ABACUS MODELS FOR PARABOLIC QUOTIENTS OF AFFINE WEYL GROUPS

CHRISTOPHER R. H. HANUSA AND BRANT C. JONES

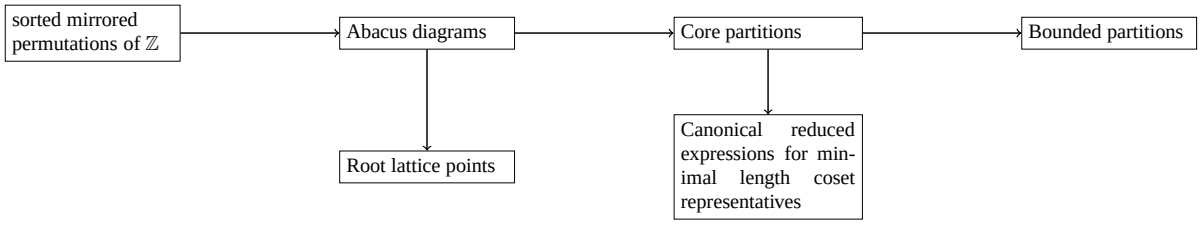
ABSTRACT. We introduce abacus diagrams that describe the minimal length coset representatives of affine Weyl groups in types $\widetilde{C}/C$, $\widetilde{B}/D$, $\widetilde{B}/B$ and $\widetilde{D}/D$. These abacus diagrams use a realization of the affine Weyl group $\widetilde{C}$ due to Eriksson to generalize a construction of James for the symmetric group. We also describe several combinatorial models for these parabolic quotients that generalize classical results in type $\widetilde{A}$ related to core partitions.

1. INTRODUCTION

Let $\widetilde{W}$ be an affine Weyl group, and W be the corresponding finite Weyl group. Then the cosets of W in $\widetilde{W}$, often denoted $\widetilde{W}/W$, have a remarkable combinatorial structure with connections to diverse structures in algebra and geometry, including affine Grassmannians [1, 2], characters and modular representations for the symmetric group [3, 4, 5], and crystal bases of quantum groups [6, 7]. Combinatorially, the elements in $\widetilde{W}$ can be understood as pairs from $\widetilde{W}/W \times W$ by the parabolic decomposition (see e.g. [8, Proposition 2.4.4]).

In type $\widetilde{A}$, these cosets correspond to a profoundly versatile combinatorial object known as an abacus diagram. From the abacus diagram, one can read off related combinatorial objects such as root lattice coordinates, core partitions, and the bounded partitions used in [9], [1] and [2]. The goal of the present paper is to extend the abacus model to types $\widetilde{B}$, $\widetilde{C}$, and $\widetilde{D}$, and define analogous families of combinatorial objects in these settings. Some of these structures are not, strictly speaking, new. Nevertheless, we believe that our development using abacus diagrams unifies much of the folklore, and we hope that it will be useful to researchers and students interested in extending results from type $\widetilde{A}$ to the other affine Weyl groups. In this sense, our paper is a companion to [10], [9] and [11].

The following diagram illustrates six families of combinatorial objects that are all in bijection. Each family has an action of $\widetilde{W}$, a Coxeter length function, and is partially ordered by the Bruhat order. We will devote one section to each of these objects and give the bijections between them using type-independent language.



We believe that the abacus diagrams and core partitions we introduce have not appeared in this generality before. In fact, we show in Theorem 5.11 that our construction answers a question of Billey and Mitchell; see Section 5.3. Several authors have used combinatorics related to bounded partitions, and we show how

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these objects are naturally related to abaci in Section 7. We also obtain some formulas for Coxeter length in Section 8 that appear to be new. To avoid interrupting the exposition, we postpone a few of the longer arguments from earlier sections to Section 9. Section 10 briefly suggests some ideas for further research.

In order to simplify the notation, we use a convention of overloading the definitions of our bijections. We let the output of the functions ($\mathcal{W}$, $\mathcal{A}$, $\mathcal{C}$, $\mathcal{R}$, $\mathcal{B}$) be the corresponding combinatorial interpretation ($\mathbb{Z}$-permutation, abacus diagram, core partition, canonical reduced expression, bounded partition, respectively), no matter the input. For example, if β is a bounded partition, then its corresponding abacus diagram is $\mathcal{A}(\beta)$.

2. George groups and $\mathbb{Z}$-permutations

2.1. **Definitions.** We follow the conventions of Björner and Brenti in [8, Chapter 8].

Definition 2.1. Fix a positive integer n and let $N = 2n + 1$. We say that a bijection $w : \mathbb{Z} \to \mathbb{Z}$ is a **mirrored $\mathbb{Z}$-permutation** if

$$w(i + N) = w(i) + N, \text{ and} \tag{2.1}$$

$$w(-i) = -w(i). \tag{2.2}$$

for all $i \in \mathbb{Z}$.

Eriksson and Eriksson [9] use these mirrored permutations to give a unified description of the finite and affine Weyl groups, based on ideas from [12]. It turns out that the collection of mirrored $\mathbb{Z}$-permutations forms a realization of the affine Coxeter group $\widetilde{C}_n$, where the group operation is composition of $\mathbb{Z}$-permutations. Since the Coxeter groups $\widetilde{B}_n$ and $\widetilde{D}_n$ are subgroups of $\widetilde{C}_n$, every element in any of these groups can be represented as such a permutation. Green [13] uses the theory of full heaps to obtain this and related representations of affine Weyl groups.

Remark 2.2. A mirrored $\mathbb{Z}$-permutation w is completely determined by its action on $\{1, 2, \dots n\}$. Also, Equations (2.1) and (2.2) imply that $w(i) = i$ for all $i = 0 \bmod N$.

We have Coxeter generators whose images $(w(1), w(2), \dots, w(n))$ are given by

$$s_i = (1, 2, \dots, i-1, i+1, i, i+2, \dots, n) \quad \text{for } 1 \leq i \leq n-1$$
$$s_0^C = (-1, 2, 3, \dots, n)$$
$$s_0^D = (-2, -1, 3, 4, \dots, n)$$
$$s_n^C = (1, 2, \dots, n-1, n+1)$$
$$s_n^D = (1, 2, \dots, n-2, n+1, n+2)$$

and we extend each of these to an action on $\mathbb{Z}$ via (2.1) and (2.2). Observe that each of these generators interchange infinitely many entries of $\mathbb{Z}$ by Equation (2.1).

Theorem 2.3. *The collection of mirrored $\mathbb{Z}$-permutations that satisfy the conditions in the second column of Table 1 form a realization of the corresponding affine Coxeter group $\widetilde{C}$, $\widetilde{B}$, or $\widetilde{D}$. The collection of mirrored $\mathbb{Z}$-permutations that additionally satisfy the sorting conditions in the third column of Table 1 form a collection of minimal length coset representatives for the corresponding parabolic quotient shown in the first column of Table 1. The corresponding Coxeter graph is shown in the fourth column of Table 1.*

Proof. Proofs can be found in [9] and [8, Section 8]. □

To describe the essential data that determines an mirrored $\mathbb{Z}$-permutation, we observe an equivalent symmetry.

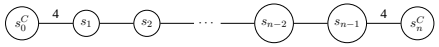
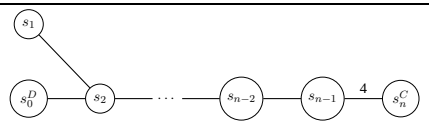
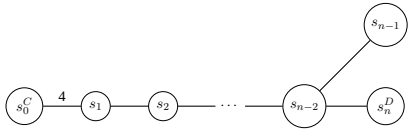
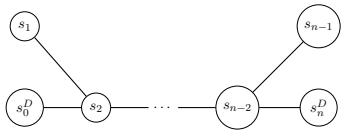
Type	Conditions on $\mathbb{Z}$-permutation for Coxeter group elements	Sorting conditions for minimal length coset representatives	Coxeter graph
$\widetilde{C}/C$		$w(1) < w(2) < \cdots < w(n) < w(n+1)$	
$\widetilde{B}/B$	$\lvert i \in \mathbb{Z} : i \leq 0, w(i) \geq 1\rvert \equiv 0 \mod 2$ (By (2.2), this is equivalent to requiring the number of negative entries lying to the right of position zero is even.)	$w(1) < w(2) < \cdots < w(n) < w(n+1)$ (Comparing with the condition on the previous row, we see that elements of $\widetilde{B}_n/B_n$ are elements of $\widetilde{C}_n/C_n$.)	
$\widetilde{B}/D$	$\lvert i \in \mathbb{Z} : i \leq n, w(i) \geq n+1\rvert \equiv 0 \mod 2$	$w(1) < w(2) < \cdots < w(n) < w(n+2)$ (Comparing with the previous conditions, we see that elements of $\widetilde{B}_n/D_n$ are *not* necessarily elements of $\widetilde{C}_n/C_n$, even though $\widetilde{B}_n$ is a subgroup of $\widetilde{C}_n$.)	
$\widetilde{D}/D$	$\lvert i \in \mathbb{Z} : i \leq 0, w(i) \geq 1\rvert \equiv 0 \mod 2$ and $\lvert i \in \mathbb{Z} : i \leq n, w(i) \geq n+1\rvert \equiv 0 \mod 2$	$w(1) < w(2) < \cdots < w(n) < w(n+2)$ (Comparing with the previous conditions, we see that elements of $\widetilde{D}_n/D_n$ are also elements of $\widetilde{B}_n/D_n$.)	

TABLE 1. Realizations of affine Coxeter groups

Lemma 2.4. (Balance Lemma) *If w is an mirrored $\mathbb{Z}$-permutation, then we have*

$$w(i) + w(N-i) = N \text{ for all } i = 1, 2, \dots n. \tag{2.3}$$

Conversely, if $w : \mathbb{Z} \to \mathbb{Z}$ is a bijection that satisfies (2.1) and (2.3), then w is an mirrored $\mathbb{Z}$-permutation.

Proof. Equations (2.1) and (2.2) imply that for all $i \in \mathbb{Z}$, $w(i) + w(N-i) = w(i) - w(i-N) = w(i) - \big(w(i) - N\big) = N$; in particular, this is true for $1 \leq i \leq n$.

To prove the converse, we must show that an infinite permutation $w : \mathbb{Z} \to \mathbb{Z}$ satisfying (2.1) and (2.3) satisfies (2.2). Equation (2.1) implies

$$w(-i - kN) = w(N-i) - (k+1)N$$

for any $i \in \{1, \dots, N\}$ and $k \geq 0$. By (2.3), we have $w(N-i) = N - w(i)$ so

$$w(-i-kN) = (N - w(i)) - (k+1)N = -w(i) - kN = -(w(i) + kN) = -w(i+kN),$$

as was to be shown. □

Given a mirrored $\mathbb{Z}$-permutation, we call the ordered sequence $[w(1), w(2), \dots, w(2n)]$ the **base window** of w. Since the set of mirrored $\mathbb{Z}$-permutations acts on itself by composition of functions, we have an action of $\widetilde{W}$ on the base window notation. The left action interchanges values while the right action interchanges positions. We have labeled the node of the Coxeter graph of $\widetilde{W}$ that is added to the Coxeter graph of W by s_0. Then, our cosets have the form $\widetilde{w}W$ (where $\widetilde{w} \in \widetilde{W}$), and the minimal length coset representatives all have s_0 as a unique right descent.

We can characterize the base windows that arise.

Lemma 2.5. *An ordered collection $[w(1), w(2), \dots, w(2n)]$ of integers is the base window for an element of $\widetilde{C}_n$ if and only if*

- *$w(1), \dots, w(2n)$ have distinct residue mod N,*
- *$w(1), \dots, w(2n)$ are not equivalent to 0 mod N, and*
- *$w(i) + w(N-i) = N$ for each $i = 1, \dots, 2n = N-1$.*

Proof. Given such a collection of integers, extend $[w(1), \dots, w(2n)]$ to a $\mathbb{Z}$-permutation w using (2.1), and set $w(iN) = iN$ for all $i \in \mathbb{Z}$. The third condition on $[w(1), \dots, w(2n)]$ ensures that w is a mirrored $\mathbb{Z}$-permutation by the Balance Lemma 2.4.

On the other hand, each of the three conditions is preserved when we apply a Coxeter generator s_i, so each element of $\widetilde{C}_n$ satisfies these conditions by induction on Coxeter length. □

With the conventions we have adopted in Table 1, we can also prove that no two minimal length coset representatives contain the same entries in their base window.

