# RESULTS AND CONJECTURES ON SIMULTANEOUS CORE PARTITIONS

DREW ARMSTRONG, CHRISTOPHER R. H. HANUSA, AND BRANT C. JONES

ABSTRACT. An $n$-core partition is an integer partition whose Young diagram contains no hook lengths equal to $n$. We consider partitions that are simultaneously $a$-core and $b$-core for two relatively prime integers $a$ and $b$, which correspond to abacus diagrams and the combinatorics of the affine symmetric group (type $A$). We observe that self-conjugate simultaneous core partitions correspond to type $C$ combinatorics, and use abacus diagrams to unite the discussion of these two sets of objects.

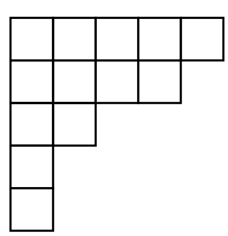
In particular, we prove that $2n$- and $(2mn+1)$-core partitions correspond naturally to dominant alcoves in the $m$-Shi arrangement of type $C_n$, generalizing a result of Fishel–Vazirani for type $A$. We also introduce a major statistic on simultaneous $n$- and $(n+1)$-core partitions and on self-conjugate simultaneous $2n$- and $(2n+1)$-core partitions that yield $q$-analogues of the type $A$ and type $C$ Coxeter-Catalan numbers.

We present related conjectures and open questions on the average size of a simultaneous core partition, $q$-analogs of generalized Catalan numbers, and generalizations to other Coxeter groups. We also discuss connections with the cyclic sieving phenomenon and $q,t$-Catalan numbers.

To the reader: Section 1 consists of a narrative introducing core partitions, a placement of our results in historical context, and intriguing related conjectures. Section 2 introduces precise definitions of abacus diagrams, which serve as the basis for the proofs of our results. The focus of Section 3 is alcoves in $m$-Shi arrangements of types $A$ and $C$. The key result is Theorem 3.5, which characterizes $m$-minimal and $m$-bounded regions as a simultaneous core condition, generalizing the result of Fishel and Vazirani [FV10] through a unified method. Theorem 4.4 gives a major statistic on simultaneous core partitions to find a $q$-analog of the type $A$ and type $C$ Catalan numbers using abacus diagrams and their bijection with lattice paths; this is the main goal of Section 4. We conclude with a few more open problems motivated by this paper. We hope you enjoy it!

## 1. CORES AND CONJECTURES

A **partition** of the integer $n \in \mathbb{N}$ is an unordered multiset of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ such that $\sum_{i=1}^{k} \lambda_i = n$. We will write this as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$, and say that the **size** of the partition is $n$ and the **length** of the partition is $k$. We will often associate a partition $\lambda$ with its **Young diagram**, which is a an array of boxes aligned up and to the left, placing $\lambda_i$ boxes in the $i$th row from the top. For example, here is the Young diagram for the partition $(5, 4, 2, 1, 1) \vdash 13$.



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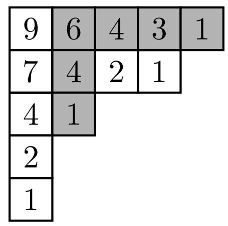
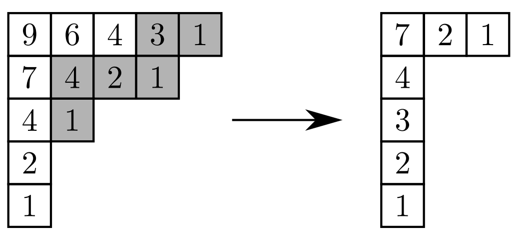
To each box $B \in \lambda$ in the partition we associate its **hook length** $h(B)$, which is the number of cells directly below and directly to the right of $B$ (including $B$ itself). For example, here we have labeled each box with its hook length, with an example hook of length 6 shaded.



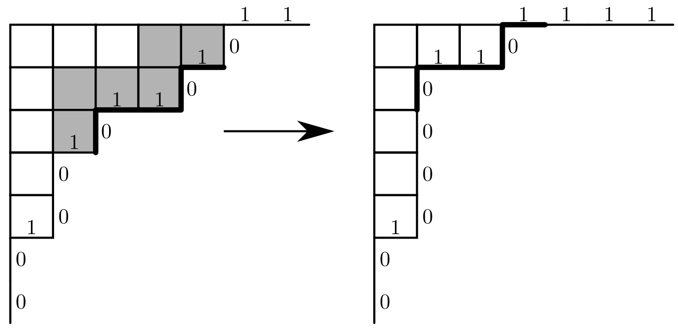

We say that an integer partiton $\lambda \vdash n$ is $a$**-core** if it has no boxes of hook length $a$. The reason for the name "$a$-core" is as follows. If a box $B \in \lambda$ has hook length $h(B) = a$, then we could also say that $\lambda$ has a **rim** $a$**-hook** consisting of the cells along the boundary of $\lambda$, traveling between the box furthest below $B$ and the box furthest to the right of $B$. The boxes of this rim hook can then be stripped away to create a smaller partition. For example, here we have stripped away the rim 6-hook from our previous example.




If we continue stripping away rim $a$-hooks from $\lambda$ we eventually arrive at a partition $\tilde{\lambda}$ that has no boxes of hook length $a$. The resulting $\tilde{\lambda}$ is called the *the* $a$-core of $\lambda$. For example, we see above that $(3, 1, 1, 1, 1) \vdash 7$ is the 6-core of $(5, 4, 2, 1, 1) \vdash 13$. Thus it makes sense to say that a partition $\lambda$ is "$a$-core" when it is equal to its own $a$-core.

However, it is not obvious from the above construction that the $a$-core of a partition is well-defined. We must show that the resulting partition is independent of the order in which we remove rim $a$-hooks. This at first seems difficult, but there is a beautiful argument of James and Kerber that makes it easy. In order to explain this we introduce the **abacus** notation for integer partitions.

First note that we can encode an integer partition as an infinite binary string beginning with 0s and ending with 1s. To do this we think of the partition sitting in an infinite corner. Then we replace vertical steps by 0 and horizontal steps by 1. The partition corresponds to the binary word we get by reading up from infinity, traversing the boundary of the partition, and then traveling right to infinity. For example, our favorite partition $(5, 4, 2, 1, 1) \vdash 13$ yields the string $\cdots 00100101101011 \cdots$. This **boundary string** contains useful information. For example, the boxes of the partition are in bijection with **inversions** in the string, i.e., pairs of symbols in which 1 appears to the left of 0. Furthermore, boxes with hook length $p$ correspond to inversions of "length $a$" (i.e., with the 1 and 0 separated by $a - 1$ intervening symbols). Using this language we see that the removal of a rim $a$-hook corresponds to converting an inversion of length $a$ into a non-inversion of length $a$. For example, in the following diagram we have replaced the substring 1011010 by 0011011.



