Applying a combinatorial determinant to count weighted cycle systems in a directed graph

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Submitted: 04 July 2007; Accepted: 21 December 2007
Mathematics Subject Classifications: Primary 05C50, 05C70; Secondary 05B20, 05C20, 11C20
Keywords: cycle system, cycle cover, hamburger graph, hamburger matrix, determinant, directed graph

Abstract

One method for counting weighted cycle systems in a graph entails taking the determinant of the identity matrix minus the adjacency matrix of the graph. The result of this operation is the sum over cycle systems of -1 to the power of the number of disjoint cycles times the weight of the cycle system. We use this fact to reprove that the determinant of a matrix of much smaller order can be computed to calculate the number of cycle systems in a hamburger graph.

This article deals with counting *cycle systems* (also called *partial cycle covers*), which are collections of vertex-disjoint directed cycles in a directed graph. The following combinatorial fact is useful in the study of cycle systems.

Theorem 1. *Let $G = (V, E)$ be a weighted, directed graph and let M be its adjacency matrix. Let $\mathcal{S}$ be the set of cycle systems of G. If $\mathcal{C}$ is a cycle system, let $|\mathcal{C}|$ denote the number of cycles in $\mathcal{C}$ an let $\mathrm{wt}(\mathcal{C})$ be the product of the weights of the edges in $\mathcal{C}$. Then*

$$\det(I - M) = \sum_{\mathcal{C}\in\mathcal{S}} (-1)^{|\mathcal{C}|}\mathrm{wt}(\mathcal{C}). \tag{1}$$

This fact is in the folklore; a brief history of its appearances can be found in [1], Section 1.4. When the graph has the structure of a *hamburger graph*—described below and presented visually in Figure 1—another simpler determinant can be used to count cycle systems efficiently, also explained below.

A hamburger graph H, introduced in [2], is made up of two acyclic graphs G_1 and G_2 and a connecting edge set E_3 with the following properties. The graph G_1 has k distinguished vertices $\{v_1, \ldots, v_k\}$ with directed paths from v_i to v_j only if $i < j$. The graph G_2 has k distinguished vertices $\{w_{k+1}, \ldots, w_{2k}\}$ with directed paths from w_i to w_j only if $i > j$. The edge set E_3 connects the vertices v_i and w_{k+i} by way of edges $e_i : v_i \to w_{k+i}$ and $e'_i : w_{k+i} \to v_i$. If desired, the graph's edges may be weighted, in which case the weight of a cycle system $\mathrm{wt}(\mathcal{C})$ is the product of the weights of the edges of $\mathcal{C}$.

The structure of a hamburger graph implies that every closed path must visit both halves of the graph. If H is a planar hamburger graph, every closed path must visit both halves of the graph exactly once, and therefore uses exactly one edge from G_2 to G_1. Following this idea further, we negate the weight of every edge from G_2 to G_1 in a hamburger graph. When H is planar and positively weighted initially, every cycle system contributes its weight positively in the sum in Equation (1). This is particularly useful if our goal is to count cycle systems in a graph.

For a cycle system $\mathcal{C}$ in a general hamburger graph H, let l be the number of edges in $\mathcal{C}$ from G_2 to G_1 and let m be the number of cycles in $\mathcal{C}$. Call a cycle system *positive* if $(-1)^{l+m} = +1$ and *negative* if $(-1)^{l+m} = -1$. Let c^+ be the sum of the weights of positive cycle systems and c^- be the sum of the weights of negative cycle systems.

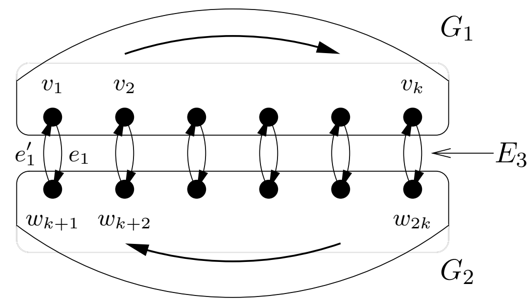

Figure 1: A hamburger graph

If $\widehat{M}$ is the adjacency matrix of H with negated weights on edges from G_2 to G_1, then Theorem 1 implies $\det(I - \widehat{M}) = c^+ - c^-$. This is a $|V| \times |V|$ determinant. In [2], the author introduced a method to calculate the same quantity by taking the determinant of a "hamburger matrix" M_H of smaller order. The hamburger matrix is a $2k \times 2k$ matrix that encodes the combinatorial information from H as follows. Define

$$M_H = \begin{bmatrix} A & D_1 \\ -D_2 & B \end{bmatrix}, \tag{2}$$

where in the $k \times k$ upper triangular matrix $A = (a_{ij})$, a_{ij} is the (weighted) number of paths from v_i to v_j in G_1 and in the $k \times k$ lower triangular matrix $B = (b_{ij})$, b_{ij} is the (weighted) number of paths from w_{k+i} to w_{k+j} in G_2. The diagonal $k \times k$ matrix D_1 has as its entries $d_{ii} = \mathrm{wt}(e_i)$ and the diagonal $k \times k$ matrix D_2 has as its entries $d_{ii} = \mathrm{wt}(e_i')$. We insist on the following restriction:

$$\mathrm{wt}(e_i)\mathrm{wt}(e_i') = 1 \text{ for } 1 \leq i \leq k.$$

Note that this implies that $D_1 = D_2^{-1}$. The weighted hamburger theorem (Theorem 2.3 from [2]) states:

Theorem 2. $\det M_H = c^+ - c^-$.

