SOLVING MULTIVARIATE FUNCTIONAL EQUATIONS

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ABSTRACT. This paper presents a new method to solve functional equations of multivariate generating functions, such as $F(r,s) = e(r,s) + xf(r,s)F(1,1) + xg(r,s)F(qr,1) + xh(r,s)F(qr,qs)$, giving a formula for $F(r,s)$ in terms of a sum over finite sequences. We use this method to show how one would calculate the coefficients of the generating function for parallelogram polyominoes, which is impractical using other methods. We also apply this method to answer a question from fully commutative affine permutations.

1. INTRODUCTION

Some generating functions are most naturally defined by a functional equation; in this article, we discuss a new method for solving certain functional equations involving multiple variables.

We set the stage with an elementary example: there are many families of combinatorial objects (binary trees, Dyck paths, triangulations of convex polygons, and many more; see [Sta11]) whose generating function satisfies the functional equation

$$C(x) = 1 + xC(x)^2.$$

The generating function $C(x) = \sum_{n \geq 0} c_n x^n$ is a formal power series in one variable x that marks some statistic (often size) on the family of combinatorial objects. We can interpret the above equation to mean that every object in the family can either be represented as an object of size zero (an empty object) or as being composed of two smaller objects from the family. The solution to the above equation is

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x},$$

and the coefficients of the power series expansion of this generating function are the Catalan numbers. This gives a conceptual reason for the prevalence of Catalan numbers in combinatorics.

When we investigate multiple statistics at the same time, the functional equations can become more complicated. Suppose we are trying to solve for the generating function $F(s, t, x, y, q)$ that satisfies a functional equation of the type

(1.1) $$F(s) = xe(s) + xf(s)F(1) + xg(s)F(qs).$$

Here we have suppressed the variables t, x, y, and q to simplify the notation; the reader should interpret $F(sq)$ as $F(sq, t, x, y, q)$. In addition, the functions e, f, and g may be formal power series in all the variables. The combinatorial significance of

a term like $F(sq)$ is that the statistic marked by q increases at the same time as the statistic marked by s as the objects are being built.

Functional equations of this type arise naturally when enumerating with statistics combinatorial objects such as polyominoes [BMB09], plane trees [BDLPP98], lattice paths [BDLPP99], and pattern-avoiding permutations [BDLPP01]. Bousquet-Mélou proved in [BM96, Lemma 2.3] that the solution to Equation (1.1) is

$$F(s) = \frac{E(s) + E(1)G(s) - E(s)G(1)}{1 - G(1)},$$

where

$$E(s) = \sum_{n \geq 0} x^{n+1} g(s)g(sq) \cdots g(sq^{n-1})e(sq^n)$$

and

$$G(s) = \sum_{n \geq 0} x^{n+1} g(s)g(sq) \cdots g(sq^{n-1})f(sq^n).$$

Bousquet-Mélou used this lemma, as well as a generalization involving derivatives of F with respect to s, to find the generating function for various classes of column-convex polyominoes. Bousquet-Mélou and Brak's survey on enumerating polyominoes and polygons [BMB09] is especially recommended reading.

Theorem 2.1 gives the solution to generating functions defined by a functional equation that simultaneously replaces multiple variables, such as in

$$F(r,s) = e(r,s) + xf(r,s)F(1,1) + xg(r,s)F(qr,1) + xh(r,s)F(qr,qs).$$

Our method applies to functional equations in an arbitrarily large number of formal variables r_1 through r_m and with arbitrarily many terms in the functional equation where each r_i can be replaced by 1 or $q^j r_i$ for $j \geq 0$.

Our proof is rather elementary—we repeatedly apply the functional equation and take the formal power series limit. Our main result (Theorem 2.1) is stated in terms of a sum over finite sequences. More importantly, in principle our method allows for the calculation of the coefficients of the generating function. This is in contrast to Bousquet-Mélou's result, which gives a quotient of q-Bessel functions, both of which are complicated infinite sums. Our method does not replace Bousquet-Mélou's method nor the powerful kernel method [BBMD$^+$02], which applies to many additional functional equations. We refer the reader interested in other solution methods to Bousquet-Mélou and Jehanne's [BMJ06].