Lemma 2.6. *Suppose $a_1, \dots, a_n$ is a collection of integers such that each a_i is equivalent to i mod N. Then, there exists a unique element $w \in \widetilde{B}_n / D_n$ that contains $a_1, \dots, a_n$ among the entries of its base window $\{w(1), \dots, w(2n)\}$.*

Proof. It follows from the Balance Lemma 2.4 that whenever a_i appears among the entries of the base window of $w \in \widetilde{B}_n$, then $N - a_i$ also appears among the entries of the base window of w. Hence, the entries of the base window $\{a_1, \dots, a_n, N-a_1, \dots, N-a_n\}$ are completely determined by the a_i.

For w to be a minimal length coset representative, we must order these entries to satisfy the condition shown in the third column of Table 1 while maintaining the condition shown in the second column of Table 1. Let $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_{2n}$ denote the entries $\{a_1, \dots, a_n, N-a_1, \dots, N-a_n\}$ of the base window arranged into increasing order so $\tilde{a}_1 < \tilde{a}_2 < \cdots < \tilde{a}_{2n}$.

By the condition shown in the third column of Table 1 and (2.1), we have that $w(n)$ is the only possible descent among the entries of the base window of w, so we have that $w(n+1) < w(n+2)$. Also, since

$w(n) < w(n+2)$, we have $N - w(n) > N - w(n+2)$ which implies $w(n+1) > w(n-1)$ by (2.1). Since $w(1) < w(2) < \cdots < w(n) < w(n+2) < w(n+3) < \cdots < w(2n)$ forms an increasing subsequence of length $2n-1$, we have $w(n+2) \leq \tilde{a}_{n+2}$ and $w(n-1) \geq \tilde{a}_{n-1}$. Putting these together, we find

$$\tilde{a}_{n-1} \leq w(n-1) < w(n+1) < w(n+2) \leq \tilde{a}_{n+2}.$$

Thus, $w(n+1)$ must be $\tilde{a}_n$ or $\tilde{a}_{n+1}$. Therefore, the entries of the base window of $w \in \widetilde{B}_n/D_n$ are either $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_{n-1}, \tilde{a}_n, \tilde{a}_{n+1}, \tilde{a}_{n+2}, \cdots, \tilde{a}_{2n}$ or $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_{n-1}, \tilde{a}_{n+1}, \tilde{a}_n, \tilde{a}_{n+2}, \cdots, \tilde{a}_{2n}$. Since $\tilde{a}_n + \tilde{a}_{n+1} = N$, precisely one of these satisfies the condition shown in the second column of Table 1. □

Corollary 2.7. *Suppose $w, w' \in \widetilde{W}/W$. If $w \neq w'$ then*

$$\{w(1), w(2), \ldots, w(2n)\} \neq \{w'(1), w'(2), \ldots, w'(2n)\}$$

as unordered sets.

Proof. This follows from the observation that the sorting conditions completely determine the ordering of elements in the base window. Considering the third column of Table 1, this is clear for $\widetilde{C}/C$ and $\widetilde{B}/B$. By Lemma 2.6, we see that this holds for $\widetilde{B}/D$ as well. Since $\widetilde{D}_n/D_n \subset \widetilde{B}_n/D_n$, it holds for $\widetilde{D}/D$. □

3. Abacus diagrams

3.1. **Definitions.** We now combinatorialize the set of integers that can appear in the base window of a mirrored $\mathbb{Z}$-permutation as an abacus diagram. These diagrams enforce precisely the conditions from Lemma 2.5.

Definition 3.1. An **abacus diagram** (or simply **abacus**) is a diagram containing $2n$ columns labeled $1, 2, \ldots, 2n$, called **runners**. Runner i contains entries labeled by the integers $mN + i$ for each **level** m where $-\infty < m < \infty$.

We draw the abacus so that each runner is vertical, oriented with $-\infty$ at the top and ∞ at the bottom, with runner 1 in the leftmost position, increasing to runner $2n$ in the rightmost position. Entries in the abacus diagram may be circled; such circled elements are called **beads**. Entries that are not circled are called **gaps**. The linear ordering of the entries given by the labels $mN + i$ (for level $m \in \mathbb{Z}$ and runner $1 \leq i \leq 2n$) is called the **reading order** of the abacus which corresponds to scanning left to right, top to bottom. (Observe that there are no entries in the abacus having labels $\{mN : m \in \mathbb{Z}\}$.)

We say that a bead b is **active** if there exist gaps (on any runner) that occur prior to b in reading order. Otherwise, we say that the bead is **inactive**. A runner is called **flush** if no bead on the runner is preceded in reading order by a gap on that same runner. We say that an abacus is **flush** if every runner is flush. We say that an abacus is **balanced** if

- there is at least one bead on every runner i for $1 \leq i \leq 2n$, and
- the sum of the labels of the lowest beads on runners i and $N - i$ is N for all $i = 1, 2, \ldots, 2n$.

We say that an abacus is **even** if there exists an even number of gaps preceding N in reading order.

Definition 3.2. Given a mirrored $\mathbb{Z}$-permutation w, we define $\mathcal{A}(w)$ to be the flush abacus whose lowest bead in each runner is an element of $\{w(1), w(2), \ldots, w(2n)\}$.

Note that this is well-defined by Lemma 2.5. Also, $\mathcal{A}(w)$ is always balanced by Lemma 2.4, so the level of the lowest bead on runner i is the negative of the level of the lowest bead on runner $N - i$. In the rest of the paper, we will implicitly assume that all abaci are balanced and flush unless otherwise noted.

Example 3.3. For the minimal length coset representative $w \in \widetilde{C}_3/C_3$ whose base window is $[-11, -9, -1, 8, 16, 18]$, the balanced flush abacus $a = \mathcal{A}(w)$ is given in Figure 1.

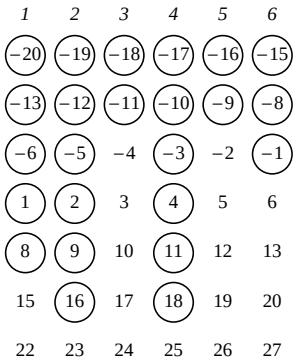


FIGURE 1. The balanced flush abacus diagram $a = \mathcal{A}(w)$ for the minimal length coset representative $w = [-11, -9, -1, 8, 16, 18] \in \widetilde{C}_3/C_3$. The circled entries are the beads; the noncircled entries are the gaps.

Type	Conditions on abaci
$\widetilde{C}/C$	balanced flush abaci
$\widetilde{B}/B$	even balanced flush abaci
$\widetilde{B}/D$	balanced flush abaci
$\widetilde{D}/D$	even balanced flush abaci

TABLE 2. Abaci for $\widetilde{W}/W$

We record some structural facts about balanced flush abaci to be used later.

Lemma 3.4. *For each $i \in \mathbb{Z}$, we have that entry $N + i$ is a gap if and only if entry $N - i$ is a bead.*

Proof. This follows from the definition together with the Balance Lemma 2.4. □

Lemma 3.5. *Fix an abacus a and consider a single row r of a. If there exists i such that the entries in columns i and $N - i$ are both beads, then the level of row r is ≤ 0. Similarly, if there exists i such that the entries in columns i and $N - i$ are both gaps, then the level of row r is > 0. In particular, we cannot have both of these conditions holding at the same time for a given row of a.*

Proof. This follows from the Balance Lemma 2.4. □

Lemma 3.6. *For each of $\widetilde{C}/C$, $\widetilde{B}/B$, $\widetilde{B}/D$ and $\widetilde{D}/D$, the map $\mathcal{A}$ is a bijection from $\widetilde{W}/W$ to the set of abaci shown in column 2 of Table 2.*

Proof. This follows from Corollary 2.7 and Lemma 2.5 for $\widetilde{C}/C$. In types $\widetilde{B}/B$ and $\widetilde{D}/D$, the condition

$$|i \in \mathbb{Z} : i \leq 0, w(i) \geq 1| \equiv 0 \mod 2$$

is equivalent to the even condition on abaci. To see this, recall that the entries in the base window of a mirrored permutation consist of the labels of the lowest beads in each runner of the abacus. Therefore, the positive entries of a mirrored permutation that appear to the left of the base window correspond to the beads lying directly above some bead in the abacus that lies to the right of N in reading order. The set of beads in the abacus succeeding N in reading order has the same cardinality as the set of gaps preceding N in reading order by Lemma 3.4.

As explained in Lemma 2.6, the condition

$$|i \in \mathbb{Z} : i \leq n, w(i) \geq n+1| \equiv 0 \mod 2$$

in type $\widetilde{B}/D$ only changes the ordering of the sorted entries in the base window, not the set of entries themselves.

The result then follows from Corollary 2.7. □

3.2. **Action of $\widetilde{W}$ on the abacus.** If we translate the action of the Coxeter generators on the mirrored $\mathbb{Z}$-permutations through the bijection $\mathcal{A}$, we find that

- s_i interchanges column i with column $i+1$ and interchanges column $2n-i$ with column $2n-i+1$, for $1 \leq i \leq n-1$
- s_0^C interchanges column 1 and $2n$, and then shifts the lowest bead on column 1 down one level towards ∞, and shifts the lowest bead on column $2n$ up one level towards $-\infty$
- s_0^D interchanges columns 1 and 2 with columns $2n-1$ and $2n$, respectively, and then shifts the lowest beads on columns 1 and 2 down one level each towards ∞, and shifts the lowest beads on columns $2n-1$ and $2n$ up one level each towards $-\infty$
- s_n^C interchanges column n with column $n+1$
- s_n^D interchanges columns $n-1$ and n with columns $n+1$ and $n+2$, respectively.

4. Root lattice points

4.1. **Definitions.** Following [14, Section 4], let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of the Euclidean space $V = \mathbb{R}^n$ and denote the corresponding inner product by $\langle \cdot, \cdot \rangle$. Define the **simple roots** α_i and the **longest root** $\widetilde{\alpha}$ for each type $W_n \in \{B_n, C_n, D_n\}$ as in [14, page 42]. The $\mathbb{Z}$-span Λ_R of the simple roots is called the **root lattice**, and we may identify V with $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda_R$ because the simple roots in types B_n, C_n and D_n are linearly independent.

There is an action of $\widetilde{W}$ on V in which s_i is the reflection across the hyperplane perpendicular to α_i for $i = 1, 2, \ldots, n$ and s_0 is the affine reflection

$$s_0(v) = v - (\langle v, \widetilde{\alpha} \rangle - 1) \frac{2}{\langle \widetilde{\alpha}, \widetilde{\alpha} \rangle} \widetilde{\alpha}.$$

Suppose w is a minimal length coset representative in $\widetilde{W}/W$ and define the **root lattice coordinate** of w to be the result of acting on $0 \in V$ by w.

Theorem 4.1. *The root lattice coordinate of an element $w \in \widetilde{W}/W$ is*

$$\sum_{i=1}^{n} \mathsf{level}_i(\mathcal{A}(w)) e_i$$

where $\mathsf{level}_i(\mathcal{A}(w))$ denotes the level of the lowest bead in column i of the abacus $\mathcal{A}(w)$. Moreover, this is a bijection to the collections of root lattice coordinates shown in Table 3.

Proof. Once we identify the Coxeter graphs from Table 1 with those in [14], it is straightforward to verify that the action of $\widetilde{W}$ on the root lattice is the same as the action of $\widetilde{W}$ on the levels of the abacus given in Section 3.2.