Finally, we can wind the boundary string around a cycle of length $a$ to obtain an **abacus diagram**. Here we read the boundary string from left to right and then proceed to the next row below. We think of the columns as **runners**, the 0s as **beads**, and the 1s as **gaps**. In this language, the removal of a rim $a$-hook corresponds to *sliding a bead up one level into a gap*.

$$\begin{array}{cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \quad \longrightarrow \quad \begin{array}{cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

We see from this that $a$-cores are well-defined: we simply push all the beads up on their runners until there are no more gaps and we say that the abacus diagram is now $a$**-flush**. This argument is due to James and Kerber [JK81, Lemma 2.7.13], although they used slightly different notation.

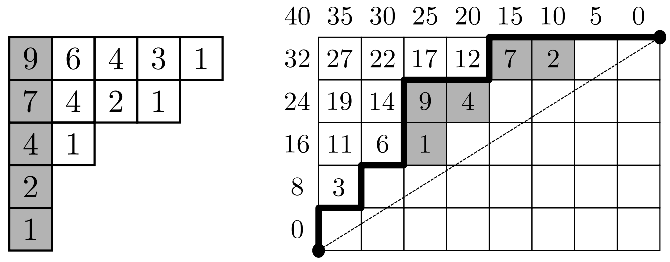
Now we turn to the main subject of the current paper: **simultaneous** core partitions. We say that an integer partition $\lambda \vdash n$ is $(a, b)$**-core** if it is simultaneously $a$-core and $b$-core. Our primary interest in $(a, b)$-cores is motivated by the following result of Jaclyn Anderson from 2002.

**Theorem 1.1.** [And02] *The total number of $(a, b)$-core partitions is finite if and only if $a$ and $b$ are coprime, in which case the number is*

$$\frac{1}{a+b}\binom{a+b}{a,b} = \frac{(a+b-1)!}{a!\,b!}. \tag{1.1}$$

Note: when $a$ and $b$ are *not* coprime, Formula (1.1) is not necessarily even an integer. We now sketch the idea and provide a proof of Anderson's bijection in Proposition 4.5. Note that an integer partition $\lambda$ is completely determined by the hook lengths of the boxes in its first column. These boxes are also in bijection with a set of beads in a certain normalized abacus diagram. If $\lambda$ is $(a, b)$-core, then its beads must be flush simultaneously in *two* ways. Anderson's construction beautifully creates a shifted abacus diagram where the $a$-flush condition is horizontal and the $b$-flush condition is vertical.

This gives a bijection between $(a, b)$-cores and lattice paths in $\mathbb{Z}^2$ from $(0, 0)$ to $(b, a)$, staying above the diagonal. We call these $(a, b)$**-Dyck paths**. For example, our favorite partition is a $(5, 8)$-core, corresponding to the following $(5, 8)$-Dyck path.

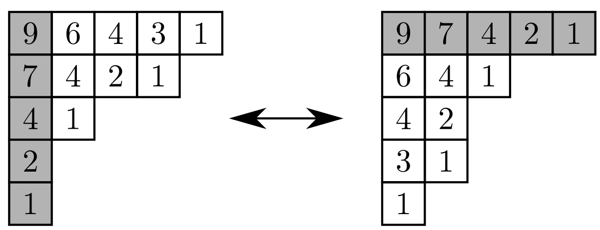



It was known since at least Bizley [Biz54] that the $(a, b)$-Dyck paths (with $a$ and $b$ coprime) are counted by Formula (1.1), which is a generalization of the classical Catalan numbers.

Next we discuss *self-conjugate* core partitions. Given an integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$, we define the **conjugate** partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_\ell) \vdash n$ by setting

$$\lambda'_j := \#\{j : \lambda_j \geq i\}.$$

Equivalently, the Young diagram of $\lambda'$ is obtained by reflecting the Young diagram of $\lambda$ across the main diagonal. For example, the partitions $(5, 4, 2, 1, 1) \vdash 13$ and $(5, 3, 2, 2, 1) \vdash 13$ are conjugate.




Observe that $\lambda$ is $(a, b)$-core if and only if $\lambda'$ is $(a, b)$-core. Thus one may be interested in studying the **self-conjugate** $(a, b)$-cores. In 2009, Ford, Mai and Sze proved the following analogue of Anderson's theorem.

**Theorem 1.2.** [FMS09] *If $a$ and $b$ are coprime, then the number of self-conjugate $(a, b)$-cores is*

$$\binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor} = \frac{(\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor)!}{\lfloor \frac{a}{2} \rfloor! \, \lfloor \frac{b}{2} \rfloor!}$$

Before discussing their proof, we make some numerological observations. Given $k \leq n \in \mathbb{N}$, we define the standard $q$**-integer**, $q$**-factorial** and $q$**-binomial coefficient**:

$$\begin{aligned} [n]_q &:= 1 + q + q^2 + \cdots q^{n-1}, \\ [n]_q! &:= [n]_q [n-1]_q \cdots [2]_q [1]_q, \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &:= \frac{[n]_q!}{[k]_q! \, [n-k]_q!}. \end{aligned}$$

Inspired by formula (1.1) we define the **rational $q$-Catalan number**:

$$\mathsf{Cat}_q(a, b) := \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a, b \end{bmatrix}_q = \frac{[a+b-1]_q!}{[a]_q! \, [b]_q!}. \tag{1.2}$$

Observe that the case $(a, b) = (n, n+1)$ corresponds to the classical "$q$-Catalan number" of MacMahon:

$$\mathsf{Cat}_q(n, n+1) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q.$$

MacMahon proved that $\mathsf{Cat}_q(n, n+1)$ is in $\mathbb{N}[q]$ by defining a statistic on lattice paths, which is now called "major index" (in honor of the fact that MacMahon held the rank of major in the British Army).

First, note that the final step of an $(n, n+1)$-Dyck path must be horizontal, and removing this last step defines a bijection between $(n, n+1)$-Dyck paths and **classical Dyck paths** from $(0,0)$ to $(n,n)$, staying *weakly* above the diagonal. Given a classical Dyck path $P$, we define the **major index** as follows. Begin at $(0,0)$ and call this vertex 0. Then $\mathsf{maj}(P)$ is the sum of $i$ such that the step $(i-1) \to i$ is horizontal and $i \to (i+1)$ is vertical (or in other words, the $i$th vertex is a "valley" of the path). MacMahon proved the following.

**Theorem 1.3.** [Mac60, page 214] *We have*

$$\sum_P q^{\mathsf{maj}(P)} = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q,$$

*where the sum is over $(n, n+1)$-Dyck paths $P$ (equivalently, classical Dyck paths $P$).*

We note that the "$q$-Catalan" numbers $\mathsf{Cat}_q(n, n+1)$ were studied by Furlinger and Hofbauer [FH85]. The following problem is open.

**Open Problem 1.4.** Given $a, b \in \mathbb{N}$ coprime, define a statistic $\mathsf{stat}$ on $(a,b)$-Dyck paths, or equivalently on $(a,b)$-cores, such that

$$\sum_P q^{\mathsf{stat}(P)} = \mathsf{Cat}_q(a,b) = \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a, b \end{bmatrix}_q.$$

Preferably we would have $\mathsf{stat} = \mathsf{maj}$ when $(a,b) = (n, n+1)$.