In Section 2, we prove Theorem 2.1 and demonstrate how it applies to some simple functional equations including Equation (1.1). In Section 3, we apply our method to the functional equation of parallelogram polyominoes and manipulate the solution to show how one would find the coefficients of the corresponding generating function. In Section 4, we apply a trivariate version of Theorem 2.1 to the study of fully commutative affine permutations, which was the original motivation for this study. An original analysis of fully commutative affine permutations in [HJ10, Lemma 3.12] involves an unwieldy *ad hoc* calculation that we are able to replace by working it into the larger framework of solving multivariate functional equations.

2. Main Result

Theorem 2.1. *Let $F(\mathbf{r})$ be a formal power series in formal variables x, q, and $\mathbf{r} = r_1, \ldots, r_m$, and let $\mathcal{J}$ be a set of m-tuples $\mathbf{j} = (j_1, \ldots, j_m)$ of integers greater than or equal to -1. Furthermore, let $e(\mathbf{r})$ and $f_{\mathbf{j}}(\mathbf{r})$ for all $\mathbf{j} \in \mathcal{J}$ be formal power series in the formal variables x, q, and $\mathbf{r}$. Suppose that $F(\mathbf{r})$ is defined by the functional equation*

$$(2.1) \qquad F(\mathbf{r}) = e(\mathbf{r}) + \sum_{\mathbf{j} \in \mathcal{J}} x f_{\mathbf{j}}(\mathbf{r}) F\left((q^{j_1} r_1)^{\varepsilon(j_1)}, \ldots, (q^{j_m} r_m)^{\varepsilon(j_m)} \right),$$

where $\varepsilon(j) = \begin{cases} 1 & if \ j \geq 0 \\ 0 & if \ j = -1 \end{cases}$. Then an explicit expression for $F(\mathbf{r})$ is

(2.2)

$$F(\mathbf{r}) = \sum_{n \geq 0} x^n \sum_{\substack{J = (\mathbf{j}^1, \ldots, \mathbf{j}^n) \\ \mathbf{j}^i \in \mathcal{J}}} e(q^{a_{1,n+1}} r_1^{b_{1,n+1}}, \ldots, q^{a_{m,n+1}} r_m^{b_{m,n+1}}) \prod_{i=1}^{n} f_{\mathbf{j}^i}(q^{a_{1,i}} r_1^{b_{1,i}}, \ldots, q^{a_{m,i}} r_m^{b_{m,i}}),$$

where the second sum is over all length n sequences J of m-tuples from $\mathcal{J}$ and where the exponents $a_{k,i}$ and $b_{k,i}$ depend on the sequence J as follows.

First, define $a_{k,1} = 0$ and $b_{k,1} = 1$ for all $1 \leq k \leq m$. Then, for $2 \leq i \leq n+1$ and $1 \leq k \leq m$, determine $a_{k,i}$ and $b_{k,i}$ in relation to the sequence $\{j_1^i, j_2^i, \ldots, j_{k-1}^i\}$: If -1 occurs in this sequence, define $b_{k,i} = 0$ and $a_{k,i}$ to be the sum of the entries of the sequence after the last occurrence of -1; otherwise define $b_{k,i} = 1$ and set $a_{k,i}$ to be the sum of all entries in this sequence.

Proof. Applying Equation (2.1) generates $|\mathcal{J}|$ new occurrences of F for every initial occurrence of F; these new occurrences are each indexed by a $\mathbf{j} \in \mathcal{J}$ and weighted by x times the function $f_{\mathbf{j}}$. The only terms that do not continue to expand in successive applications of Equation (2.1) are those with a terminal $e(\mathbf{r})$ function. As such, the new non-expanding terms generated by applying the functional equation an $(n+1)$-st time consist of one term for every sequence of length n on the alphabet $\mathcal{J}$, each of which is multiplied by x^n. Taking the formal power series limit of this iterative procedure, the powers of x become arbitrarily large as to cause no occurrences of F to remain.