For example, in $\widetilde{B}_n/B_n$, we have $\alpha_n = e_n$ so

$$\begin{aligned} s_n(a_1 e_1 + \cdots + a_n e_n) &= (a_1 e_1 + \cdots + a_n e_n) - (a_n - 0) \frac{2}{\langle \alpha_n, \alpha_n \rangle} \alpha_n \\ &= a_1 e_1 + \cdots + a_{n-1} e_{n-1} - a_n e_n \end{aligned}$$

and this corresponds to interchanging columns n and $n+1$ in the abacus by the Balance Lemma 2.4. Similarly, reflection through the hyperplane orthogonal to a root of the form $e_{n-1}+e_n$ corresponds to interchanging columns $n-1$ and n with columns $n+1$ and $n+2$, respectively. The generators s_0^C and s_0^D are affine reflections, so we need to shift the level by 1 as described in Section 3.2. For example, in $\widetilde{C}_n/C_n$, we have $\widetilde{\alpha}=2e_1$ so

$$\begin{aligned} s_0(a_1e_1+\cdots+a_ne_n) &= (a_1e_1+\cdots+a_ne_n)-(2a_1-1)\frac{2}{\langle\widetilde{\alpha},\widetilde{\alpha}\rangle}\widetilde{\alpha} \\ &= (-a_1+1)e_1+a_2e_2\cdots+a_ne_n. \end{aligned}$$

Because the number of beads to the right of N in an abacus is $\sum_{i=1}^{n}|\mathsf{level}_i(\mathcal{A}(w))|$, it follows from the proof of Lemma 3.6 that the correspondence between abaci and root lattice coordinates is a bijection to the images shown in Table 3. □

Type	Set of root lattice coordinates
$\widetilde{C}/C$	$(a_1,\ldots,a_n)\in\mathbb{Z}^n$
$\widetilde{B}/B$	$(a_1,\ldots,a_n)\in\mathbb{Z}^n$ such that $\sum_{i=1}^{n}\lvert a_i\rvert$ is even.
$\widetilde{B}/D$	$(a_1,\ldots,a_n)\in\mathbb{Z}^n$
$\widetilde{D}/D$	$(a_1,\ldots,a_n)\in\mathbb{Z}^n$ such that $\sum_{i=1}^{n}\lvert a_i\rvert$ is even.

TABLE 3. Root lattice points for $\widetilde{W}/W$

Example 4.2. For the minimal length coset representative $w=[-11,-9,-1,8,16,18]\in\widetilde{C}_3/C_3$, the root lattice coordinates $e_1+2e_2-2e_3$ can be read directly from the levels of the lowest beads in the first three runners of the abacus in Figure 1.

Shi [15] has worked out further details about the relationship between root system geometry and mirrored $\mathbb{Z}$-permutations.

5. CORE PARTITIONS

5.1. **Definitions.** A **partition** is a sequence $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_k\geq 0$ of weakly decreasing integers. Each partition has an associated **diagram** in which we place λ_i unit boxes on the i-th row of the diagram, where the first row is drawn at the top of the diagram. The **hook length** of a box B in λ is the sum of the number of boxes lying to the right of B in the same row and the number of boxes lying below B in the same column, including B itself. The **main diagonal** of a partition diagram is the set of all boxes with position coordinates $(i,i)\in\mathbb{N}^2$; for non-zero integers j, the **j-th diagonal** of a partition diagram is the set of all boxes with position coordinates $(i,i+j)\in\mathbb{N}^2$. We use the notation $p\|q$ to denote the integer in $\{0,1,\ldots,q-1\}$ that is equal to p mod q.

Definition 5.1. We say that a partition λ is a $(2n)$-**core** if it is impossible to remove $2n$ consecutive boxes from the southeast boundary of the partition diagram in such a way that the result is still a partition diagram. Equivalently, λ is a $(2n)$-core if no box in λ has a hook length that is divisible by $2n$. We say that a partition λ is **symmetric** if the length of the i-th row of λ is equal to the length of the i-th column of λ, for all i. We say that a partition λ is **even** if there are an even number of boxes on the main diagonal of λ.

Every abacus diagram a determines a partition λ, as follows.

Definition 5.2. Given an abacus a, create a partition $\mathcal{C}(a)$ whose southeast boundary is the lattice path obtained by reading the entries of the abacus in reading order and recording a north-step for each bead, and recording an east-step for each gap.

If we suppose that there are M active beads in a, then $\mathcal{C}(a)$ can be equivalently described as the partition whose i-th row contains the same number of boxes as gaps that appear before the $(M-i+1)$st active bead in reading order. In this way each box in the partition corresponds to a unique bead-gap pair from the abacus in which the gap occurs before the active bead in reading order.

For an active bead b in an abacus a, we define the **symmetric gap** $g(b)$ to be the gap in position $2N-b$ that exists by Lemma 3.4. Then, for $b > N$ the bead-gap pair $(b, g(b))$ in a corresponds to the box on the main diagonal on the row of $\mathcal{C}(a)$ corresponding to b.

Example 5.3. For the minimal length coset representative $w \in \widetilde{C}_3/C_3$ whose base window is $[-11,-9,-1,8,16,18]$, the partition $\lambda = \mathcal{C}(w) = (10,9,6,5,5,3,2,2,2,1)$ is given in Figure 2. To find this partition from the abacus diagram in Figure 1, follow the abacus in reading order, recording a horizontal step for every gap (starting with the gap in position -4) and a vertical step for every bead (ending with the bead in position 18).

0	*1*	*2*	*3*	*2*	*1*	*0*	*1*	*2*	*3*
1	*0*	*1*	*2*	*3*	*2*	*1*	*0*	*1*	
2	*1*	*0*	*1*	*2*	*3*				
3	*2*	*1*	*0*	*1*					
2	*3*	*2*	*1*	*0*					
1	*2*	*3*							
0	*1*								
1	*0*								
2	*1*								
3									

FIGURE 2. The 6-core partition $\lambda = \mathcal{C}(w) = (10,9,6,5,5,3,2,2,2,1)$ for the minimal length coset representative $w = [-11,-9,-1,8,16,18] \in \widetilde{C}_3/C_3$. The numbers inside the boxes are the residues, described below.

Proposition 5.4. *The map $\mathcal{C}$: {abaci} $\to$ {partitions} is a bijection from and onto the sets shown in Table 4 on Page 12.*

Proof. A balanced flush abacus a determines a partition $\mathcal{C}(a)$ by Definition 5.2. The fact that a is flush by construction implies that $\mathcal{C}(a)$ is a $(2n)$-core. By Lemma 3.4, the sequence of gaps and beads is inverted when reflected about position N, so $\mathcal{C}(a)$ is symmetric.

We can define an inverse map. Starting from a $(2n)$-core partition λ, encode its southeast boundary lattice path on the abacus by recording each north-step as a bead and each east-step as a gap, placing the midpoint of the lattice path from λ to lie between entries $N-1$ and $N+1$ on the abacus. The resulting abacus will be flush because λ is a $(2n)$-core. The resulting abacus will be balanced because whenever position i is the lowest bead on runner $(i\|N)$, then by symmetry $2N-i$ is the highest gap on runner $(N-i\|N)$ so $N-i$ is the lowest bead on runner $(N-i\|N)$.

Moreover, we claim that the map $\mathcal{C}$ restricts to a bijection between even abaci and even core partitions. In the correspondence between abaci and partitions, the entry N corresponds to the midpoint of the boundary lattice path. In particular, this entry lies at the corner of a box on the main diagonal. Therefore, the number of gaps preceding N in the reading order of a is equal to the number of horizontal steps

lying below the main diagonal of $\lambda = \mathcal{C}(a)$, which is exactly the number of boxes contained on the main diagonal of λ. $\square$

5.2. **Residues for the action of $\widetilde{W}$.** If we translate the action of the Coxeter generators on abaci through the bijection $\mathcal{C}$, we obtain an action of $\widetilde{W}$ on the symmetric $(2n)$-core partitions.

To describe this action, we introduce the notion of a **residue** for a box in the diagram of a symmetric $(2n)$-core partition. The idea that motivates the following definitions is that s_i should act on a symmetric $(2n)$-core λ by adding or removing all boxes with residue i. In contrast with the situation in types $\widetilde{A}$ and $\widetilde{C}$, it will turn out that for types $\widetilde{B}$ and $\widetilde{D}$ the residue of a box in λ may depend on λ and not merely on the coordinates of the box.

To begin, we orient $\mathbb{N}^2$ so that (i, j) corresponds to row i and column j of a partition diagram, and define the **fixed residue** of a position in $\mathbb{N}^2$ to be

$$\mathsf{res}(i,j) = \begin{cases} (j-i) \| (2n) & \text{if } 0 \leq (j-i) \| (2n) \leq n \\ 2n - \big((j-i) \| (2n)\big) & \text{if } n < (j-i) \| (2n) < 2n. \end{cases}$$

Then, the fixed residues are given by extending the pattern illustrated below.

$$\begin{array}{ccccccccccccc}
0 & 1 & 2 & \cdots & n-1 & n & n-1 & \cdots & 2 & 1 & 0 & 1 & 2 \\
1 & 0 & 1 & 2 & \cdots & n-1 & n & n-1 & \cdots & 2 & 1 & 0 & 1 \\
2 & 1 & 0 & 1 & 2 & \cdots & n-1 & n & n-1 & \cdots & 2 & 1 & 0 \\
\vdots & 2 & 1 & 0 & 1 & 2 & \cdots & n-1 & n & n-1 & \cdots & 2 & 1 \\
n-1 & \vdots & 2 & 1 & 0 & 1 & 2 & \cdots & n-1 & n & n-1 & \cdots & 2 \\
n & n-1 & \vdots & 2 & 1 & 0 & 1 & 2 & \cdots & n-1 & n & n-1 & \cdots \\
n-1 & n & n-1 & \vdots & 2 & 1 & 0 & 1 & 2 & \cdots & n-1 & n & n-1 \\
\vdots & n-1 & n & n-1 & \vdots & 2 & 1 & 0 & 1 & 2 & \cdots & n-1 & n
\end{array}$$

We define an **escalator** to be a connected component of the entries (i, j) in $\mathbb{N}^2$ satisfying

$$(j-i) \| (2n) \in \{n-1, n, n+1\}.$$

If an escalator lies above the main diagonal $i = j$, then we say it is an **upper escalator**; otherwise, it is called a **lower escalator**. Similarly, we define a **descalator** to be a connected component of the entries (i, j) in $\mathbb{N}^2$ satisfying

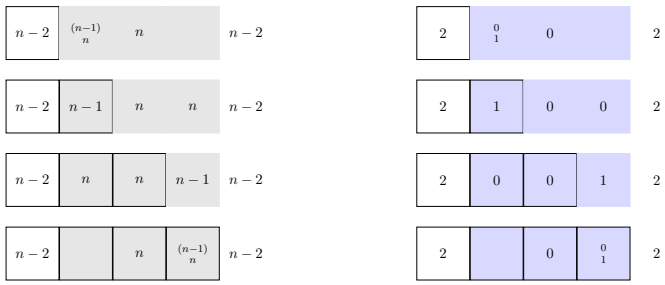
$$(j-i) \| (2n) \in \{-1, 0, 1\}.$$

If a descalator lies above the main diagonal $i = j$, then we say it is an **upper descalator**; if a descalator lies below the main diagonal $i = j$, it is called a **lower descalator**. There is one **main descalator** that includes the main diagonal $i = j$ which is neither upper nor lower.

Let λ be a symmetric $(2n)$-core partition. We define the **residues of the boxes in an upper escalator** lying on row i depending on the number of boxes in the i-th row of λ that intersect the escalator, as shown in Figure 3(a).

Here, the outlined boxes represent entries that belong to the row of λ, while the shaded cells represent entries in an upper escalator. The schematic in Figure 3(a) shows all ways in which these two types of entries can overlap, and we have written the residue assignments that we wish to assign for each entry.

Note that in the first case, where λ is adjacent to but does not intersect the upper escalator, we view the first box of the escalator as being simultaneously $(n-1)$-addable and n-addable. In this case, the rightmost cell of the upper escalator has undetermined residue, and is neither addable nor removable. A similar situation occurs when a row of λ ends with three boxes in the upper escalator. In all other cases, the entries of an upper escalator have undefined residue.



(a) (b)

FIGURE 3. Residue assignments for upper escalators and descalators

We similarly define the **residues of the boxes in a lower escalator** lying on column j depending on the number of boxes in the j-th column of λ that intersect the escalator. The precise assignment is simply the transpose of the schematic in Figure 3(a).

Moreover, we similarly define the **residues of the boxes in an upper descalator** on row i depending on the number of boxes in the i-th row of λ that intersect the descalator, as shown in Figure 3(b). The transpose of this schematic gives the assignment of **residues of the boxes in a lower descalator**.

Finally, the residues of the descalator containing the main diagonal are fixed, and we define the **residue of an entry** (i, j) **lying on the main descalator** to be

$$\mathsf{mres}(i,j) = \begin{cases} 0 & \text{if } (j-i) = 0 \\ 1 & \text{if } (j-i) \in \{1,-1\} \text{ and } (j+i) \equiv 1 \mod 4 \\ 0 & \text{if } (j-i) \in \{1,-1\} \text{ and } (j+i) \equiv 3 \mod 4 \end{cases}$$

where the upper left most box has coordinates $(i, j) = (1, 1)$.