We would even like an elementary proof that $\mathsf{Cat}_q(a,b)$ is a polynomial in $\mathbb{N}[q]$ when $a, b \in \mathbb{N}$ are coprime. (The only known proof of this fact [GG12, Section 1.12] uses the representation theory of rational Cherednik algebras.) We conjecture a solution to Problem 1.4 below.

Here is another open problem.

**Open Problem 1.5.** One can verify the following evaluation at $q = -1$:

$$\left. \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a, b \end{bmatrix}_q \right|_{q=-1} = \binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor}.$$

Is this an example of a "cyclic sieving phenomenon"? (See the survey [Sag11] for details.) That is, does there exist a cyclic group action on $(a,b)$-cores such that "rotation by $180°$" corresponds to conjugation of the partition? Can one use this to view the results of Anderson and Ford-Mai-Sze as two special cases of a more general theorem?

We also state a related conjecture.

**Conjecture 1.6.** *Let $a, b \in \mathbb{N}$ be coprime. Then the average size of an $(a,b)$-core and the average size of a self-conjugate $(a,b)$-core are both equal to*

$$\frac{(a+b+1)(a-1)(b-1)}{24}. \tag{1.3}$$

*Olsson and Stanton* [OS07] *proved that there is a unique $(a,b)$-core of maximum size (which happens to be self-conjugate), and this size is*

$$\frac{(a^2-1)(b^2-1)}{24}.$$

*Thus we can rephrase our conjecture by stating that the average ratio between an $(a,b)$-core and the* largest *$(a,b)$-core is*

$$\frac{(a+b+1)}{(a+1)(b+1)}.$$

This conjecture was observed experimentally by the first author in 2011 and has been publicized informally since then. While the second author has been able to prove Formula (1.3) for small values of $a$, a framework to prove the conjecture in general has been met with frustratingly little progress. The fact that the same average size holds for both $(a,b)$-cores and self-conjugate $(a,b)$-cores makes it seem that the conjecture may be related to Problem 1.5.

Now we return to a discussion of the Ford-Mai-Sze theorem. Their proof is an ingenious bijection between self-conjugate $(a,b)$-cores and lattice paths in a $\lfloor \frac{a}{2} \rfloor \times \lfloor \frac{b}{2} \rfloor$ rectangle. But perhaps too ingenious, since it does not involve abacus diagrams, and it makes no connection to Anderson's theorem. In the current paper we give a more natural interpretation of the Ford-Mai-Sze result using the type $C$ abacus model recently developed by the second and third authors [HJ12]. In this language we will see that there is a natural framework in which Anderson's result corresponds to the affine Weyl group of type $A$ and the Ford-Mai-Sze result corresponds to the affine Weyl group of type $C$.

We will also describe an analogous relationship between the **Shi hyperplane arrangements** of types $A$ and $C$. Given a finite cystallographic root system $\Phi \subseteq V$ with positive roots $\Phi^+$, the $m$-Shi arrangement consists of the hyperplanes

$$\mathsf{Shi}^m(\Phi) = \cup_{\alpha\in\Phi^+} \{H_{\alpha,-m+1}, H_{\alpha,-m+2}, \cdots H_{\alpha,m-1}, H_{\alpha,m}\},$$

where $H_{\alpha,k} = \{x \in V : \langle x,\alpha\rangle = k\}$. Each chamber of the $m$-Shi arrangement is a union of alcoves, which can be thought of as elements of the corresponding affine Weyl group. These alcoves, in turn, can be encoded by abacus diagrams.

Fishel and Vazirani considered the $m$-Shi arrangement of type $A_{n-1}$. They used abacus diagrams to construct a bijection between *minimal* alcoves in all the chambers of $\mathsf{Shi}^m(A_{n-1})$ and $(n, mn+1)$-core partitions [FV10], and a bijection between *maximal* alcoves in the *bounded* chambers of $\mathsf{Shi}^m(A_{n-1})$ and $(n, mn-1)$-core partitions [FV09]. In Section 3 of this paper we use the type $C$ abacus diagrams of Hanusa and Jones to prove the following new result: There is a bijection between *minimal* alcoves of the chambers of $\mathsf{Shi}^m(C_n)$ and self-conjugate $(2n, 2mn+1)$-cores, and there is a bijection between *maximal* alcoves in the *bounded* chambers of $\mathsf{Shi}^m(C_n)$ and self-conjugate $(2n, 2mn-1)$-cores. It is an open problem to extend the theory to types $B$ and $D$. The study of $(a,b)$-cores when $b = \pm 1 \bmod a$ is called the "Fuss-Catalan" level of generality. It is also an open problem to extend these results on Shi arrangements to more general $b$.

Finally, we return to the problem of $q$-Catalan numbers. For any finite reflection group $G$ with "degrees" $d_1 \le d_2 \le \cdots \le d_\ell =: h$, one can define a $q$**-Catalan number**

$$\mathsf{Cat}_q(G) = \prod_{i=1}^{\ell} \frac{[h+d_i]_q}{[d_i]_q}.$$

For definitions see Section 2.7 of [Arm09]. It is known that $\mathsf{Cat}_q(G) \in \mathbb{N}[q]$ (see [GG12]), but combinatorial interpretations of this fact are missing in almost all cases. The (symmetric) group of type $A_{n-1}$ has degrees $2, 3, \ldots, n$, and so

$$\mathsf{Cat}_q(A_{n-1}) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q,$$

which we discussed above. The group of type $C_n$ (the hyperoctahedral group) has degrees $2, 4, 6, \ldots, 2n$, and so

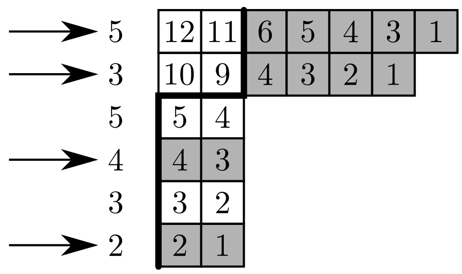
$$\mathsf{Cat}_q(C_n) = \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}.$$

In Section 4 of this paper we will describe explicit "major index" type statistics on $(n, n+1)$-cores and self-conjugate $(2n, 2n+1)$-cores which explain the numbers $\mathsf{Cat}_q(A_{n-1})$ and $\mathsf{Cat}_q(C_n)$. These are obtained by transfering the standard "major index" on lattice paths via the bijections of Anderson and Ford-Mai-Sze. The new observation is that the statistics are so natural to express in the language of abacus diagrams. It is an open problem to define a similar statistic on general $(a,b)$-cores (see Problem 1.4). However, we will now state a conjecture that may solve the problem.

Given an $(a,b)$-core $\lambda$, we say that its $b$**-boundary** consists of the skew-subdiagram of boxes with hook lengths $< b$. We define the $a$**-rows** of $\lambda$ as follows. Consider the boxes in the first column of $\lambda$ and reduce their hook lengths modulo $a$. Consider the highest row in each residue class. These are the $a$-rows of the diagram.

