same number of combinatorial types. We do not offer a formula. We expect it to be complicated—unlike the proof.

Theorem 6. *On any board, let $\mathbb{P}$ be any rider with 3 basic moves. The number of combinatorial types of q nonattacking unlabelled copies of $\mathbb{P}$ on dilations of the board is independent of the basic move set.*

Proof. We again rely on Lemma 2 to assume pieces can be placed anywhere in the board $\mathcal{B}$ so long as they do not attack. Since we need not worry about integral points, a projective transformation does not change the problem.

Every 3-move rider can be transformed by a projective transformation so that its slopes are $0, 1, \infty$. It follows that all 3-move riders are projectively equivalent. The theorem follows. □

We were led to this theorem by applying Kotěšovec's quasipolynomial formulas to count combinatorial types of $q = 4$ of two different 3-move pieces: the semiqueen [3, p. 732], which has the queen's moves except for one diagonal move, and the trident (his "bishop + semirook") [3, p. 730], which has the queen's moves without the horizontal. We substituted $n = -1$ in both formulas and found the same result: 151 combinatorial types. Kotěšovec has also empirically calculated the counting quasipolynomials for nonattacking configurations of up to 6 semiqueens and 6 triangular rooks, pieces that are equivalent to semiqueens on a triangular board. As Theorem 6 predicts, the numbers of combinatorial types are the same for both boards even though the complete counting formulas differ. See the numbers in Table 1 and Kotěšovec's formulas in [4, Sequences A202654–A202657] for 3 to 6 semiqueens on the square board and [4, Sequences A193981–A193984] for 3 to 6 triangular rooks. This is the only comparison we were able to make for $q > 3$ and $r \geq 3$ of two pieces having the same number of moves.

5. Do fours fail?

With four or more pieces and moves, is there still a single formula in terms of r, or in other words, is the number of combinatorial types independent of the actual moves? The problem is much more complicated than those we solved here. We studied the simplest case: four 4-move pieces. Initially we expected the set of moves to matter, but a more detailed analysis, though inconclusive, cast doubt on that. The question remains open.

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