The exponents $a_{k,i}$ and $b_{k,i}$ arise as the actions of $F((q^{j_1} r_1)^{\varepsilon(j_1)}, \ldots, (q^{j_m} r_m)^{\varepsilon(j_m)})$ are followed. The initial values $a_{k,1}$ and $b_{k,1}$ are so defined since the first expansion is of $F(r_1, \ldots, r_m)$. Subsequently, the precise sequence $J = (\mathbf{j}^1, \ldots, \mathbf{j}^n)$ determines the exponents of q and the presence of the variable r_k. The variable r_k will not be present after $j_k^i = -1$ while the exponent of q in entry k is increased by positive j_k^i, resetting to zero when $j_k^i = -1$. $\square$

Independently, a 1-dimensional analogue of our Equation (2.2) was found by Mansour and Song in [MS09, Lemma 2.1]. Their work uses the kernel method to solve systems of functional equations that are of a different type than our Equation (2.1). Both types of functional equation can be considered generalizations of Equation (1.1).

Example 2.2. We start with the simple functional equation
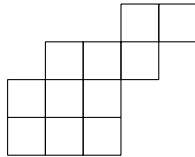
$$F(r) = r + xF(r) + xF(qr).$$

FIGURE 1. A parallelogram polyomino with area $a = 11$, left height $l = 2$, right height $r = 1$, width $w = 5$, and height $h = 4$. Its contribution to the generating function $F(q, s, t, x, y)$ would be $q^{11}s^2tx^5y^4$.

The coefficients $c_{n,h}$ of the generating function $F(1, x, q) = \sum c_{n,h}x^n q^h$ count the number of ways to produce h heads when flipping n coins. To apply Theorem 2.1, we note $\mathcal{J} = \{0, 1\}$, $e(r) = r$, $f_0(r) = 1$, and $f_1(r) = 1$, so that:

$$F(r, x, q) = \sum_{n \geq 0} x^n \sum_{\substack{J=(j^1,\ldots,j^n) \\ j^i \in \{0,1\}}} q^{a_{n+1}} r^{b_{n+1}} = \sum_{n \geq 0} x^n \sum_{\substack{J=(j^1,\ldots,j^n) \\ j^i \in \{0,1\}}} q^{\sum_{i=1}^n j^i} r = \sum_{n \geq 0} (1+q)^n r x^n.$$

As expected, $F(1, x, q) = \sum_{n \geq 0}(1+q)^n x^n$. Alternatively, because $F(r) = \frac{r}{1-x} + \frac{x}{1-x}F(qr)$, we can expand directly to find the equivalent expression, $F(x, q, r) = \frac{r}{1-x-qx}$.

Example 2.3. Suppose $F(s)$ satisfies $F(s) = xe(s) + xf_{-1}(s)F(1) + xf_1(s)F(qs)$, as in Equation (1.1). Theorem 2.1 implies that

$$F(s) = \sum_{n \geq 0} x^{n+1} \sum_{\substack{J=(j^1,\ldots,j^n) \\ j^i \in \{-1,1\}}} e(q^{a_{n+1}} s^{b_{n+1}}) \prod_{i=1}^{n} f_{j^i}(q^{a_i} s^{b_i}),$$

where a_i is the number of consecutive 1s in J ending with entry $i - 1$ and $b_i = 1$ if $j_1 = \cdots = j_{i-1} = 1$ and $b_i = 0$ otherwise.

Example 2.4. A multivariate functional equation with nice symmetry is

(2.3)
$$\begin{aligned} F(r, s) = {}& e(r, s) + xf_{-1,-1}(r, s)F(1, 1) + xf_{-1,1}(r, s)F(1, qs) \\ & + xf_{1,-1}(r, s)F(qr, 1) + xf_{1,1}(r, s)F(qr, qs). \end{aligned}$$

When solving Equation (2.3) in the form of Equation (2.2), the exponents $a_{k,i}$ have a simple description. Given a sequence $J = (\mathbf{j}^1, \ldots, \mathbf{j}^n)$ on the alphabet

$$\mathcal{J} = \{\{-1, -1\}, \{-1, 1\}, \{1, -1\}, \{1, 1\}\},$$

define $J_k = (j_k^1, \ldots, j_k^n)$ for $k \in \{1, 2\}$. The exponent $a_{k,i}$ is the number of consecutive 1s in J_k ending with entry $i - 1$.