**Definition 1.7.** Let $\lambda$ be an $(a,b)$-core partition with $a < b$ coprime. The **skew length** $s\ell(\lambda)$ is the number of boxes of $\lambda$ that are simultaneously in the $b$-boundary and the $a$-rows of $\lambda$.

For example, the partition $(7,6,2,2,2,2) \vdash 21$ is a $(7,8)$-core, and its skew length is 13.




Recall that the **length** of an integer partition $\ell(\lambda)$ is its number of nonzero rows. Then we make the following conjecture.

**Conjecture 1.8.** *Let $a < b$ be coprime. Then we have*

$$\sum_{\lambda} q^{\ell(\lambda)+s\ell(\lambda)} = \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a,b \end{bmatrix}_q,$$

*where the sum is over $(a,b)$-cores $\lambda$.*

For example, the $(7,8)$-core shown above has $\ell(\lambda) + s\ell(\lambda) = 6 + 13 = 19$. We might be tempted to define a statistic $\mathsf{maj}(\lambda) = \ell(\lambda) + s\ell(\lambda)$. *Unfortunately*, in the classical Catalan case of $(n, n+1)$-cores, the statistic $\ell + s\ell$ is **not** clearly related to any of the known "major index" type statistics.

We expect Conjecture 1.8 to be difficult. In fact, it is just a shadow from the more general subject of $q,t$-Catalan combinatorics. To illustrate this, define the **co-skew-length** of an $(a,b)$-core by $s\ell'(\lambda) = (a-1)(b-1)/2 - s\ell(\lambda)$.

**Conjecture 1.9.** *Let $a < b$ be coprime. Then we have*

$$\sum_{\lambda} q^{\ell(\lambda)} t^{s\ell'(\lambda)} = \sum_{\lambda} t^{\ell(\lambda)} q^{s\ell'(\lambda)}, \tag{1.4}$$

*where the sum is over $(a,b)$-cores $\lambda$.*

Either side of equation (1.4) should be regarded as a "rational $q,t$-Catalan number". Then Conjecture 1.9 is a generalization of the "symmetry problem" for $q,t$-Catalan numbers. This problem is quite hard (see, for example, [Hag08]). Thus, even partial progress is welcome.

To end this section we point to some related work. The "rational $q,t$-Catalan numbers" (1.4) have been independently defined and studied by Gorsky and Mazin [GM13, GM12], and they will also appear soon in a paper of Armstrong, Loehr, and Warrington [ALW13]. The paper [ALW13] will explore three different interpretations of the "skew length" statistic. (In addition to the interpretation here, the other two are due to Gorsky–Mazin and Loehr–Warrington.) Finally, the general subject of "rational Catalan numbers" and related structures has been studied by Armstrong, Rhoades ,and Williams [ARW13].

## 2. Abacus diagrams

An **abacus diagram** (or simply **abacus**) is a diagram containing $R$ columns labeled $1, 2, \ldots, R$, called **runners**. Runner $i$ contains entries labeled by the integers $mR + i$ for each **level** $m$ where $-\infty < m < \infty$.

We draw the abacus so that each runner is vertical, oriented with $-\infty$ at the top and $\infty$ at the bottom, with runner 1 in the leftmost position, increasing to runner $R$ in the rightmost position. Entries in the abacus diagram may be circled; such circled elements are called **beads**. Entries that are not circled are called **gaps**. We refer to the collection consisting of the lowest beads in each runner as the **defining beads** of the abacus. We say that an abacus is $j$**-flush** if whenever position $e$ is a bead in the abacus we have that $e - j$ is also a bead.

As this construction essentially defines a labeling on the infinite binary string from Section 1, two abacus diagrams are equivalent if the infinite sequence of beads and gaps are the same (so that their entries only differ by a constant position). Two common ways to standardize an abacus diagram are to make it *normalized* or *balanced*. We say that an abacus is **balanced** if the sum of the levels of the defining beads of the abacus is zero. We say that an abacus is **normalized** if the first gap occurs in position 0. (And when normalized, we may relabel runner $R$ as runner 0 and place it on the left.)

Balanced, $n$-flush abacus diagrams with $n$ runners are in bijection with minimal length coset representatives of type $\widetilde{A}_n/A_n$ by interpreting the defining beads of the abacus (written in increasing order) as the entries in the base window of the corresponding affine permutation.

Further, the classical argument of James [JK81] described in Section 1 gives a bijection between the set of balanced $n$-flush abacus diagrams and the set of $n$-core partitions: Given an abacus, we create a partition whose southeast boundary is the lattice path obtained by reading the entries of the abacus in increasing order and recording a north-step for each bead, and recording an east-step for each gap.

**Example 2.1.** The 4-core partition $\lambda = (3, 3, 1, 1, 1)$ corresponds to a 4-flush abacus and to the element $[-4, 1, 6, 7]$ in $\widetilde{A}_3/A_3$. The corresponding abacus presented here is both normalized (the first gap is in position 0) and balanced (the levels of the defining beads on runners 1 through 4 are 0, 1, 1, $-2$, which sum to 0). On the right is the alternative method of drawing this same normalized abacus on runners 0 through 3.

| | | |
|---|---|---|
| 7 | 3 | 2 |
| 6 | 2 | 1 |
| 3 | | |
| 2 | | |
| 1 | | |

| | | | |
|---|---|---|---|
| $-7$ | $-6$ | $-5$ | $-4$ |
| $-3$ | $-2$ | $-1$ | 0 |
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 |

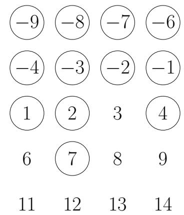
| | | | |
|---|---|---|---|
| $-8$ | $-7$ | $-6$ | $-5$ |
| $-4$ | $-3$ | $-2$ | $-1$ |
| 0 | 1 | 2 | 3 |
| 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 |

In [HJ12], the second and third authors introduced an abacus diagram model with $R = 2n$ runners to represent minimal length coset representatives of type $\widetilde{C}_n/C_n$. In this type $C$ abacus model, we use $N = 2n + 1$ implicit labels per row so that the linear ordering of the entries of the abacus are given by the labels $mN + i$ for level $m \in \mathbb{Z}$ and runner $1 \leq i \leq 2n$. (Under these conventions, there are no entries in any type $C$ abacus having labels $\{mN : m \in \mathbb{Z}\}$.)

We also impose a stricter definition of **balanced** on a type $C$ abacus—the level of the defining bead on runner $i$ is the negative of the level of the defining bead on runner $N - i$. This imposes an antisymmetry on type $C$ abaci where entry $N - b$ is a bead if and only if entry $N + b$ is a gap. Restricting James's bijection gives a bijection between the set of type $C$ balanced $2n$-flush abacus diagrams and the set of self-conjugate $2n$-core partitions. Under this construction, we can then interpret the defining beads of abacus written in increasing order as the corresponding minimal length coset representative of $\widetilde{C}_n/C_n$ written in one-line notation as a mirrored $\mathbb{Z}$-permutation, just as in type $A$.