Example 5.5. In Figure 4, we consider residues using all of the features discussed above. It will turn out that this corresponds to type $\widetilde{D}_5/D_5$ and that the assignment of residues in each of the other types uses a subset of these features as described in the fourth column of Table 4.

Here, the unshaded entries of Figure 4 are fixed residues given by $\mathsf{res}(i, j)$ for $n = 5$; the gray shaded entries represent an upper and lower escalator, and the blue shaded entries represent an upper and lower descalator. The descalator containing the main diagonal has fixed residues given by $\mathsf{mres}(i, j)$. The precise residues of the boxes in the upper/lower escalators/descalators for a particular partition depend on how the partition intersects the shaded regions.

Definition 5.6. Suppose λ is a symmetric $(2n)$-core partition representing an element of type $\widetilde{W}_n/W_n \in \{\widetilde{C}_n/C_n, \widetilde{B}_n/B_n, \widetilde{B}_n/D_n, \widetilde{D}_n/D_n\}$, and embed the digram of λ in $\mathbb{N}^2$ as above. Then we assign the **residue** of $(i, j) \in \mathbb{N}^2$ as described above, using the features listed in the fourth column of Table 4.

Definition 5.7. Given a symmetric $(2n)$-core partition λ with residues assigned, we say that two boxes from $\mathbb{N}^2$ are i**-connected** whenever they share an edge and have the same residue i. We refer to the i-connected components of boxes from $\mathbb{N}^2$ as i**-components**.

We say that an i-component C is **addable** if adding the boxes of C to the diagram of λ results in a partition, and that C is **removable** if removing the boxes of C from the diagram of λ results in a partition.

We are now in a position to state the action of $\widetilde{W}$ on symmetric $(2n)$-core partitions in terms of residues.

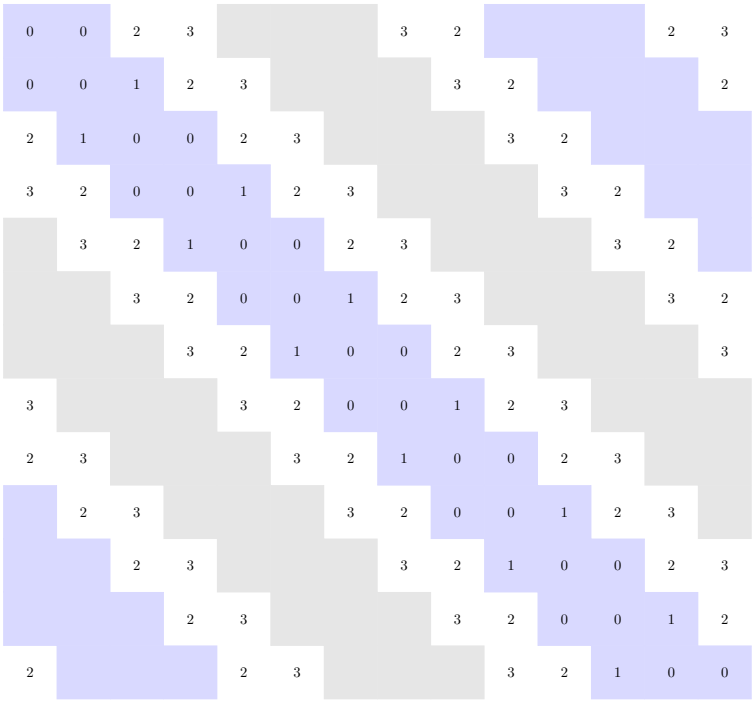


FIGURE 4. The fixed residues in $\widetilde{D}_5/D_5$

Type	Abaci	Partitions	Features for residue assignment
$\widetilde{C}/C$	balanced flush abaci	symmetric $(2n)$-cores	fixed residues only
$\widetilde{B}/B$	even balanced flush abaci	even symmetric $(2n)$-cores	fixed residues with descalators
$\widetilde{B}/D$	balanced flush abaci	symmetric $(2n)$-cores	fixed residues with escalators
$\widetilde{D}/D$	even balanced flush abaci	even symmetric $(2n)$-cores	fixed residues with escalators and descalators

TABLE 4. Core partitions for $\widetilde{W}/W$

Theorem 5.8. *Let $w \in \widetilde{W}/W$ and suppose $\lambda = \mathcal{C}(w)$. If s_i is an ascent for w then s_i acts on λ by adding all addable i-components to λ. If s_i is a descent for w then s_i acts by removing all removable i-components from λ. If s_i is neither an ascent nor a descent for w then s_i does not change λ.*

The proof of this result is postponed to Section 9.

Example 5.9. Consider the minimal length coset representative $w = [-12, -7, -5, 2, 3, 8, 9, 16, 18, 23] \in \widetilde{D}_5/D_5$. The corresponding core partition $\lambda = \mathcal{C}(w) = (11, 8, 7, 4, 3, 3, 3, 2, 1, 1, 1)$ is pictured in Figure 5. The known residues are placed in their corresponding boxes.

If we were to apply the generator s_0 to λ, this would remove four boxes—the two boxes in the upper right corner and the two boxes in the lower left corner. The box on the main diagonal with residue 0 is not removed because we cannot remove only part of its connected component of residue 0 boxes.

From λ, we determine that s_0 and s_4 are descents (as they would remove boxes), s_1 and s_3 are ascents (as they would add boxes), and s_2 and s_5 are neither ascents nor descents (as they would leave the diagram unchanged).

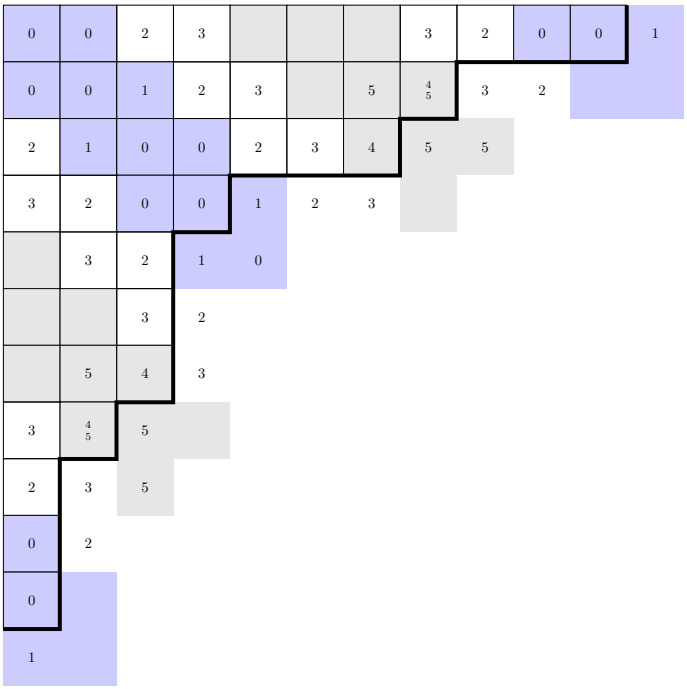


FIGURE 5. The core partition $\lambda = \mathcal{C}(w) = (11, 8, 7, 4, 3, 3, 3, 2, 1, 1, 1)$ corresponding to the minimal length coset representative $w = [-12, -7, -5, 2, 3, 8, 9, 16, 18, 23] \in \widetilde{D}_5/D_5$. The numbers indicate the residues of the boxes.

5.3. **Bruhat order on symmetric $(2n)$-cores.** In this section, we use an argument of Lascoux [11] to show that Bruhat order on the minimal length coset representatives $\widetilde{W}/W$ corresponds to a modified containment order on the corresponding core partition diagrams. This affirmatively answers a question of Billey and Mitchell [2, Remark 12].

Definition 5.10. Let λ and μ be two symmetric $(2n)$-cores. Suppose that every box of μ that does not lie on an escalator or descalator is also a box of λ, and that:

- Whenever the i-th row of μ intersects an upper escalator, upper descalator or the main descalator in 1 box, then the i-th row of λ intersects the given region in 1 or 3 boxes.
- Whenever the i-th row of μ intersects an upper escalator or upper descalator or the main descalator in 2 boxes, then the i-th row of λ intersects the given region in 2 or 3 boxes.
- Whenever the i-th row of μ intersects an upper escalator or upper descalator or the main descalator in 3 boxes, then the i-th row of λ intersects the given region in 3 boxes.

In this situation, we say that λ **contains** μ, denoted $\lambda \trianglerighteq \mu$.

Theorem 5.11. *Let $w, x \in \widetilde{W}/W$. Then $w \geq x$ in Bruhat order if and only if $\mathcal{C}(w) \trianglerighteq \mathcal{C}(x)$.*

Proof. We proceed by induction on the number of boxes in w. When the Coxeter length of w is 1, then $x = w$, $x = e$, or x is not related to w in Bruhat order. In each case, the result is clear.

Let s_i be a Coxeter generator such that $s_i w < w$ in $\widetilde{W}/W$. If $x \leq w$ then the Lifting Lemma [8, Proposition 2.2.7] implies that $s_i w \geq \min(x, s_i x)$ in $\widetilde{W}$. Every reduced expression for a nontrivial element of $\widetilde{W}/W$ ends in s_0, so if $s_i x < x$ then $s_i x \in \widetilde{W}/W$. Therefore, $s_i w \geq \min(x, s_i x)$ in $\widetilde{W}/W$.

Conversely, if $w > s_i w \geq \min(x, s_i x)$ in $\widetilde{W}/W$ then we find that $w \geq x$; this follows directly when $x < s_i x$ or $x = s_i x$, and follows by another application of the Lifting Lemma when $s_i x < x$.

We therefore have the equivalence $x \leq w$ if and only if $\min(x, s_i x) \leq s_i w$. Hence, we reduce to considering the pair $w' = s_i w$, $x' = \min(x, s_i x)$ in $\widetilde{W}/W$, and we need to show that $\mathcal{C}(w') \trianglerighteq \mathcal{C}(x')$ if and only if $\mathcal{C}(w) \trianglerighteq \mathcal{C}(x)$ to complete the proof by induction.

Suppose $\mathcal{C}(w) \trianglerighteq \mathcal{C}(x)$. Then we pass to $\mathcal{C}(s_i w)$ by removing boxes with residue i from the end of their rows and columns. By Definition 5.10, every such box from $\mathcal{C}(w)$ is either removable in $\mathcal{C}(x)$ or else absent from $\mathcal{C}(x)$. Hence, $\mathcal{C}(w') \trianglerighteq \mathcal{C}(x')$. Similarly, it follows from Definition 5.10 that if $\mathcal{C}(w') \trianglerighteq \mathcal{C}(x')$ then every addable box with residue i in $\mathcal{C}(x')$ is either addable in $\mathcal{C}(w')$ or already present in $\mathcal{C}(w')$. Hence, $\mathcal{C}(w) \trianglerighteq \mathcal{C}(x)$. □

6. Reduced expressions from cores

6.1. **The upper diagram.** Let $\lambda = \mathcal{C}(w)$. We now define a recursive procedure to obtain a canonical reduced expression for w from λ. Recall that the first diagonal is the diagonal immediately to the right of the main diagonal.

Definition 6.1. Define the **reference diagonal** to be

$$\begin{cases} \text{the main diagonal} & \text{in types } \widetilde{C}/C \text{ and } \widetilde{B}/D \\ \text{the first diagonal} & \text{in types } \widetilde{B}/B \text{ and } \widetilde{D}/D \end{cases}$$

Given a core $\lambda = \lambda^{(1)}$ we define the **central peeling procedure** recursively as follows. At step i, we consider $\lambda^{(i)}$ and set $d^{(i)}$ to be the number of boxes on the reference diagonal of $\lambda^{(i)}$. Suppose B_i is the box at the end of the $d^{(i)}$-th row of $\lambda^{(i)}$ and r_i is the residue of this box. In the case when B_i is both n-removable and $(n-1)$-removable, we set $r_i = n$. Apply generator s_{r_i} to $\lambda^{(i)}$ to find $\lambda^{(i+1)}$.

We define $\mathcal{R}(\lambda)$ to be the product of generators $s_{r_1} s_{r_2} \cdots s_{r_\ell}$.