3. APPLICATION: PARALLELOGRAM POLYOMINOES

A *parallelogram polyomino* (also called a *staircase polyomino*) is a horizontally and vertically convex union of 1×1 lattice squares that touches the bottom-left and top-right corners of its bounding rectangle. (An example is given in Figure 1.) Parallelogram polyominoes are one of the first classes of polyominoes to have been enumerated—Pólya [Pól69] found the generating function for parallelogram polyominoes by perimeter and area. One remarkable feature is that the number of parallelogram polyominoes with perimeter $2n$ is the Catalan number c_{n-1}. Indeed, Delest and

Viennot [DV84] give a bijection between parallelogram polyominoes with perimeter $2n + 2$ and Dyck paths of length $2n$ that at the same time accounts for area.

For each parallelogram polyomino P, we can determine its area a, the height of its leftmost column l, the height of its rightmost column r, and the width w and height h of its bounding rectangle. Mireille Bousquet-Mélou [BM96] used a layered approach (constructing the polyominoes column by column) to prove that the parallelogram polyomino generating function $F(q, s, t, x, y) = \sum_P q^{a(P)} s^{l(P)} t^{r(P)} x^{w(P)} y^{h(P)}$ satisfies the functional equation

$$F(s) = \frac{xstyq}{1 - styq} + \frac{xsq}{(1 - sq)(1 - syq)} F(1) + \frac{-xsq}{(1 - sq)(1 - syq)} F(sq).$$

This is of the form in Equation (1.1), discussed in Example 2.3. As such,

$$F(s) = \sum_{n \geq 0} x^{n+1} \sum_{J \in \{-1,1\}^n} (-1)^{o(J)} \frac{q^{a_{n+1}+1} s^{b_{n+1}} ty}{1 - q^{a_{n+1}+1} s^{b_{n+1}} ty} \prod_{i=1}^{n} \frac{q^{a_i+1} s^{b_i}}{(1 - q^{a_i+1} s^{b_i})(1 - q^{a_i+1} s^{b_i} y)},$$

where the sum is sequences J of length n over the alphabet $\{-1, 1\}$ and $o(J)$ is the number of entries of J equal to 1. Expanding the middle factor and simplifying,
(3.1)

$$F(s) = \sum_{n \geq 0} x^{n+1} \sum_{J \in \{-1,1\}^n} (-1)^{o(J)} \sum_{m \geq 1} (tyq^{a_{n+1}+1} s^{b_{n+1}})^m \prod_{i=1}^{n} \frac{q^{a_i+1} s^{b_i}}{(1 - q^{a_i+1} s^{b_i})(1 - q^{a_i+1} s^{b_i} y)}$$

$$= \sum_{n \geq 0} \sum_{m \geq 1} x^{n+1} t^m \sum_{J \in \{-1,1\}^n} (-1)^{o(J)} (yq^{a_{n+1}+1} s^{b_{n+1}})^m \prod_{i=1}^{n} \frac{q^{a_i+1} s^{b_i}}{(1 - q^{a_i+1} s^{b_i})(1 - q^{a_i+1} s^{b_i} y)}.$$

We conclude that the coefficient of $x^{n+1} t^m$ is

$$\sum_{J \in \{-1,1\}^n} (-1)^{o(J)} (yq^{a_{n+1}+1} s^{b_{n+1}})^m \prod_{i=1}^{n} \frac{q^{a_i+1} s^{b_i}}{(1 - q^{a_i+1} s^{b_i})(1 - q^{a_i+1} s^{b_i} y)}.$$

Extraction with respect to s or y is possible by conditioning on the length of the initial run of 1's in J.

4. APPLICATION: AFFINE PERMUTATIONS

The methods developed in Section 2 provide an alternative to a calculation of Hanusa and Jones [HJ10], who enumerated fully commutative affine permutations by number of inversions. We do not require the theory of affine permutations; we prove an equivalent result on $(L)(M)(R)$-words. We recall that a permutation w is 321-avoiding if there do not exist entries $i < j < k$ such that $w_i > w_j > w_k$.