Throughout this paper, we work in types $A$ and $C$ simultaneously by letting $N$ be the number of implicit labels used on each row of the abacus, so $N = n$ in type $A$ and $N = 2n + 1$ in type $C$. We also let $R$ be the number of runners in the abacus, so $R = n$ in type $A$ and $R = 2n$ in type $C$.

**Example 2.2.** In $\widetilde{C}_2$, consider the mirrored $\mathbb{Z}$-permutation determined by $[w(1), w(2), w(3), w(4)] = [-2, 1, 4, 7]$. The corresponding abacus diagram is




and the corresponding self-conjugate 4-core partition is $(2, 1)$.

## 3. Regions of the $m$-Shi arrangement

Consider the root system of type $A_{n-1}$ or type $C_n$ embedded in a Euclidean space $V = \mathbb{R}^n$ with inner product $(\cdot, \cdot)$ and orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$. Then the *$m$-Shi hyperplane arrangement* consists of $v \in V$ such that

$$-m < (v, \alpha) \leq m \text{ for all positive roots } \alpha.$$

For example, the hyperplanes in type $A_n$ consist of $v = \sum_{i=1}^{n} v_i \varepsilon_i$ such that

$$v_i - v_j \in \{0, 1, \ldots, m\} \text{ for } 1 \leq i < j \leq n,$$

while for type $C_n$ we additionally have

$$v_i + v_j \in \{0, 1, \ldots, m\} \text{ for } 1 \leq i < j \leq n$$

$$2v_i \in \{0, 1, \ldots, m\} \text{ for } 1 \leq i \leq n\,.$$

In this work, we restrict to the dominant cone $\{v \in V : (v, \alpha) \geq 0 \text{ for all positive roots } \alpha\}$. Pictures from rank 2 are shown in Figures 1 and 2. A bar over a number represents the negative of that number.

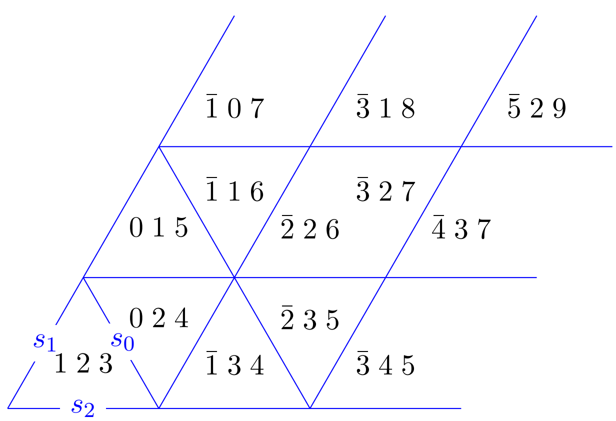



FIGURE 1. The $m = 2$ Shi arrangement in $A_2$

Recall from [Hum90, Chapter 4] that the affine Weyl group acts on $V$ by reflections, and so $V$ decomposes into **alcoves** that are the connected components of the complement of the set of hyperplanes orthogonal to positive roots together with all their parallel translates. Denote the fundamental alcove by

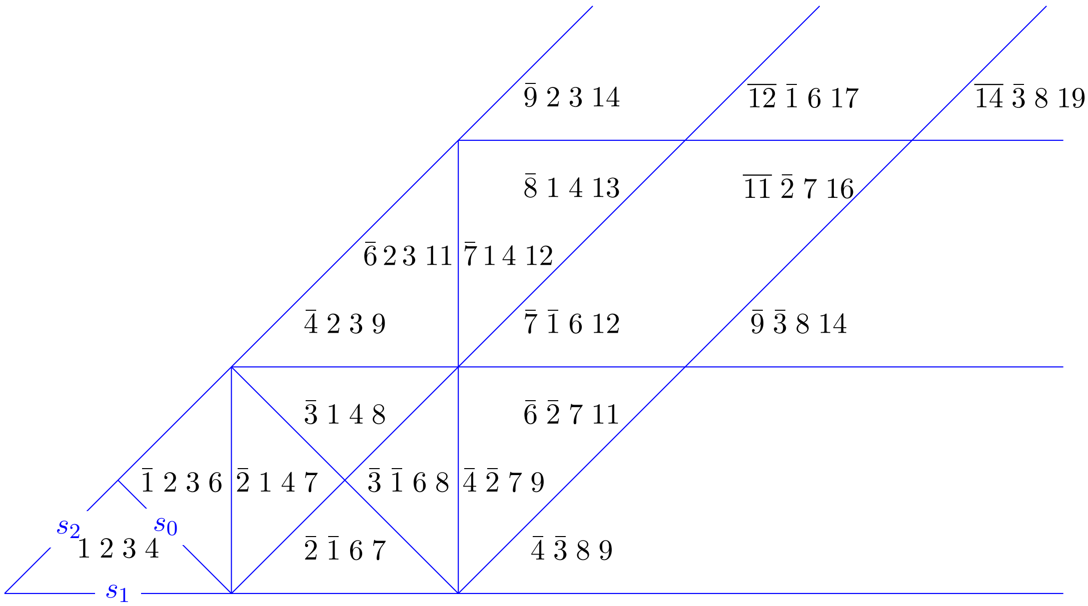



FIGURE 2. The $m = 2$ Shi arrangement in $C_2$

$A_\circ$. Then there is a simply transitive action of the affine Weyl group on the set of alcoves. In fact, there are two actions: the left action reflects an alcove (non-locally) across one of the defining hyperplanes, while the right action reflects an alcove (locally) across one of its bounding hyperplanes. Then, we have that $wA_\circ$ corresponds to a dominant alcove if and only if $w$ is "left grassmannian" in the sense that $D_L(w) \subseteq \{s_0\}$, where $D_L$ denotes the left descent set.

We can read off the right (local) descents and the left (non-local) inner products of $wA_\circ$ simultaneously from the abacus, a result of Shi [Shi99, Theorem 4.1].

**Lemma 3.1.** *Let $w(i)$ denote the $i$th entry of the one-line notation for $w$, or equivalently, the position of the $i$th defining bead in the abacus for $w$ in type $A$ or $C$. Then the inner product of any point in $wA_\circ$ with a positive root $\alpha$ satisfies*

$$\left\lfloor \frac{w(j) - w(i)}{N} \right\rfloor < (wA_\circ, \alpha) < \left\lfloor \frac{w(j) - w(i)}{N} \right\rfloor + 1,$$

*for some $1 \le i < j \le R$.*

*Proof.* This follows from the symmetry of the abacus and [HJ12, Theorem 4.1]. It is straightforward to work out the correspondence between differences and positive roots. For example, the inner product with $e_i + e_j$ in type $C$ can be realized as $\lfloor \frac{w(n+j)-w(i)}{N} \rfloor$. □

**Lemma 3.2.** *In types $A$ and $C$, $w$ has $s_i$ as a right descent if and only if the position of the defining bead in column $i+1$ is at least $N$ plus the position of the defining bead in column $i$ (where column $0$ is interpreted as column $2n$ in type $C$, and column $n$ in type $A$).*

*Proof.* This follows from [HJ12, Section 3.2]. □

As a consequence of these lemmas, it is possible to read off the "Shi coordinates" of an alcove that specify the number of translations in each positive root direction.