We also define the **upper diagram**, denoted U_λ, to be the union of the boxes B_i encountered in the central peeling procedure respecting the following conditions: In $\widetilde{B}/D$ and $\widetilde{D}/D$, the application of s_n^D removes two boxes from the $d^{(i)}$-th row and we record in U_λ the box *not* on the n-th diagonal; In $\widetilde{B}/B$ and $\widetilde{D}/D$, the application of s_0^D removes two boxes from the $d^{(i)}$-th row and we record in U_λ the box *not* on the main or $2n$-th diagonal.

Proposition 6.2. *We have that $\mathcal{R}(\lambda)$ is a reduced expression for w and the number of boxes in U_λ equals the Coxeter length of $\mathcal{W}(\lambda)$.*

Proof. At each step, the central peeling procedure records a descent so this follows from Proposition 5.8. □

We now present two examples of the central peeling procedure, one in $\widetilde{C}_3/C_3$ and another in $\widetilde{D}_4/D_4$.

Example 6.3. The steps of the central peeling procedure applied to $\lambda = \mathcal{C}([-11, -9, -1, 8, 16, 18]) \in \widetilde{C}_3/C_3$ are presented in Figure 6. The collection of gray boxes tallied during the central peeling procedure is U_λ; Figure 7 shows U_λ superimposed over λ. We can read off the canonical reduced expression $\mathcal{R}(\lambda)$ starting in the center of λ and working our way up, reading the gray boxes from right to left. We have $\mathcal{R}(\lambda) = s_0 s_1 s_0 s_3 s_2 s_1 s_0 s_2 s_3 s_2 s_1 s_0 s_2 s_3 s_2 s_1 s_0$.

Example 6.4. Let $w = [-15, -11, -10, 4, 5, 19, 20, 24] \in \widetilde{D}_4/D_4$. The steps of the central peeling procedure applied to $\lambda = \mathcal{C}(w)$ are presented in Figure 8; notice that no tallied boxes are on the main diagonal or the fourth diagonal. These gray boxes make up U_λ, which in turn is superimposed over λ in Figure 9. The canonical reduced expression is $\mathcal{R}(\lambda) = s_0 s_4 s_2 s_1 s_4 s_3 s_2 s_0 s_4 s_3 s_2 s_1 s_4 s_3 s_2 s_0$.

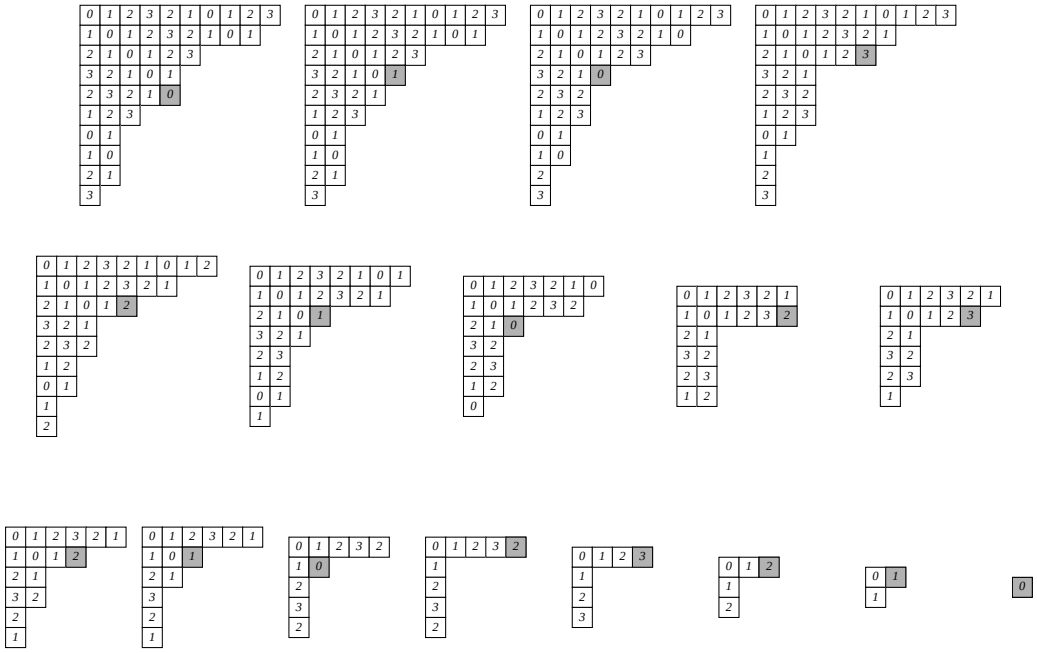


FIGURE 6. The application of the central peeling procedure described in Definition 6.1 on the core partition $\lambda = \mathcal{C}(w)$ for $w = [-11, -9, -1, 8, 16, 18] \in \widetilde{C}_3/C_3$. The shaded boxes are the boxes B_i tallied by the algorithm; they make up the upper diagram U_λ, shown in Figure 7.

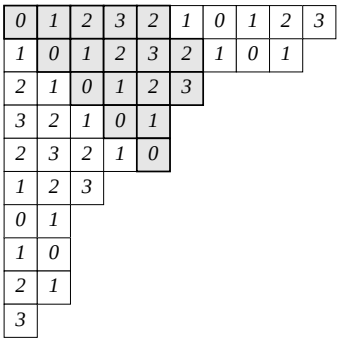


FIGURE 7. The core partition $\lambda = \mathcal{C}(w)$ for $w = [-11, -9, -1, 8, 16, 18] \in \widetilde{C}_3/C_3$, with its upper diagram U_λ superimposed. Reading the shaded boxes from the bottom to the top from right to left gives the canonical reduced expression $\mathcal{R}(\lambda) = s_0s_1s_0s_3s_2s_1s_0s_2s_3s_2s_1s_0s_2s_3s_2s_1s_0$.

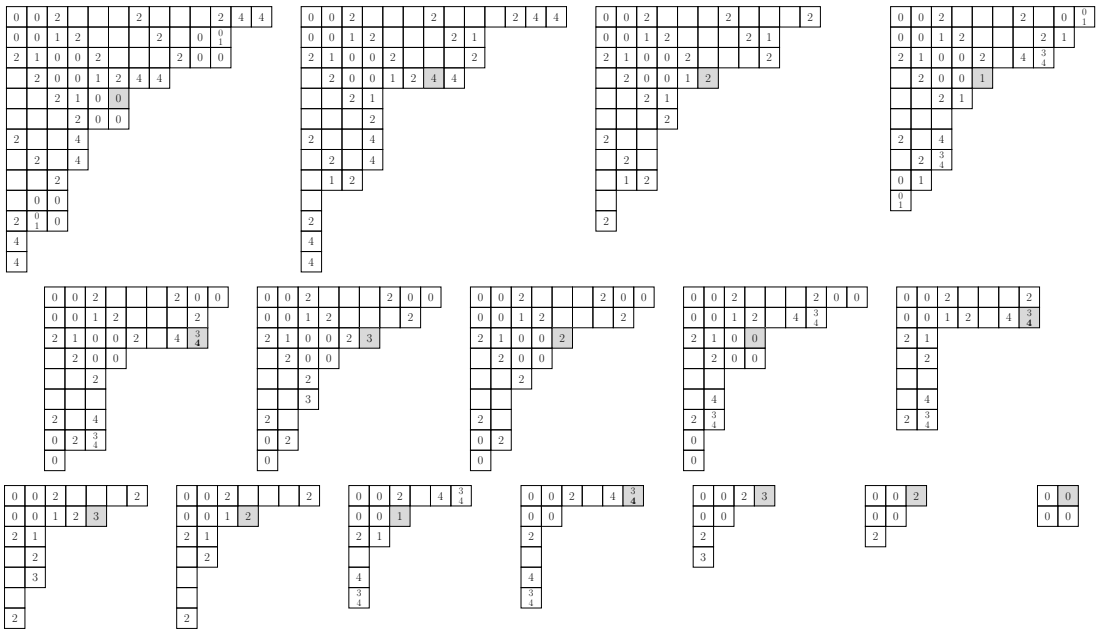


FIGURE 8. The application of the central peeling procedure described in Definition 6.1 on the core partition $\lambda = \mathcal{C}(w)$ for $w = [-15, -11, -10, 4, 5, 19, 20, 24] \in \widetilde{D}_4/D_4$. The shaded boxes are the boxes B_i tallied by the algorithm; they make up the upper diagram U_λ, shown in Figure 9.

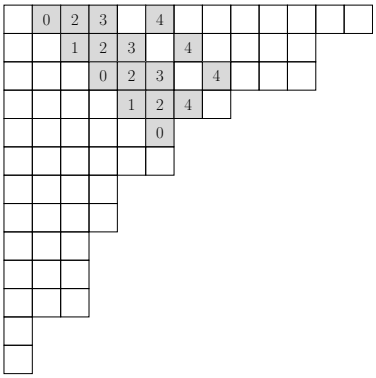


FIGURE 9. The core partition $\lambda = \mathcal{C}(w)$ for $w = [-15, -11, -10, 4, 5, 19, 20, 24] \in \widetilde{D}_4/D_4$, with its upper diagram U_λ shaded. (U_λ was calculated in Figure 8.) Reading the shaded boxes from the bottom to the top from right to left gives the canonical reduced expression $\mathcal{R}(\lambda) = s_0 s_4 s_2 s_1 s_4 s_3 s_2 s_0 s_4 s_3 s_2 s_1 s_4 s_3 s_2 s_0$.

6.2. **The bounded diagram.** We now describe a nonrecursive method to determine the upper diagram of any core λ. This method generalizes a bijection of Lapointe and Morse [1] in $\widetilde{A}_n/A_n$.

We say that a box in λ having hook length $< 2n$ is **skew**. The skew boxes on any particular row form a contiguous segment lying at the end of the row. Consider the collection of boxes $\nu \subset \lambda$ defined row by row as the segment that begins at the box lying on the main diagonal and then extends to the right for the same number of boxes as the number of skew boxes in the row. In $\widetilde{B}/B$ and $\widetilde{D}/D$, remove from ν all boxes along the main diagonal. In $\widetilde{B}/D$ and $\widetilde{D}/D$, remove from ν all boxes along the n-th diagonal. We call this collection of boxes ν the **bounded diagram** of λ, denoted $\widetilde{U}_\lambda$.

Theorem 6.5. *Fix a symmetric $(2n)$-core partition λ. Then, the bounded diagram $\widetilde{U}_\lambda$ is equal to the upper diagram U_λ.*

The proof of this theorem is postponed to Section 9.2.

Example 6.6. For the minimal length coset representative $w = [-11, -9, -1, 8, 16, 18] \in \widetilde{C}_3/C_3$ and its corresponding core partition $\lambda = \mathcal{C}(w)$, a visualization of the construction of $\widetilde{U}_\lambda$ is given in Figure 10. This agrees with U_λ as found in Figure 7.

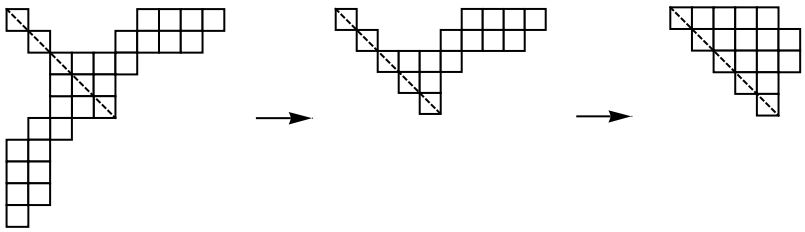


FIGURE 10. For $w = [-11, -9, -1, 8, 16, 18] \in \widetilde{C}_3/C_3$, we left-justify the skew boxes to the boxes on the main diagonal to find the bounded diagram.

7. BOUNDED PARTITIONS

If we left-adjust the boxes of the upper diagram along with the residues they contain, we obtain a structure that has appeared in other contexts including: Young walls in crystal bases of quantum groups [16], Eriksson and Eriksson's partitions in [9], the affine partitions of Billey and Mitchell [2], and the k-bounded partitions of Lapointe and Morse [1]. Lam, Schilling and Shimozono [17] develop combinatorics analogous to the k-bounded partitions for type $\widetilde{C}$, and [18] contains similar constructions for types $\widetilde{B}$ and $\widetilde{D}$. In this section, we explain how these objects are related to the abacus diagrams and core partitions we have introduced.