Definition 4.1. Let L, M, and R be positive integers. We define an $(L)(M)(R)$-word to be a 321-avoiding permutation of the set $\{1, 2, \ldots, L + M + R\}$ where the numbers $\{1, \ldots, L\}$ are in increasing order, the numbers $\{L + M + 1, \ldots, L + M + R\}$ are in increasing order, and every number in the set $\{1, \ldots, L\}$ is to the left of every number in the set $\{L + M + 1, \ldots, L + M + R\}$.

When the exact value of an integer in a word is of no consequence, we may replace integers in $\{1, \ldots, L\}$ with a letter L, integers in $\{L + 1, \ldots, L + M\}$ with the letter M, and integers in $\{L + M + 1, \ldots, L + M + R\}$ with the letter R.

Let $w = w_1 w_2 \ldots w_n$ be an $(L)(M)(R)$-word. We say that (i, j) is an *inversion* if $i < j$ and $w_i > w_j$ and we say that w has an M-*descent* in position i if the i-th M is larger than the $(i+1)$-st M.

Example 4.2. The $(4)(7)(2)$-word

$$w = 1, 2, \mathbf{7}, 3, \mathbf{8}, 4, \mathbf{5}, \mathbf{10}, \mathbf{6}, \mathbf{12}, \mathbf{9}, 13, 14, \mathbf{11}$$

may be written LLMLMLMMMMMRRM. Note that w is 321-avoiding, has a total of 11 inversions and has M-descents in positions 2 ($8 \searrow 5$), 4 ($10 \searrow 6$), and 6 ($12 \searrow 9$). (Note: the M's are the bolded numbers in the sequence.)

We define the following combinatorial statistics for an $(L)(M)(R)$-word w with at least one M-descent.

- $n(w) = L + M + R$ is the size of the word (marked by variable x). It is also helpful to define $m(w) = M$.
- $l(w)$ is the number of inversions of w (marked by variable q),
- $\alpha(w)$ is the number of M's after the last M-descent and before the leftmost R (marked by variable r),
- $\beta(w)$ is the number of R's before the last M (marked by variable s), and
- $\gamma(w)$ is the number of M's to the right of the leftmost R (marked by variable t).

In Example 4.2, $\alpha(w) = 1$, $\beta(w) = 2$, and $\gamma(w) = 1$.

Hanusa and Jones [HJ10, Lemma 3.12] determine the generating function $\sum x^{n(w)} q^{l(w)}$ for all $(L)(M)(R)$-words w with at least two M-descents. Their formula requires calculating coefficients of a generating function that is a quotient of q-Bessel functions and subsequently inserting those coefficients into another generating function. In Theorem 4.3, we present an alternative method involving two multivariate functional equations. Applying Theorem 2.1 would allow the calculation of an expression for $\sum x^{n(w)} q^{l(w)}$ that does not require the original awkward substitution.

We recall that the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ equals $\frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$ and enumerates the ways to intersperse k smaller numbers with $n - k$ larger numbers, keeping track of the number of inversions incurred using the variable q.

Theorem 4.3. *Suppose L and R are positive integers. Let*

$$G(x, q, r, s, t) = \sum_w x^{m(w)} q^{l(w)} r^{\alpha(w)} s^{\beta(w)} t^{\gamma(w)}$$

be the generating function for the $(L)(M)(R)$-words with at least two M-descents, and let $G(x, q, r, s, t) = F_0(x, q, r) + F_1(x, q, r, s, t)$, where F_0 (respectively F_1) enumerates words with zero M's (respectively at least one M) to the right of the leftmost R. Then $F_0(x, q, r)$ satisfies the functional equation

$$(4.1) \qquad F_0(r) = \frac{E_0(r)}{1 - xr} + \frac{xqr}{(1-qr)(1-xr)} F_0(1) - \frac{xqr}{(1-qr)(1-xr)} F_0(qr)$$

where

$$E_0(r) = \sum_{m \geq 0} \sum_{i=1}^{m-1} x^{m+1} \begin{bmatrix} L+i \\ L \end{bmatrix}_q \left(\begin{bmatrix} m \\ i \end{bmatrix}_q - 1 \right) \left(\sum_{k=1}^{m-i-1} q^k r^k \right)$$