**Example 3.3.** The alcove labeled $\bar{1}16$ in Figure 1 lies

- $\lfloor \frac{1-(-1)}{3} \rfloor = 0$ translates past the hyperplane labeled $s_1$
- $\lfloor \frac{6-1}{3} \rfloor = 1$ translate past the hyperplane labeled $s_2$
- $\lfloor \frac{6-(-1)}{3} \rfloor = 2$ translates past the hyperplane labeled $s_0$.

If we reflected the labeling of the boundary of $A_\circ$ by $w$, we would find that $s_2$ is the unique right descent of $\bar{1}16$, corresponding to the fact that 6 is the position of the defining bead in column 3 and this lies at least $N = 3$ positions past the defining bead in column 2.

**Example 3.4.** The alcove labeled $\bar{4}\bar{2}79$ in Figure 2 lies

- $\lfloor \frac{(-2)-(-4)}{5} \rfloor = 0$ translates past the hyperplane labeled $s_1$
- $\lfloor \frac{7-(-2)}{5} \rfloor = 1$ translate past the hyperplane labeled $s_2$
- $\lfloor \frac{9-(-4)}{5} \rfloor = 2$ translates past the hyperplane labeled $s_0$
- $\lfloor \frac{7-(-4)}{5} \rfloor = 2$ translates past the hyperplane perpendicular to the remaining positive root $e_1+e_2$.

The unique right descent for this element is $s_1$.

We say that a dominant alcove is $m$**-minimal** if it is the unique alcove of minimal length in its region of the $m$-Shi arrangement. We say that a dominant alcove is $m$**-bounded** if it is the unique alcove of maximal length in its region of the $m$-Shi arrangement. The uniqueness of these alcoves was shown in [Ath05]. Our main result in this section is the following theorem. A less explicit proof for type $A$ is given in [FV10].

**Theorem 3.5.** *In types $A$ and $C$, a dominant alcove is $m$-minimal if and only if the corresponding abacus diagram is $(Rm + 1)$-flush. Moreover, a dominant alcove is $m$-bounded if and only if the corresponding abacus diagram is $(Rm - 1)$-flush.*

*Proof.* Let $wA_\circ$ be a dominant alcove. Then $wA_\circ$ is $m$-minimal if it is the minimal length alcove in its region of the $m$-Shi arrangement. Equivalently, for each descent $s_i$ of $w$, we must have that $w$ and $ws_i$ are separated by an $m$-Shi hyperplane. Contrapositively, there do not exist two defining beads in the abacus for $w$ that form a right descent and contribute a left inner product with a positive root that is greater than $m$.

By Lemmas 3.1 and 3.2, this means that the one-line notation for $w$ never contains $1 \leq i < j \leq R$ with

$$\left\lfloor \frac{w(j) - w(i)}{N} \right\rfloor > m$$

and

$$\big(w(j) \bmod N\big) \equiv \big(w(i) \bmod N\big) + 1 \pmod R$$

But this is precisely equivalent to requiring that the abacus be $(Rm + 1)$-flush.

Similarly, $wA_\circ$ is $m$-bounded if it is the maximal length alcove in its region of the $m$-Shi arrangement. This is equivalent to requiring that there do not exist two defining beads in $w$ that form a right ascent and have a left inner product with the corresponding positive root that is greater than $m$. Once again by Lemmas 3.1 and 3.2, this means that the one-line notation for $w$ never contains $1 \leq i < j \leq R$ with

$$\left\lfloor \frac{w(j) - w(i)}{N} \right\rfloor > m$$

and

$$\big(w(j) \bmod N\big) \equiv \big(w(i) \bmod N\big) - 1 \pmod R$$

This is precisely equivalent to requiring that the abacus be $(Rm - 1)$-flush. $\square$

**Example 3.6.** In type $A_2$, $m = 1$, the $m$-minimal alcoves correspond to

$$123, 024, 015, \bar{1}34, \bar{2}26.$$

The element $\bar{2}35$ is not minimal because the $\bar{2}$ and $5$ are off by $2 > m$ levels, and they form a descent since $\bar{2}$ lies on column $(-2 \bmod 3) = 1$ of the abacus while $5$ lies on column $(5 \bmod 3) = 2$.

The element $\bar{2}26$ is minimal because although $\bar{2}$ and $6$ are off by $2 > m$ levels, we find that these two entries do not form a descent on the abacus.

**Example 3.7.** In type $C_2$, $m = 1$, the $m$-minimal alcoves correspond to

$$1234, \bar{1}236, \bar{2}147, \bar{2}\bar{1}67, \bar{4}239, \bar{7}\bar{1}6\,12.$$

**Corollary 3.8.** *The $m$-minimal alcoves in the $m$-Shi arrangement of type $C_n$ are in bijection with self-conjugate partitions that are $(2n)$-core and $(2nm+1)$-core. The $m$-bounded alcoves in the $m$-Shi arrangement of type $C_n$ are in bijection with self-conjugate partitions that are $(2n)$-core and $(2nm-1)$-core.*

Applying Ford, Mai, and Sze's formula [FMS09], we recover that there are $\binom{nm+n}{n}$ dominant regions and $\binom{nm+n-1}{n}$ bounded regions in the $m$-Shi arrangement of type $C_n$. This agrees with Athanasiadis's [Ath04, Corollary 1.3].

## 4. A major statistic on simultaneous core partitions

In this section, we define a major statistic that gives the $q$-analog of the Coxeter-Catalan numbers, $\mathsf{Cat}_q(A_{n-1})$ and $\mathsf{Cat}_q(C_n)$, for simultaneous core partitions in types $A$ and $C$, respectively.

**Definition 4.1.** Let $\lambda$ be a simultaneous $(n, n+1)$-core partition. Create the sequence $x = (x_0, \ldots, x_{n-1})$ where $x_i$ equals the number of boxes in the first column of $\lambda$ with hook length equal to $i \bmod n$. (Note $x_0 = 0$ always.) Define

$$\mathsf{maj}(\lambda) = \sum_{i\,:\,x_{i-1} \geq x_i} (2i - x_i). \tag{4.1}$$

Let $\lambda$ be a self-conjugate simultaneous $(2n, 2n+1)$-core partition. We define the set $\mathcal{W}$ of **diagonal arm lengths** $\{w_1, w_2, \ldots, w_k\}$ where $w_i$ is one more than the number of boxes to the right of the $i$-th box on the diagonal of $\lambda$. Create the sequence $x = (x_0, x_1, \ldots, x_n)$ where $x_0 = 0$ and

$$x_i = |\{w \in \mathcal{W} : w \bmod 2n \equiv i\}| - |\{w \in \mathcal{W} : w \bmod 2n \equiv 2n - i + 1\}|$$

for $1 \leq i \leq n$. Define

$$\mathsf{maj}(\lambda) = 2 \sum_{i\,:\,x_{i-1} \geq x_i} (2i - x_i - 1). \tag{4.2}$$

**Remark.** Similar to the definition of the major statistic of a permutation, these sums are over the positions of the (weak) descents in a sequence. In terms of the abacus diagram, $x_i$ is the level of the defining bead on runner $i$ (in type $A$ the abacus must be normalized first), so this definition of $\mathsf{maj}$ can also be applied directly to the corresponding abacus diagram.