Remark 3. For a general graph H, calculation is required to calculate the entries of M_H. Also, the matrix $I - \widehat{M}$ matrix is sparse compared to M_H. These properties make it unclear if the calculation of $\det(M_H)$ takes less processor time to compute than $\det(I - \widehat{M})$, even if the former is of smaller order than the latter.

The original proof of the weighted hamburger theorem was based on an involution-like argument with terms canceling in the permutation expansion of the determinant of M_H. We now will reprove the weighted hamburger theorem in a different, simpler way. For one, any weighted hamburger graph H with $2k$ distinguished vertices has an equivalent *complete weighted hamburger graph* K with exactly $2k$ vertices. That is, K has one weighted directed edge between each pair of vertices v_i and v_j $(i < j)$ in the upper half and between each pair of vertices w_{k+i} and w_{k+j} $(i > j)$ in the lower half. The weights on the edges of these new edges in K are determined by the weights of the edges in H. More precisely, the weight of the edge from v_i to v_j in K is equal to the sum of the weights of the paths from v_i to v_j in H *not passing through* any other distinguished vertex along the way. (See Figure 2 for an example.) In our conversion from H to K, we do not modify the weights on the edges in E_3.

By construction, this conversion from H to K preserves the weighted cycle system sum $c^+ - c^-$, because we can think of every weighted cycle system in K as a family of weighted cycle systems in H that visit the same distinguished vertices in order. We will now apply Theorem 1 to find the weighted cycle sum on K.

The weighted adjacency matrix $\widetilde{M}$ of K is of the form

$$\widetilde{M} = \begin{bmatrix} \widetilde{A} & D_1 \\ -D_2 & \widetilde{B} \end{bmatrix},$$

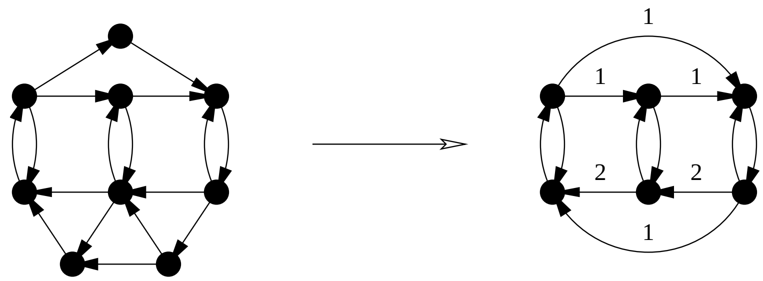

Figure 2: Conversion from a hamburger graph H to a complete weighted hamburger graph K

where by the structure of K, we know $\widetilde{A}$ is an upper-triangular matrix and $\widetilde{B}$ is a lower-triangular matrix. Compare this matrix to the hamburger matrix M_H in Equation (2). The entries of A in M_H are the sum of the weights of the paths from v_i to v_j in H with no routing restriction, while the entries of $\widetilde{A}$ are the sum of the weights of the paths from v_i to v_j in H without visiting any other distinguished vertices. We now apply another well-known combinatorial fact, that in an weighted acyclic directed graph with (weighted) adjacency matrix N, the number of paths from v_i to v_j is equal to the (i, j) entry of $(I-N)^{-1}$. (This appears for example in [1], Theorem 1.9.) This theorem applied to graphs G_1 and G_2 implies that $A = (I - \widetilde{A})^{-1}$ and $B = (I - \widetilde{B})^{-1}$, respectively. We use this fact in the calculation below.

Theorem 1 tells us that the weighted cycle sum of K is equal to the determinant of

$$I - \widetilde{M} = \begin{bmatrix} I - \widetilde{A} & -D_1 \\ D_2 & I - \widetilde{B} \end{bmatrix}. \tag{3}$$

Since A and B both are triangular matrices with 1's along the diagonal, multiplying on the left by the block matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ does not change the determinant. This implies

$$\det(I - \widetilde{M}) = \det \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I - \widetilde{A} & -D_1 \\ D_2 & I - \widetilde{B} \end{bmatrix} = \det \begin{bmatrix} I & -AD_1 \\ BD_2 & I \end{bmatrix} = \det \begin{bmatrix} AD_1 & I \\ -I & BD_2 \end{bmatrix}.$$

The last equivalence is because negating the last k columns of the matrix and interchanging k columns of the matrix (c_i with c_{k+i}) both contribute a sign of $(-1)^k$ to the determinant. When $D_1 = D_2^{-1}$, we can multiply this result on the right by $\det \begin{bmatrix} D_1^{-1} & 0 \\ 0 & D_2^{-1} \end{bmatrix} = 1$, yielding

$$\det(I - \widetilde{M}) = \det \begin{bmatrix} A & D_2^{-1} \\ -D_1^{-1} & B \end{bmatrix} = \det \begin{bmatrix} A & D_1 \\ -D_2 & B \end{bmatrix} = \det(M_H).$$

This reproves the weighted hamburger theorem.

Acknowledgments

I would like to thank Mireille Bousquet-Mélou for introducing me to Theorem 1 and an anonymous referee for pointing me to its history. I also thank two anonymous referees for their careful reading and suggestions on clarifying and streamlining the article.

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