Definition 7.1. Given a core λ, apply the central peeling procedure to find the upper diagram U_λ. We define the **bounded partition** $\mathcal{B}(\lambda)$ to be the partition β whose i-th part equals the number of boxes in row i of U_λ.

In $\widetilde{B}/D$ (and $\widetilde{D}/D$), if there exists a part of size n ($n-1$, respectively), and the last part β_k of this size has rightmost box with residue $n-1$, then adorn β_k with a star decoration.

By construction, the number of boxes in a bounded partition is the Coxeter length of the corresponding element in $\widetilde{W}/W$. Besides being historical, the motivation for the name "bounded partition" is that part sizes in $\mathcal{B}(w)$ are bounded in the different types by $2n$, $2n-1$, or $2n-2$, as shown in Table 5.

Type	Bounded partition structure
$\widetilde{C}/C$	parts with size $\leq 2n$, where parts of size $1, \ldots, n$ may occur at most once.
$\widetilde{B}/B$	parts with size $\leq 2n-1$, where parts of size $1, \ldots, n-1$ may occur at most once.
$\widetilde{B}/D$	parts with size $\leq 2n-1$, where parts of size $1, \ldots, n-1$ may occur at most once, and one of the parts of size n may be starred.
$\widetilde{D}/D$	parts with size $\leq 2n-2$, where parts of size $1, \ldots, n-2$ may occur at most once, and one of the parts of size $n-1$ may be starred.

TABLE 5. Bounded partitions of $\widetilde{W}/W$

Remark 7.2. The need for a decoration in $\widetilde{B}/D$ and $\widetilde{D}/D$ arises because it is possible that two core partitions yield the same bounded partition. For example, consider elements $w_1 = s_0 s_1 \cdots s_{n-2} s_{n-1}$ and $w_2 = s_0 s_1 \cdots s_{n-2} s_n$ of $\widetilde{B}/D$. They both have the same Coxeter length and correspond to a bounded partition with one part of size n; however, these elements are distinct and have different abaci and cores. We would say that $\mathcal{B}(w_1) = (n^*)$ and $\mathcal{B}(w_2) = (n)$. The reader should interpret the star as arising from a length n sequence of generators (length $n-1$ in type $\widetilde{D}/D$) that ends with s_{n-1} instead of s_n.

Recall that Definition 5.2 gives a correspondence between rows of λ and active beads in $\mathcal{A}(\lambda)$. Together with Theorem 6.5, this presents a method to determine the bounded partition from an abacus diagram.

Lemma 7.3. *For a row r of λ corresponding to an active bead $b > N$ in $\mathcal{A}(\lambda)$, the number of skew boxes in row r on or above the main diagonal equals the number of gaps between b and $\max(b-N, g(b))$, inclusive.*

Proof. By definition 5.2, the skew boxes on a row corresponding to an active bead b are the gaps between $b-N$ and b. Also, the diagonal box on this row corresponds to the symmetric gap $g(b)$, and the result follows. □

Define **offsets** x_0 and x_n corresponding to the occurrence of generators s_0^D and s_n^D as follows. These offsets record whether a fork occurs on the left or right sides of the Coxeter graph, respectively.

$$x_0 = \begin{cases} 0 & \text{in types } \widetilde{C}/C \text{ and } \widetilde{B}/D \\ -1 & \text{in types } \widetilde{B}/B \text{ and } \widetilde{D}/D. \end{cases} \qquad x_n = \begin{cases} 0 & \text{in types } \widetilde{C}/C \text{ and } \widetilde{B}/B \\ -1 & \text{in types } \widetilde{B}/D \text{ and } \widetilde{D}/D. \end{cases}$$

Proposition 7.4. *Given an abacus diagram a, the bounded partition $\mathcal{B}(a)$ is found by creating one part for each bead $b > N$ in descending order, as follows:*

- *For each bead $b > N+n$, create a part of size equal to the number of gaps between b and $b-N$ plus $1 + x_0 + x_n$.*
- *For each bead $N+1 \leq b \leq N+n$, create a part of size equal to $b-N$ plus x_0. If $b = N+n$ in types $\widetilde{B}/D$ and $\widetilde{D}/D$, then star the resulting part.*

Proof. When $b > N+n$, the number of skew boxes equals the number of gaps between b and $b-N$ by Lemma 7.3. We add one for the main diagonal, and apply offsets for the main diagonal and n-th diagonal. When $N+1 \leq b \leq N+n$, the number of boxes in row r equals the number of gaps between b and $g(b)$ by Lemma 7.3, which equals $b-N$ by Lemma 3.4. None of these parts enter the n-th diagonal, so we only apply offset x_0. Starring conditions follow from Definition 7.1. □

We are now in a position to prove the main result of this section.

Theorem 7.5. *The map $\mathcal{B}$ is a bijection of $\widetilde{W}/W$ onto the partitions described in Table 5.*

Type	Residue fillings
$\widetilde{C}/C$	Fill all boxes in column $1 \leq i \leq n+1$ with residue $i-1$ and all boxes in column $n+2 \leq i \leq 2n$ with residue $2n+1-i$.
$\widetilde{B}/B$	Fill all boxes in columns 1 and $2n-1$ with residues 0 and 1 alternating, starting at the top with 0. Fill all boxes in column $2 \leq i \leq n$ with residue i and all boxes in column $n+1 \leq i \leq 2n-2$ with residue $2n-i$.
$\widetilde{B}/D$	Fill all boxes in column $1 \leq i \leq n-1$ with residue $i-1$, all boxes in column $n+1$ with residue n, and all boxes in column $n+2 \leq i \leq 2n-1$ with residue $2n-i$. In the last row of length n (if one exists), if the part is starred, fill the box in column n with residue $n-1$; otherwise fill it with residue n. In other rows of length n, place residues n and $n-1$ alternating up from there. Fill all other boxes in column n with residue $n-1$.
$\widetilde{D}/D$	Fill all boxes in columns 1 and $2n-2$ with residues 0 and 1 alternating, starting with 0 at the top. Fill all boxes in column $2 \leq i \leq n-2$ with residue i, all boxes in column n with residue n, and all boxes in column $n+1 \leq i \leq 2n-3$ with residue $2n-i-1$. In the last row of length $n-1$ (if one exists), if the part is starred, fill the box in column $n-1$ with residue $n-1$; otherwise fill it with residue n. In other rows of length $n-1$, place residues n and $n-1$ alternating up from there. Fill all other boxes in column $n-1$ with residue $n-1$.

TABLE 6. Fillings of bounded partitions

Proof. We present the inverse map to $\mathcal{B}$ as described in Proposition 7.4. This inverse takes a bounded partition β and gives an abacus $a = \mathcal{A}(\beta)$. We write $\beta = (\beta_1, \ldots, \beta_K, \beta_{K+1}, \ldots, \beta_{K+k})$, where each part $\beta_i \geq n+1+x_0+x_n$ for $1 \leq i \leq K$ (and unstarred if applicable), and $\beta_i \leq n+x_0$ for $K+1 \leq i \leq K+k$ (where $n+x_0$ is starred in types $\widetilde{B}/D$ and $\widetilde{D}/D$).

For each part β_{K+i} for $1 \leq i \leq k$, place a bead in position $N+\beta_{K+i}-x_0$. In types $\widetilde{B}/B$ and $\widetilde{D}/D$, if $K+k$ is odd, place a bead in position $N+1$. After this, if position $N+j$ for $1 \leq j \leq n$ is a gap, place a bead in position $N-j$.

We now insert beads into the abacus one at time for parts β_K through β_1 in this order. For each part, we consider the possible positions for placing the next bead to be the positions in reading order after the current lowest bead that have a bead one level above. For part β_K of size $n+j+x_0+x_n$, we place a bead in the j-th possible position. For a part β_i for $i < K$, place a bead b in the $\beta_i - \beta_{i+1}+1$-th possible position. By this construction, the number of gaps between b and $b-N$ is exactly β_i+1.

Once we have finished placing beads as described, fill in beads above $n+1$ as necessary to make the abacus balanced and flush.

Observe that the abacus enforces the structural conditions given in Table 5. □

It is natural to fill the boxes of the bounded partitions with the residues present in the corresponding boxes of the upper diagrams before left adjusting. As these residues are inherited from the residues in the core partition, they follow a predictable pattern.

Proposition 7.6. *Fix a bounded partition $\beta = \mathcal{B}(w)$ and fill the boxes of β with residues as described in Table 6. The canonical reduced expression $\mathcal{R}(w)$ is obtained by reading these residues from β, right to left in rows and from bottom to top.*

Proof. This follows from Definition 7.1 and Theorem 6.5. □

Example 7.7. The bounded partition $\beta = (8,8,5,5,5,4,2)$ could be a member of any one of $\widetilde{B}_5/B_5$, $\widetilde{B}_5/D_5$, or $\widetilde{D}_5/D_5$. The different fillings given by Proposition 7.6 are shown in Figure 11. Since no part

of β is starred, the last row of length 5 in $\widetilde{B}_5/D_5$ contains a residue of 5 instead of a 4, and the residues in this column are determined from there. Similarly, the row of length 4 in $\widetilde{D}_5/D_5$ contains a residue of 5. To read the reduced expression for $\mathcal{R}(\beta)$, read from right to left in rows from bottom to top; for example in $\widetilde{D}_5/D_5$ we would have

$$\mathcal{R}(\beta) = s_2s_0s_5s_3s_2s_1s_5s_4s_3s_2s_0s_5s_4s_3s_2s_1s_5s_4s_3s_2s_0s_1s_2s_3s_5s_4s_3s_2s_1s_0s_2s_3s_5s_4s_3s_2s_0.$$

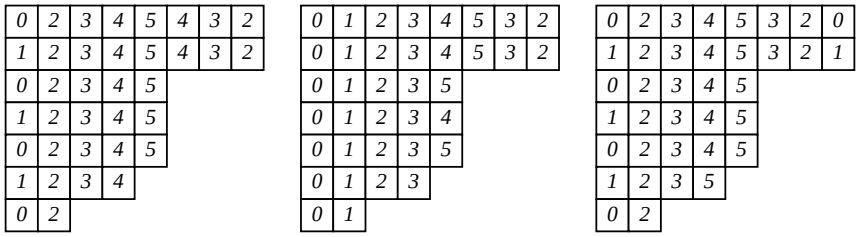


FIGURE 11. Example of the bounded partition $\beta = (8, 8, 5, 5, 5, 4, 2)$ and the residues which populate β when β represents an element of $\widetilde{B}_5/B_5$, $\widetilde{B}_5/D_5$, and $\widetilde{D}_5/D_5$, respectively.

8. COXETER LENGTH FORMULAS

If we sum contributions to the Coxeter length from beads in the same runners in Proposition 7.4, we obtain the following formula for the Coxeter length of a minimal length coset representative defined by way of an abacus.

Corollary 8.1. *Let a be the abacus corresponding to a minimal length coset representative $w \in \widetilde{W}/W$. For $1 \leq i \leq n$, choose the lowest bead $B(i)$ occurring in either runner i or $N - i$, and the bead $n + 1 \leq b(i) \leq N + n$ occurring in $B(i)$'s runner. Let g_i be the number of gaps between $B(i)$ and $b(i)$ in a. Then*

$$\ell(w) = \sum_{1 \leq i \leq n} g_i + (1 + x_0 + x_n)\big(\#\text{ of beads} > N + n\big) + \sum_{N+1 \leq b \leq N+n} (b - N + x_0).$$

Example 8.2. In our example of $w = [-11, -9, -1, 8, 16, 18] \in \widetilde{C}_3/C_3$ with abacus $a = \mathcal{A}(w)$ in Figure 1, we create a part of the bounded partition $\mathcal{B}(w)$ for each bead in a greater than $N = 7$. The beads in question are 18, 16, 11, 9, and 8. The first three beads are greater than $7 + 3$, so we count the number of gaps between b and $b - N$ and add one, giving 5, 5, and 4. The last two beads are between 8 and 10, so we take $b - N$, giving 2 and 1. This agrees with the previously found $\mathcal{B}(w) = (5, 5, 4, 2, 1)$. In the terminology of Corollary 8.1, $B_1 = b_1 = 8$, $B_2 = 16$, $b_2 = 9$, $B_3 = 18$, and $b_3 = 4$. We then calculate $g_1 = 0$, $g_2 = 4$, $g_3 = 7$ and see that $\ell(w) = 11 + (1)(3) + 3 = 17$, agreeing with the number of generators in $\mathcal{R}(w)$ found in Example 6.3.