**Example 4.2.** For the $(7, 8)$-core partition $\lambda = (7, 6, 2, 2, 2, 2)$, the hook lengths of the boxes in the first column of $\lambda$ are $\{12, 10, 5, 4, 3, 2\}$, which modulo 7 equals $\{5, 3, 5, 4, 3, 2\}$. We conclude that $x = (0, 0, 1, 2, 1, 2, 0)$ with weak descents in positions 1, 4, and 6. As such,

$$\mathsf{maj}(\lambda) = (2 \cdot 1 - 0) + (2 \cdot 4 - 1) + (2 \cdot 6 - 0) = 21.$$

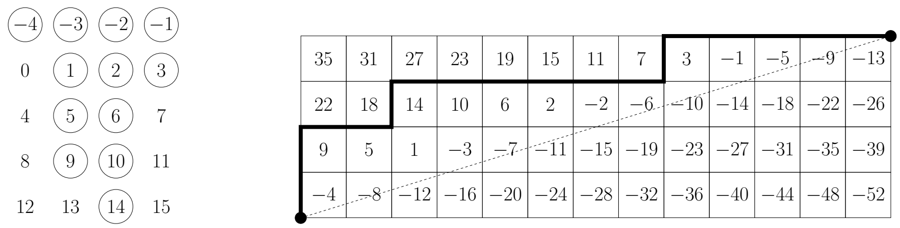


FIGURE 3. The $(4, 13)$-core partition $\lambda = (7, 4, 4, 2, 2, 1, 1, 1)$ corresponds to a 4-flush and 13-flush abacus $a$ and a lattice path from $(0, 0)$ to $(13, 4)$ staying below $y = \frac{4}{13}x$. The boxes to the right of the lattice path correspond to the beads in the abacus.

**Example 4.3.** For the self-conjugate $(14, 15)$-core partition

$$\mu = (19, 19, 16, 12, 9, 9, 9, 7, 7, 4, 4, 4, 3, 3, 3, 3, 2, 2, 2),$$

the set of diagonal arm lengths is $\{19, 18, 14, 9, 5, 4, 3\}$, which modulo 14 gives $\{5, 4, 14, 9, 5, 4, 3\}$. Therefore we have $x = (0, -1, 0, 1, 2, 2, -1, 0)$ with weak descents in positions 1, 5, and 6. We find that

$$\mathsf{maj}(\mu) = 2\big[\big(2 \cdot 1 - 1 - (-1)\big) + \big(2 \cdot 5 - 1 - 2\big) + \big(2 \cdot 6 - 1 - (-1)\big)\big] = 42.$$

The major result in this section is that

**Theorem 4.4.** *The major statistic defined above gives a $q$-analog of the type $A$ and type $C$ Catalan numbers. In particular,*

$$\sum_{\substack{\lambda \text{ is an} \\ (n,n+1)\text{-core}}} q^{\mathsf{maj}(\lambda)} = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q \qquad \text{and} \qquad \sum_{\substack{\lambda \text{ is a self-conj.} \\ (2n,2n+1)\text{-core}}} q^{\mathsf{maj}(\lambda)} = \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}.$$

The proof of this result is given below; it uses the bijections between simultaneous core partitions, abacus diagrams, and lattice paths. In type $A$, we quickly revisit and then apply Anderson's bijection [And02, Proposition 1]. In type $C$, Ford, Mai, and Sze [FMS09] developed a lattice path method to count self-conjugate $(a, b)$-core partitions for $a < b$ relatively prime. Hanusa and Jones's abacus model for type $C$ [HJ12] streamlines this bijection and helps to develop further intuition about it.

**Proposition 4.5** (Proposition 1, [And02])**.** *The following is a bijection:*

$$\mathcal{L} : \left\{ \begin{array}{c} a\text{-flush and } b\text{-flush} \\ \text{abacus diagrams} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} N\text{-}E \text{ lattice paths} \\ (0,0) \to (b,a) \\ \text{on or above } y = \frac{a}{b}x \end{array} \right\}.$$

[The reader will remark that we have applied a vertical reflection to Anderson's original bijection.]

*Proof.* The bijection organizes integers into a square lattice, where the $a$-flush condition on an abacus is read horizontally and the $b$-flush condition is read vertically, so that a $N$-$E$ lattice path specifies a *normalized* abacus that is both $a$-flush and $b$-flush by marking the dividing line between the set of beads and the set of gaps.

More specifically, in the box with corners $(i, j)$ and $(i + 1, j + 1)$, place the integer $-a(i + 1) + bj$. For a $N$-$E$ lattice path $L : (0, 0) \to (b, a)$, interpret the number to the right of an up step to be the defining bead on the $a$-flush abacus, and the number below a right step to be the defining bead on the $b$-flush abacus. (See Figure 3.) $\square$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| −23 | −22 | −21 | −20 | −19 | −18 | −17 | −16 |
| −15 | −14 | −13 | −12 | −11 | −10 | −9 | −8 |
| −7 | −6 | −5 | −4 | −3 | −2 | −1 | 0 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 94 | 86 | 78 | 70 | 62 | 54 | 46 | 38 | 30 | 22 | 14 | 6 | −2 |
| 81 | 73 | 65 | 57 | 49 | 41 | 33 | 25 | 17 | 9 | 1 | −7 | −15 |
| 68 | 60 | 52 | 44 | 36 | 28 | 20 | 12 | 4 | −4 | −12 | −20 | −28 |
| 55 | 47 | 39 | 31 | 23 | 15 | 7 | −1 | −9 | −17 | −25 | −33 | −41 |
| 42 | 34 | 26 | 18 | 10 | 2 | −6 | −14 | −22 | −30 | −38 | −46 | −54 |
| 29 | 21 | 13 | 5 | −3 | −11 | −19 | −27 | −35 | −43 | −51 | −59 | −67 |
| 16 | 8 | 0 | −8 | −16 | −24 | −32 | −40 | −48 | −56 | −64 | −72 | −80 |
| 3 | −5 | −13 | −21 | −29 | −37 | −45 | −53 | −61 | −69 | −77 | −85 | −93 |

FIGURE 4. The placement of integers in boxes $(i, j)$ for 8-flush and 13-flush abaci for $0 \leq i \leq 7$, $0 \leq j \leq 12$. The flush conditions force $-6$ to be a bead and 7 to be a gap. Because of the symmetry, a lattice path is completely defined by the portion from $(0, 0)$ to $(4, 6)$. An example of an antisymmetric abacus a its corresponding lattice path is also shown.