MLLMLM$^\searrow$MM$^\searrow$MM$^\searrow$$_\bullet$M$_\bullet$M$_\bullet$M$_\bullet$RRRR

FIGURE 2. When placing an $(M+1)$-st M into an $(L)(M)(R)$-word with at least two M-descents and no M's to the right of the leftmost R, the available positions that will not form a 321-pattern are between the last M-descent and the leftmost R.

and $F_1(x, q, t, r, s)$ satisfies the functional equation

$$F_1(r, s, t) = E_1(r, s, t) + \frac{x}{1 - qr} F_1(1, s, qt) - \frac{xqr}{1 - qr} F_1(qr, s, qt)$$

(4.2)

$$- \frac{xt(qs)^{R+1}}{1 - qs} F_1(r, 1, t) + \frac{xt}{1 - qs} F_1(r, qs, t),$$

where

$$E_1(r, s, t) = xt \left(\sum_{b=1}^{R} (qs)^b \right) F_0(r) +$$

$$\sum_{m \geq 0} \sum_{i=1}^{m-1} x^{m+1} \begin{bmatrix} L + i \\ L \end{bmatrix}_q \left(\begin{bmatrix} m \\ i \end{bmatrix}_q - 1 \right) \left(\sum_{k=1}^{m-i-1} \sum_{c=1}^{k} \sum_{b=1}^{R} q^{k+b+c-1} r^{k-c} s^b t^c \begin{bmatrix} b + c - 2 \\ b - 1 \end{bmatrix}_q \right).$$

Proof. Similar to the proof in [HJ10], we use West's generating tree method [Wes90] to enumerate $(L)(M)(R)$-words with at least two M-descents. We fix L and R and determine all the ways in which we may insert an $(M+1)$-st M into an $(L)(M)(R)$-word (incrementing each number larger than $L + M$ by one), keeping track of how the combinatorial statistics change in each case.

When we restrict to enumerating $(L)(M)(R)$-words with at least two M-descents in which there is no M to the right of an R, there is one contiguous set of positions into which we may insert $L + M + 1$ that does not create a 321-pattern—between the rightmost M-descent and the leftmost R. (See Figure 2.) Condition on k, the number of M's to the right of where we insert $L + M + 1$. When $k = 0$, $m(w)$ and $\alpha(w)$ increase by one and $l(w)$ stays the same. As k ranges from 1 to $\alpha(w)$, then $m(w)$ increases by one, $l(w)$ increases by k, and $\alpha(w)$ becomes k. From this we see that the $(L)(M+1)(R)$-words of this type generated from $(L)(M)(R)$-words of this type have generating function

$$\sum_w x^{m(w)+1} q^{l(w)} r^{\alpha(w)+1} + \sum_w \sum_{k=1}^{\alpha(w)} x^{m(w)+1} q^{l(w)+k} r^k$$

$$= xr \sum_w x^{m(w)} q^{l(w)} r^{\alpha(w)} + x \sum_w x^{m(w)} q^{l(w)} \sum_{k=1}^{\alpha(w)} (qr)^k$$

$$= xr \sum_w x^{m(w)} q^{l(w)} r^{\alpha(w)} + \frac{xqr}{1 - qr} \sum_w x^{m(w)} q^{l(w)} \left(1 - (qr)^{\alpha(w)} \right)$$

$$= xr F_0(r) + \frac{xqr}{1 - qr} F_0(1) - \frac{xqr}{1 - qr} F_0(qr).$$

We also need to enumerate the $(L)(M+1)(R)$-words of this type that do not arise in this way. More specifically, we enumerate $(L)(M+1)(R)$-words w such that removing

$$\text{LMMLLMM}^\smallsetminus\bullet\text{MMM}^\smallsetminus\bullet\text{M}_\bullet\text{M}_\bullet\text{RRMMRM}_\bullet\text{R}_\bullet\text{R}_\bullet$$

FIGURE 3. When placing an $(M+1)$-st M into an $(L)(M)(R)$-word with at least two M-descents and at least one M to the right of the leftmost R, there are two contiguous sets of possible positions.