Given a core partition λ, Theorem 6.5 gives a simple method to find the corresponding bounded partition, from which its Coxeter length can be read directly. It is possible to determine the Coxeter length of the corresponding minimal length coset representative in another way, analogous to the method given in type $\widetilde{A}_n/A_n$ in Proposition 3.2.8 in [10] by translating Proposition 8.1 into the language of core partitions.

If we fill in the boxes of a core along the diagonals with the numbers 1 through $2n$ instead of the residues, the number at the end of each row corresponds to the runner number of the bead in the abacus

which corresponds to that row. This filling allows us to visualize which rows of λ correspond to beads in the same runner. Involved in the statement of the proposition below is the determination of the rows $R(i)$ and $r(i)$ of λ that correspond to the beads $B(i)$ and $b(i)$.

Proposition 8.3. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a symmetric $(2n)$-core partition. If all parts of λ are less than or equal to n, then*

$$\ell(\mathcal{W}(\lambda)) = \sum_{1 \le i \le k} \max(0, \lambda_i - i + 1 + x_0).$$

Otherwise, define $R(i)$ to be the longest row of λ to have runner number i or $N - i$ labeling its rightmost box. Then follow the boundary of λ extended n steps out in each direction from its center, a path which will involve n vertical steps and n horizontal steps. Define $r(i)$ to be the unique row of λ ending with one of the n vertical steps and whose rightmost box is labeled by the same choice of i or $N - i$ as for $R(i)$. Define d to be the number of rows of λ whose box on the main diagonal has hook length greater than $2n$. Then,

$$\ell\big(\mathcal{W}(\lambda)\big) = \sum_{i=1}^{n} \big(\lambda_{R(i)} - \lambda_{r(i)}\big) + (1 + x_0 + x_i)d + \sum_{d+1 \le i \le k} \max(0, \lambda_i - i + 1 + x_0).$$

Proof. In a core of the first type, all boxes are skew. The sum counts the number of boxes in the bounded partition, because the number of boxes between the diagonal and λ_i inclusive is $\lambda_i - i + 1$.

In a core of the second type, there exists a box in λ with hook length at least $2n$. Hence the boundary of λ extends more than n steps out in each direction from its center, and $r(i)$ is well defined since in the abacus $a = \mathcal{A}(\lambda)$, bead $b(i)$ exists between $n + 1$ and $N + n$ on runner $B(i) \| N$, the runner number of row $R(i)$. The first two terms in the formula translate directly from Proposition 8.1 and correspond to contributions from beads $b > N + n$. Beads $b \le N + n$ correspond to rows of λ starting with row $d + 1$; again the number of boxes between the diagonal and λ_i inclusive is $\lambda_i - i + 1$. □

Example 8.4. Consider the core $\lambda = (12, 12, 8, 8, 7, 5, 5, 4, 2, 2, 2, 2) \in \widetilde{B}_3/D_3$, pictured in Figure 12 with runner numbers.

1	2	3	4	5	6	1	2	3	4	5	6
6	1	2	3	4	5	6	1	2	3	4	5
5	6	1	2	3	4	5	6				
4	5	6	1	2	3	4	5				
3	4	5	6	1	2	3					
2	3	4	5	6							
1	2	3	4	5							
6	1	2	3								
5	6										
4	5										
3	4										
2	3										

FIGURE 12. The core partition $\lambda = (12, 12, 8, 8, 7, 5, 5, 4, 2, 2, 2, 2) \in \widetilde{B}_3/D_3$. The numbers in the boxes are the runner numbers.

This partition has parts of size larger than 3, so we apply the second half of Proposition 8.3. The runners corresponding to $i = 1$, 2, and 3 are have runner numbers 6, 5, and 3. Consequently, $\lambda_{R(1)} = \lambda_1 = 12$, $\lambda_{R(2)} = \lambda_2 = 12$, $\lambda_{R(3)} = \lambda_5 = 7$, $\lambda_{r(1)} = \lambda_6 = 5$, $\lambda_{r(2)} = \lambda_7 = 5$, $\lambda_{r(3)} = \lambda_5 = 7$, and $d = 4$. We conclude that $\ell\big(\mathcal{W}(\lambda)\big) = (12 - 5) + (12 - 5) + (7 - 7) + (1 + 0 - 1) \cdot 4 + (3) = 17$.

Another formula for Coxeter length can be obtained summing contributions from the longest rows in $\mathcal{C}(w)$ ending with a given runner number, and then subtracting terms to account for boxes that are counted twice in the peeling process. This gives a second analogue of Proposition 3.2.8 in [10].

Definition 8.5. Let λ be a symmetric $(2n)$-core. Let D be the lowest box on the main diagonal of λ. For $1 \leq i \leq n$, let $R(i)$ be index of the longest row $\lambda_{R(i)}$ of λ having rightmost box from runner i or $N-i$.

The **rim** of λ consists of the boxes from λ that have no box lying directly to the southeast. We define the **rim walk** $W(i)$ from the rightmost box B of $\lambda_{R(i)}$ to be the collection of boxes encountered when walking along the rim of λ from B towards D, ending at the last box encountered from runner $\min(i, N-i)$ prior to D. We let $h(i)$ denote the **height** of $W(i)$, defined to be the number of rows of λ that intersect $W(i)$ and end in a box having a runner number that is different from the last box of $\lambda_{R(i)}$.

Theorem 8.6. *Let λ be a symmetric $(2n)$-core. The Coxeter length of $\mathcal{W}(\lambda)$ is*

$$\ell(\mathcal{W}(\lambda)) = \sum_{1 \leq i \leq n} \big(\lambda_{R(i)} - R(i) - h(i) + 1\big) + x_0 d_0 + x_n d_n.$$

where d_i is the number of boxes on the i-th diagonal of λ.

Proof. By Theorem 6.5, the Coxeter length is equal to the number of boxes in the bounded diagram $\widetilde{U}_\lambda$ of λ. There is one row in $\widetilde{U}_\lambda$ for each bead in $\mathcal{A}(\lambda)$ succeeding N in reading order, and the number of boxes on each row in $\widetilde{U}_\lambda$ is the number of skew boxes in the row, together with entries from the 0-th and n-th diagonals, depending on the Coxeter type.

We claim that the number of skew boxes from all rows ending with a box from runner i is equal to $\lambda_{R(i)} - R(i) + 1 - h(i)$. Depending on the Coxeter type, we will also subtract boxes corresponding to $x_0 d_0$ and $x_n d_n$ in the last step of the construction of $\widetilde{U}_\lambda$.

To see why the claim is true, consider the last bead b in reading order lying on runner i. This bead has $\lambda_{R(i)} - R(i) + 1$ gaps lying weakly between $g(b)$ and b. Each gap lying between N and b corresponds to a skew box for precisely one bead lying on runner i to the right of N in reading order.

Since $N+i$ is the earliest bead on runner i in reading order that corresponds to a row lying above the main diagonal, we must therefore subtract the gaps prior to position $\max((N+i)-N, g(N+i)) = \max(i, N-i)$ that are not of the form $g(b')$ for any bead b' on runner i.

By Lemma 3.4, such gaps correspond to beads lying between $\min(i, N-i)$ and b that do not lie on runner i. This quantity is equal to the height of our rim walk beginning at the last box of row $R(i)$, which corresponds to B.

Including contributions for each runner i yields the sum given in the formula. □

Example 8.7. Consider the core $\lambda = (12, 12, 8, 8, 7, 5, 5, 4, 2, 2, 2, 2) \in \widetilde{B}_3/D_3$, pictured in Figure 12 with runner numbers.

We have $R_1 = 1$, $R_2 = 2$ and $R_3 = 5$. The rim walk from the last box B of λ_1 consists of all 12 boxes on the rim lying between B and D, and there are 3 rows (namely, 2, 4, and 5) that do not end in runner 6. Hence, $h(1) = 3$. The rim walk from the last box B' of λ_2 consists of the 10 boxes on the rim lying between B' and the box immediately right of D. There are 2 rows (namely, 3 and 5) that do not end in runner 5, so $h(2) = 2$. The rim walk from the last box B'' of λ_5 consists of the single box B'' because there are no other boxes from runner 3 lying on the rim between B'' and D. Hence, $h(3) = 0$.

Thus, we compute $\ell(\mathcal{W}(\lambda))$ as

$$\big(\big((\lambda_{R(1)} - R(1) + 1) - h(1)\big) + \big((\lambda_{R(2)} - R(2) + 1) - h(2)\big) + \big((\lambda_{R(3)} - R(3) + 1) - h(3)\big)\big) - d_n$$
$$= ((12-3) + (11-2) + (3-0)) - 4 = 17.$$

9. PROOFS

9.1. **Residues for core partitions.** We now turn to the proof of Theorem 5.8.

Proof. [of Theorem 5.8] Given $w \in \widetilde{W}_n/W_n$, suppose that $a = \mathcal{A}(w)$ and $\lambda = \mathcal{C}(w)$. Let the residues be assigned to entries of $\mathbb{N}^2$ near the southeast boundary of the diagram of λ as in Definition 5.6. Consider the application of a generator s_i.

By Lemma 3.4, the midpoint of the boundary lattice path of λ occurs at entry N in the abacus, and this corresponds to the outermost corner of the lowest box on the main diagonal in λ. In every type, the boxes on the main diagonal are assigned residue 0.

Because the assignment of fixed residues is constant along northwest-southeast diagonals, the assignment of fixed residue to entry (i, j) is the same as the assignment of fixed residue to entry $(i - u, j + v)$ whenever $u, v \geq 0$ and $u + v = 2n$. From this it follows that every bead on a given runner is assigned the same fixed residue. Moreover, we find that all of the beads in runner j correspond to boxes with residue

$$\begin{cases} j - 1 & \text{if } 1 \leq j \leq n + 1 \\ 2n - j + 1 & \text{if } n + 2 \leq j \leq 2n. \end{cases}$$

We observe that the connected components of boxes with fixed residue are always single boxes. A box with fixed residue i is removable if and only if it lies at the end of its row and column, which therefore occurs if and only if it corresponds to an active bead on runner $i+1$ or $2n-i+1$ with a gap immediately preceding it in the reading order of the abacus. Similarly, a box with fixed residue i is addable if and only if it corresponds to a gap on runner $i + 1$ or $2n - i + 1$ with an active bead immediately preceding it in the reading order of the abacus. The action of s_i in type $\widetilde{C}_n$ swaps runners i and $i + 1$ as well as $2n - i$ and $2n - i + 1$ which therefore interchanges all of the i-addable and i-removable boxes.

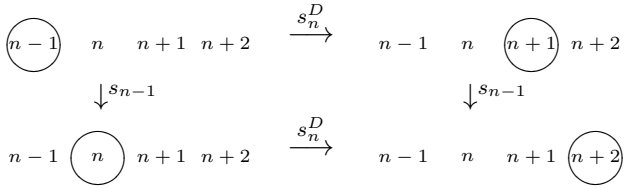
Since the abacus is flush, exchanging runners i with $i + 1$ and $2n - i$ with $2n - i + 1$ (mod N) either adds some set of boxes with residue i to the diagram of λ in the case that s_i is an ascent, or else removes a set of boxes with residue i in the case that s_i is a descent. If s_i is neither an ascent nor a descent then the levels of the lowest beads in the relevant columns are the same, so the abacus and corresponding core remain unchanged.

This proves the result in the case when s_i ($0 \leq i \leq n$) is a generator of type $\widetilde{C}$. The generators s_{n-1} and s_n^D apply to entries in an escalator, which we consider below. The generators s_1 and s_0^D apply to entries in an descalator, and the argument is entirely similar so we omit it.

To verify the result for s_{n-1} and s_n^D, we consider the action of these generators on all possible abaci. It suffices to consider the action on a single row of the abacus since this translates into a connected segment of the lattice path boundary of the core partition. Using the results for type $\widetilde{C}$ generators that we have already shown above, we observe that the lattice path always begins on the boundary of a box with residue $n - 2$ by induction.

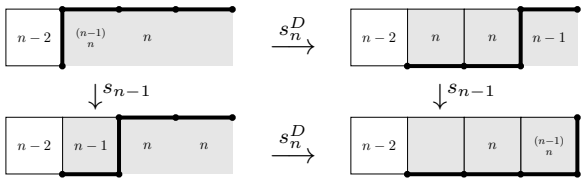
Case: The abacus row contains a single bead among the positions $\{n - 1, n, n + 1, n + 2\}$.