**Proposition 4.6.** *The following is a bijection:*

$$\mathcal{L} : \left\{ \begin{array}{c} \text{antisymmetric } a\text{-flush and} \\ b\text{-flush abacus diagrams} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} N\text{-}E \text{ lattice paths} \\ (0,0) \to (\lfloor \frac{b}{2} \rfloor, \lfloor \frac{a}{2} \rfloor) \end{array} \right\}.$$

*Proof.* When an antisymmetric abacus is $a$-flush and $b$-flush, this gives conditions on the possible sets of beads and gaps. For one, when $a$ (or $b$) is odd, then there must be a bead in position $\frac{1-a}{2}$ (or $\frac{1-b}{2}$) and a gap in position $\frac{a+1}{2}$ (or $\frac{b+1}{2}$) since they are an antisymmetric pair and in the same runner.

When $a$ and $b$ are of opposite parity, then there must be a bead in position $\frac{1-b-a}{2}$ and a gap in position $\frac{1+b+a}{2}$ because they are an antisymmetric pair and the inverse assignment would create an impossibility for position $\frac{1+b-a}{2}$ in terms of being both $a$-flush and $b$-flush.

Organize the integers into a square lattice depending on the parity of $a$. When $a$ is even, place the integer $\frac{1+b-a}{2} - ai + bj$ in the box with corners $(i, j)$ and $(i + 1, j + 1)$. When $a$ is odd, instead insert $\frac{1+2b-a}{2} - ai + bj$. (See Figure 4.)

As in type $A$ case, the rows correspond to the abacus with $a$ runners and the columns correspond to the abacus with $b$ runners. The conditions discussed above imply that a $a$-flush and $b$-flush abacus corresponds to a $N$-$E$ lattice path from $(0, 0)$ to $(b, a)$ which is symmetric about the point $\left(\frac{b}{2}, \frac{a}{2}\right)$ that separates the gaps (to the left and above) from the beads (to the right and below). The inherent symmetry implies that we need only consider the lattice path from $(0, 0)$ to $\left(\lfloor \frac{b}{2} \rfloor, \lfloor \frac{a}{2} \rfloor\right)$. □

**Remark.** The diagonal hook lengths discussed by Ford, Mai, and Sze can be recovered by analyzing the set of positive beads. An antisymmetric bead-gap pair for a positive bead $x$ corresponds to a diagonal hook of length $2x - 1$. Indeed, we recover the numbers in the lattice of Ford, Mai, and Sze after matching their indexing conventions and applying the transformation $f(x) = 2x - 1$ to the numbers in our lattice.

*Proof of Theorem 4.4.* In type $A$, the bijection of Anderson [And02, Proposition 1] specifies to a bijection

$$\mathcal{L} : \left\{ \begin{array}{c} n\text{-flush and } (mn+1)\text{-flush} \\ \text{abacus diagrams} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} N\text{-}E \text{ lattice paths} \\ (0,0) \to (n,n) \\ \text{on or above } y = x \end{array} \right\},$$

where the abacus diagram is normalized and drawn on runners 0 through $n-1$. In type $C$, the bijection in Proposition 4.6 restricts to the bijection

$$\mathcal{L} : \left\{ \begin{array}{c} \text{antisymmetric } 2n\text{-flush and} \\ (2n+1)\text{-flush abacus diagrams} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} N\text{-}E \text{ lattice paths} \\ (0,0) \to (n,n) \end{array} \right\}.$$

We will show that $\mathsf{maj}(\lambda)$ equals $\mathsf{maj}(L)$ in type $A$ and $2\,\mathsf{maj}(L)$ in type $C$ where $\mathsf{maj}(L)$ is the usual major statistic

$$\mathsf{maj}(L) = \sum_{i:(L_i, L_{i+1})=(E,N)} i$$

for lattice paths. The desired result follows from a classical result of MacMahon; see for example [Hag08, Chapter 1].

We must determine the positions of East steps followed by North steps in $L$. Since the position of the lowest bead in runner $i$ corresponds to the $i$-th North step, an East step before the $i$-th North step occurs if the level of the lowest bead in runner $i$ is less than or equal to the lowest bead in runner $i-1$, which is exactly the condition that the sequence $x$ has a descent in position $i$.

Now we must determine the step along $L$ where this North step occurs to see what its contribution to $\mathsf{maj}(\lambda)$ should be. A North step always corresponds to changing runners in the abacus. An East step corresponds to walking up the levels in the runner. So if the weak descent of $x$ occurs in position $i$ with a bead on level $x_i$, then this corresponds to having traversed $i$ North steps (in type $C$, $(i-1)$ North steps) and $(i-x_i)$ East steps, which contributes $2i - x_i$ to $\mathsf{maj}(\lambda)$ (in type $C$, $2(2i - x_i - 1)$). The sum over all descents gives Equations (4.1) and (4.2). □

## 5. Further questions

In this section, we present some directions for future research.

**Question 5.1.** The dominant regions of the Shi arrangement form a set of representatives for certain orbits inside the set of all (not necessarily dominant) Shi regions. For example, in type $A$ one can label the regions of the Shi arrangement by a Dyck path (representing some dominant Shi region) together with a permutation that is a minimal length coset representative for a quotient that is defined by the choice of Dyck path.

If we instead use a simultaneous core to represent the dominant Shi region in type $A$ or type $C$, what additional data would we need to add as a decoration in order to parameterize the full set of (not necessarily dominant) Shi regions? Recent work of Mészáros [Més13] contains relevant combinatorics for type $\widetilde{C}$.

**Question 5.2.** From the perspective of abacus diagrams, the simultaneous cores we have studied are defined entirely in terms of conditions that have the form “If a bead exists at position $i$ then a bead exists at position $f(i)$,” where $f(i)$ is the function $i - j$ with $j$ constant. As we have seen, these “convexity” conditions conspire to produce a finite set of abacus diagrams when $j$ is relatively prime to $n$.

It is natural to consider more general types of functions $f(i)$. For example, when defining abacus diagrams that correspond to dominant regions of the type $\widetilde{B}$ or type $\widetilde{D}$ Shi arrangements, we must impose distinct flush conditions depending on the column containing $i$ in the abacus.

Which functions $f$ produce finite sets of abaci? Is it possible to enumerate these sets directly from the abacus diagram and the conditions imposed by $f$? Are there other natural classes of partitions that are defined in terms of convex conditions on abaci?

**Question 5.3.** Our argument that the dominant regions of the $m$-Shi arrangement correspond to simultaneous core partitions does not generalize to types $\widetilde{B}$ and $\widetilde{D}$ because the condition for $s_i$ to be a right descent on the abacus analogous to Lemma 3.2 involves non-adjacent columns.

Starting from the fact that that the set of dominant alcoves in these types correspond to even $(2n)$-core partitions, it would be natural to look for a simple criterion on these partitions that selects the subset of minimal or bounded $m$-Shi alcoves. The conditions of being $(2mn \pm 1)$-core do not produce the correct subsets.

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