the number $L + M + 1$ (and decrementing each larger element by one) leaves an $(L)(M)(R)$-word w' with one M-descent and no M to the right of an R. Suppose that w''s one M-descent occurs in position i. There are $\left(\begin{bmatrix} M \\ i \end{bmatrix}_q - 1\right)$ ways to arrange the M-entries in this way and $\begin{bmatrix} L+i \\ L \end{bmatrix}_q$ ways to intertwine the L's with the first i M's. We then reinsert $L + M + 1$ to the right the existing M-descent. If there are k M's to the left of $L + M + 1$ (where k ranges from 1 to $m - i - 1$), then $l(w) = l(w') + k$ and $\alpha(w) = k$. Summing over all m gives $E_0(r)$.

Combining these new $(L)(M+1)(R)$-words with the $(L)(M+1)(R)$-words created using the generating tree and subsequently solving for $F_0(r)$ gives Equation (4.1).

When there is an M to the right of an R, there are two disjoint contiguous sets of positions into which we may insert $L + M + 1$. (See Figure 3.)

First, we may insert $L + M + 1$ into a position between the rightmost M-descent and the leftmost R. Suppose that there are k M's to the right of where we insert $L + M + 1$ and to the left of the leftmost R, where k ranges from 0 to $\alpha(w)$. In this case, $m(w)$ increases by one, $l(w)$ increases by $k + \gamma(w)$, and $\alpha(w)$ becomes k, while $\beta(w)$ and $\gamma(w)$ remain unchanged. This provides a contribution of

$$\sum_w \sum_{k=0}^{\alpha(w)} x^{m(w)+1} q^{l(w)+k+\gamma(w)} r^k s^{\beta(w)} t^{\gamma(w)}$$

$$= x \sum_w x^{m(w)} q^{l(w)} s^{\beta(w)} (qt)^{\gamma(w)} \sum_{k=0}^{\alpha(w)} (qr)^k$$

$$= \frac{x}{1-qr} \sum_w x^{m(w)} q^{l(w)} s^{\beta(w)} (qt)^{\gamma(w)} \left(1 - (qr)^{\alpha(w)+1}\right)$$

$$= \frac{x}{1-qr} F(x, q, 1, s, qt) - \frac{xqr}{1-qr} F(x, q, qr, s, qt).$$

Alternatively, we may insert $L + M + 1$ into a position after the rightmost M. Suppose that there are k R's to the left of where we insert $L + M + 1$, where k ranges from $\beta(w)$ to R. In this case, $m(w)$ and $\gamma(w)$ increase by one, $l(w)$ increases by k,

$\alpha(w)$ stays the same, and $\beta(w)$ becomes k. This provides a contribution of

$$\sum_w \sum_{k=\beta(w)}^{R} x^{m(w)+1} q^{l(w)+k} r^{\alpha(w)} s^k t^{\gamma(w)+1}$$

$$= xt \sum_w x^{m(w)} q^{l(w)} r^{\alpha(w)} t^{\gamma(w)} \sum_{k=\beta(w)}^{R} (qs)^k$$

$$= \frac{xt}{1-qs} \sum_w x^{m(w)} q^{l(w)} r^{\alpha(w)} t^{\gamma(w)} \left((qs)^{\beta(w)} - (qs)^{R+1} \right)$$

$$= \frac{xt}{1-qs} F(x,q,r,qs,t) - \frac{xt(qs)^{R+1}}{1-qs} F(x,q,r,1,t).$$

We also must enumerate $(L)(M+1)(R)$-words w with at least two M-descents and at least one M to the right of an R such that removing element $L+M+1$ (and decrementing each larger element by one) gives an $(L)(M)(R)$-word w' that is not in this class. This can occur in two ways—either $L+M+1$ is the only M to the right of the leftmost R in w or $L+M+1$ is to the left of the leftmost R and w' has only one M-descent.

In the former case we insert $L+M+1$ into any position after the leftmost R in an $(L)(M)(R)$-word with at least two M-descents and no M's to the right of an R. Conditioning on b, the number of R's before $L+M+1$, gives a contribution of $xt \left(\sum_{b=1}^{R} (qs)^b \right) F_0(r)$.