Then, the abaci satisfy the following commutative diagram.

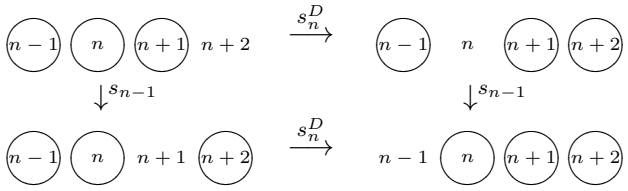


We translate these entries of the abaci into a segment of the boundary lattice path. Observe that Lemma 3.5 implies that we are above the main diagonal, so we assign residues to boxes in the upper

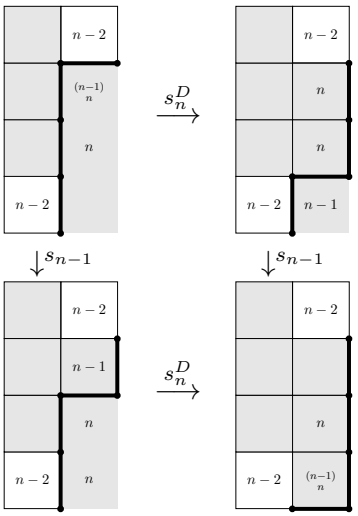
escalator horizontally. Then, it is straightforward to verify that the action of s_i does add (or remove) all addable (removable, respectively) components with residue i, for $i \in \{n-1, n\}$, as shown below.



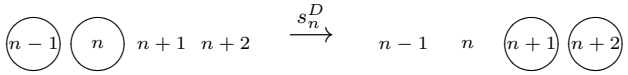
Case: The abacus row contains a single gap among the positions $\{n-1, n, n+1, n+2\}$.
Then, the abaci satisfy the following commutative diagram.



We translate these entries of the abaci into a segment of the boundary lattice path. Observe that Lemma 3.5 implies that we are below the main diagonal, so we assign residues to boxes in the lower escalator vertically. Then, it is straightforward to verify that the action of s_i does add (or remove) all addable (removable, respectively) components with residue i, for $i \in \{n-1, n\}$, as shown below.

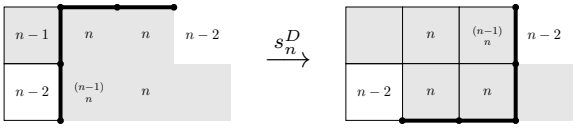


Case: The abacus row contains two beads in one of the configurations shown below.

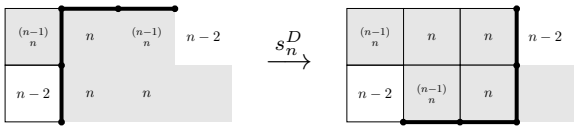


We translate these entries of the abaci into a segment of the boundary lattice path.

If we are above the main diagonal, then we assign residues to the boxes in the upper escalator horizontally, as shown below.

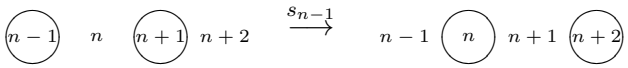


If we are below the main diagonal, then we assign residues to the boxes in the lower escalator vertically, as shown below.



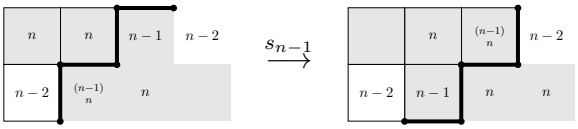
In each subcase, it is straightforward to verify that the action of s_i does add (or remove) all addable (removable, respectively) components with residue i, for $i \in \{n-1, n\}$.

Case: The abacus row contains two beads in one of the configurations shown below.

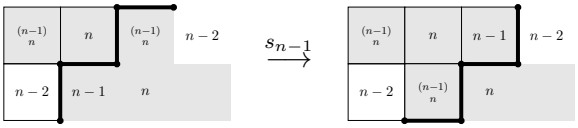


We translate these entries of the abaci into a segment of the boundary lattice path.

If we are above the main diagonal, then we assign residues to the boxes in the upper escalator horizontally, as shown below.



If we are below the main diagonal, then we assign residues to the boxes in the lower escalator vertically, as shown below.



In each subcase, it is straightforward to verify that the action of s_i does add (or remove) all addable (removable, respectively) components with residue i, for $i \in \{n-1, n\}$.

Case: The abacus positions $\{n-1, n, n+1, n+2\}$ are all beads or all gaps.

In this case, both s_{n-1} and s_n^D fix the boundary lattice path segment. It is straightforward to see that the corresponding core partitions have no addable nor removable i-boxes for $i \in \{n-1, n\}$.

Observe that Lemma 3.5 prohibits abaci in either of the configurations shown below.

This exhausts the cases. □

9.2. **The upper partition.** We now turn to the proof of Theorem 6.5.

Proof. [of Theorem 6.5] Fix a core partition λ and its associated abacus $a = \mathcal{A}(\lambda)$. Recall that a box in λ having hook length $< 2n$ is called **skew**. Define the bounded diagram $\widetilde{U}_\lambda$ as in Section 6.2. We say that a box of λ lying in the bounded diagram $\widetilde{U}_\lambda$ is **bounded**. We will prove that the bounded boxes are the boxes peeled in the central peeling procedure.

To prove Theorem 6.5, we work by induction on Coxeter length, investigating the application of one step of the central peeling process. For the remainder of this proof, let B be the rightmost box in the row r of λ containing the lowest box on the reference diagonal, b be its corresponding active bead in $\mathcal{A}(\lambda)$, and let i be the residue of B. When we apply s_i to remove B, we claim that:

(1) B is bounded.
(2) We remove no other bounded box.
(3) If a box B' was bounded in λ, then B' is also bounded in $s_i(\lambda)$.

Proving these claims complete the proof of Theorem 6.5 because they show that during the central peeling procedure the bounded boxes of λ remain bounded boxes in intermediary steps and that exactly one bounded box of U_λ is peeled in each step.

In terms of the abacus, b is essentially the first bead succeeding N in reading order. However, if we are in a type that uses s_0^D then there may be a bead in position $N+1$ that cannot be moved to the left because position $N+2$ is a gap. In this case, b is the next bead in reading order after $N+1$. In all cases, $b \leq 2N+1$ by the Balance Lemma 2.4.

We first prove (1) using the definition of $(2n)$-core. When we are in a type that uses s_0^C, then r is the row containing the lowest box on the main diagonal. All boxes in row r starting from the first diagonal to B all have hook length less than $2n$, so they are all skew and consequently bounded as well, including B. When we are in a type that uses s_0^D, then r is the row containing the lowest box on the first diagonal. All boxes in row r starting from the first diagonal to B all have hook length less than or equal to $2n$; equality occurs only when r has a box in the $2n$-th diagonal and row $r+1$ has a box on the main diagonal. The skew boxes in r are therefore all boxes between the second and $2n$-th diagonal; this implies that the bounded boxes in r are all boxes between the main and $(2n-1)$-st diagonal, this last box being B by Definition 6.1.

To prove (2), consider some box B' that is removed when applying s_i. We will show that there are non-skew boxes in the row containing B' which imply that B' is not bounded. Define $\widehat{B}$ to be the lowest box on the r-th column. This is the reflection of B by the main diagonal and in the abacus this column corresponds to the symmetric gap $g(b)$. Suppose that we apply generator s_i for $1 \leq i \leq n-1$ or $s_i = s_n^C$. We note that both $\widehat{B}$ and B' have residue i and are in non-consecutive diagonals containing this residue. Therefore the box that is at the same time above $\widehat{B}$ and to the left of B' has hook length at least $2n$ and is therefore not skew.

When we apply the generator s_0^C, the box B is on the main diagonal and the box B' is on the j-th diagonal for $j \geq 2n$. Therefore the box that is at the same time above B and to the left of B' has hook length at least $2n$ and is therefore not skew.

When we apply the generator s_0^D, we remove two boxes in B''s row; we prove the existence of the two required gaps. In this case, B is on the first diagonal, with a box directly below, a box to the left, and the box $\widehat{B}$ to the lower-left, all four of which are removed when s_0^D is applied. The two boxes to the left of B' that are above B and $\widehat{B}$ both have hook length at least $2n$.

When we apply the generator s_n^D, we also remove two boxes in B''s row. If B' is on the j-th diagonal where $j \geq N+n-1$, then the box above $\widehat{B}$ and to the left of B' has hook length at least $2n$, as does its right neighbor. The last case is that B' is in diagonal $n+1$ while B is in diagonal $n-1$. In this case, B and B' are in the same block of four boxes which are removed upon the application of s_n^D. There are n bounded boxes in B''s row, not including B'. This exhausts the cases.

To prove (3), we show that the number of skew boxes on each row of λ above B is equal to the number of skew boxes on each row of $s_i(\lambda)$ above B. Left-justifying these skew boxes to the main diagonal then gives the same bounded region.

We consider first generators s_i where $1 \leq i \leq n-1$ and s_n^C. Let b' be a bead lying to the right of b in reading order. If $b' \leq N+n$, then all boxes between the main diagonal and B' are skew. No boxes in this row are removed upon application of s_i because the residue of B' is different from the residue of B. If $b' > N+n$, then the number of skew boxes is the number of gaps between b' and $b'-N$. If the action of s_i fixes runner $b' \| N$, then the number of gaps between b' and $b'-N$ does not change.

Otherwise b' is on runner $b\|N$ or runner $(N-b+1)\|N$. If b' is on runner $b\|N$, then since $b-1$ is a gap, then so is $b'-N-1$. If b' is on runner $(N-b+1)\|N$, then we notice that because b is a bead then $N-b$ is a gap, and consequently so is $b'-N-1$. In both cases, if we restrict our attention to the entries between b' and $b'-N$, we see that in the application of s_i, we lose a gap in position $b'-1$ but gain a gap in position $b'-N+1$, so there is no net change.

In the case of s_0^C, both b' and b are on runner 1, so because $b-2=2n$ is a gap, so is $b'-N-2$, and we see that in the application of s_0^C, we lose a gap in position $b'-2$ but gain a gap in position $b'-N+2$.

If b' is involved in a transposition under s_0^D, then there are two cases. Either b is in position $N+2$ or b is in position $2N+1$. In the former case, there is a bead in position $N+1$, and there is a gaps in positions $2n-1$ and $2n$. Consequently, b' is in either runner 1 or 2, and there are gaps g' and g'' to the left of b' on runners $2n-1$ and $2n$, as well as on the level below. When we apply s_0^D, we therefore lose the gaps in runners $2n-1$ and $2n$, but then gain them back in runners 1 and 2. In the latter case, the only beads lower than N are in runner 1, and there is a gap in position $2n$. Consequently, b' is also in runner 1, and there are gaps in positions $b'-2$ and $b'-N-2$. When we apply s_0^D, we lose the gap $b'-2$ but then gain back $b'-N+2$.

Finally, if b' is involved in a transposition under s_n^D, then b is in position $N+n+1$ or $N+n+2$, and there are gaps in positions $N+n-1$ and $N+n$. Consequently, b' is in either runner $n+1$ or $n+2$, and there are gaps g' and g'' to the left of b' on runners n and $n-1$, as well as on the level below. When we apply s_n^D, we therefore lose the gaps in runners n and $n-1$, but then gain them back in positions $g'-N+2$ and $g''-N+2$. This completes the proof of (3). □

10. Future work

There are various results ([19], [20], [21], [22], to name some recent examples) involving type $\widetilde{A}$ objects that we expect have analogues in the other types.

Another family of combinatorial objects in bijection with $\widetilde{C}_n/C_n$ is the lecture hall partitions of Bousquet-Mélou and Eriksson [23]. It may be interesting to see whether some analogue of these partitions exists in the other affine types.

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DEPARTMENT OF MATHEMATICS, QUEENS COLLEGE (CUNY), 65-30 KISSENA BLVD., FLUSHING, NY 11367
E-mail address: `christopher.hanusa@qc.cuny.edu`
URL: `http://people.qc.cuny.edu/faculty/christopher.hanusa/`

DEPARTMENT OF MATHEMATICS AND STATISTICS, MSC 1911, JAMES MADISON UNIVERSITY, HARRISONBURG, VA 22807
E-mail address: `brant@math.jmu.edu`
URL: `http://www.math.jmu.edu/~brant/`