In the latter case we follow a similar argument to that given for $E_0(r)$, except that after placing $L+M+1$ to the left of k M's (for a contribution of q^k), we intertwine any positive number of R's with between 1 and k M's. If we intertwine b R's with c M's, we first pass one R past c M's and one M past $b-1$ R's for a contribution of q^{b+c-1}, and there are $\left[\begin{smallmatrix} b+c-2 \\ b-1 \end{smallmatrix} \right]_q$ ways to intertwine the remaining M's and R's. With c M's to the right of the leftmost R, there are $k-c$ M's after the last M-descent and before the leftmost R; this accounts for the $r^{k-c} s^b t^c$. $\qquad\square$

Finally, we note that Equation (4.1) has the form of Equation (1.1), which is solved in Example 2.3, and that Equation (4.2) is a functional equation of the form

$$F(r,s,t) = e(r,s,t) + f_{-1,0,1} F(1,s,qt) + f_{1,0,1} F(qr,s,qt) + f_{0,-1,0} F(r,1,t) + f_{0,1,0} F(r,qs,t),$$

to which Theorem 2.1 also applies directly. The coefficients of the generating function $\sum x^{n(w)} q^{l(w)}$ calculated using these functional equations agree with the formula given in [HJ10].

References

[BBMD+02] Cyril Banderier, Mireille Bousquet-Mélou, Alain Denise, Philippe Flajolet, Danièle Gardy, and Dominique Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Mathematics*, 246(1-3):29 – 55, 2002. Formal Power Series and Algebraic Combinatorics 1999.

[BDLPP98] Elena Barcucci, Alberto Del Lungo, Elisa Pergola, and Renzo Pinzani. A methodology for plane tree enumeration. In *Proceedings of the 7th Conference on Formal Power Series and Algebraic Combinatorics (Noisy-le-Grand, 1995)*, volume 180, pages 45–64, 1998.

[BDLPP99] Elena Barcucci, Alberto Del Lungo, Elisa Pergola, and Renzo Pinzani. Some combinatorial interpretations of q-analogs of Schröder numbers. *Ann. Comb.*, 3(2-4):171–190, 1999. On combinatorics and statistical mechanics.

[BDLPP01] Elena Barcucci, Alberto Del Lungo, Elisa Pergola, and Renzo Pinzani. Some permutations with forbidden subsequences and their inversion number. *Discrete Math.*, 234(1-3):1–15, 2001.

[BM96] Mireille Bousquet-Mélou. A method for the enumeration of various classes of column-convex polygons. *Discrete Math.*, 154(1-3):1–25, 1996.

[BMB09] Mireille Bousquet-Mélou and Richard Brak. Exactly solved models. In *Polygons, polyominoes and polycubes*, volume 775 of *Lecture Notes in Phys.*, pages 43–78. Springer, Dordrecht, 2009.

[BMJ06] Mireille Bousquet-Mélou and Arnaud Jehanne. Polynomial equations with one catalytic variable, algebraic series and map enumeration. *J. Combin. Theory Ser. B*, 96(5):623–672, 2006.

[DV84] Marie-Pierre Delest and Gérard Viennot. Algebraic languages and polyominoes enumeration. *Theoret. Comput. Sci.*, 34(1-2):169–206, 1984.

[HJ10] Christopher R. H. Hanusa and Brant C. Jones. The enumeration of fully commutative affine permutations. *European J. Combin.*, 31(5):1342–1359, 2010.

[MS09] Toufik Mansour and Chunwei Song. Kernel method and system of functional equations. *J. Comput. Appl. Math.*, 224(1):133–139, 2009.

[Pól69] G. Pólya. On the number of certain lattice polygons. *J. Combinatorial Theory*, 6:102–105, 1969.

[Sta11] Richard P. Stanley. Catalan addendum. `http://math.mit.edu/~rstan/ec/catadd.pdf`, 2011.

[Wes90] Julian West. *Permutations with forbidden sequences; and, stack-sortable permutations*. PhD thesis, MIT, 1990.

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