ENSURING EVERY CANDIDATE WINS UNDER POSITIONAL VOTING

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Abstract. Given a fixed set of voter preferences, different candidates may win outright given different scoring rules. We investigate how many voters are able to allow all n candidates to win for some scoring rule. We will say that these voters impose a disordering on these candidates. The minimum number of voters it takes to impose a disordering on 3 candidates is 9. For 4 candidates, 6 voters are necessary, for 5 candidates, 4 voters are necessary, and it takes only 3 voters to disorder 9 candidates. In general, we prove that m voters can disorder n candidates when m and n are both greater than or equal to 3, except when $m = 3$ and $n \leq 8$, when $n = 3$ and $m \leq 8$, and when $n = 4$ and m equals 4 or 5.

1. Background Information

Saari, in his paper "Millions of election outcomes from a single profile" [3], proved for n candidates that it is possible to create a profile leading to $(n-1)(n-1)!$ different strict positional election rankings. (Recall that tallying votes using a positional rule assigns specific weights to candidates based on how they are positioned on the ballot. For example, the plurality vote is when the top-ranked candidate is given a weight of one and every other candidate is given a weight of zero. The Borda count is when the top-ranked candidate is given a weight of n, the second-ranked candidate is given a weight of $n-1$, through the last-ranked candidate, who receives a weight of 1.) In other words, even if the voter's preferences remain the same, it is possible that different scoring rules can produce a large number of different election outcomes. Saari's result shows that even with as few as 10 candidates, the number of possible election outcomes for one profile can be over one million! This highlights that the choice of the positional rule used in the election is very important.

Yet how do we know that this result is relevant to everyday elections? If the number of voters necessary to produce a profile with different outcomes is large, then we can dismiss Saari's result as irrelevant to most organizations. While in this article I do not explain how many voters are necessary to create a profile with $(n-1)(n-1)!$ election outcomes, I characterize how many voters are required to create a profile under which every candidate will win outright under some positional rule. I show that the required number of voters is smaller than one might expect. For example, if there are nine or more voters, then they can vote in such a way that any of three candidates might win under a different positional rule. A particularly surprising conclusion is that as the number of candidates increases, the number of voters required to construct such a profile decreases! A complete explanation for this counter-intuitive result is given in the paper, but the basic intuition behind this assertion is that as the number of candidates increases, the number of degrees of freedom in the normalized positional rule increases. With three candidates, there is but one degree of freedom; whereas

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Number of candidates	Number of voters
$n = 3$	$m \geq 9$
$n = 4$	$m \geq 6$
$5 \leq n \leq 8$	$m \geq 4$
$n \geq 9$	$m \geq 3$

Table 1. For each given combination of number of candidates and corresponding number of voters, a profile exists where all candidates win under some scoring rule.

with n candidates, there are $n-2$ degrees of freedom. Table 1 provides a summary of the results. Additional references which discuss finding minimal numbers of voters for paradoxes include Weber's "How many voters are needed for paradoxes?" [5] and Saari's "Mathematical structure of voting paradoxes: II. Positional voting" [4]. The geometrical theory behind positional rules can be found in Saari's "Basic Geometry of Voting" [2].

I now formalize the necessary background knowledge and then prove the aforementioned results. Consider m voters expressing their preferences for n candidates. Assume that each voter ranks all n candidates with no ties. We will call this set of preferences a *profile*. A *scoring rule* is be a vector with monotonically nonincreasing entries of the form $\mathbf{x} = (x_1, x_2, x_3, \ldots, x_{n-1}, x_n)$. This scoring rule is *normalized* when $x_1 = 1$ and $x_n = 0$. For example, the scoring rule given above for the Borda count is

$$\mathbf{x} = (n, n-1, \ldots, 2, 1)\,;$$

after a linear transformation, its normalized scoring rule is

$$\mathbf{x} = \left(1, \frac{n-2}{n-1}, \ldots, \frac{1}{n-1}, 0\right).$$

A linear scaling of a scoring rule does not change the outcome of the election; however, it provides the framework for the topic of this article. One may decide to vary the point values given to the $n-2$ middle candidates in order to change the weight that a first-place, second-place, third-place, etc. finish should carry. A different candidate may win under different scoring rules. (There will be many examples of this throughout this article.) Given a scoring rule $\mathbf{x}$, we will represent the score of a candidate c by the notation $\mathbf{x}(c)$.

The author wishes to determine the minimal number of voters who can collude to rank the candidates in such a way that *each* candidate will win outright with some scoring rule. We will call such a set of voter preferences a *disordering profile*, and say that (m, n) is a *disordering pair* if a set of m voter preferences disordering n candidates exists. We will say that a pair of candidates c_1 and c_2 are *disordered* by a set of voter preferences if two scoring rules $\mathbf{x}_1$ and $\mathbf{x}_2$ exist where $\mathbf{x}_1(c_1) > \mathbf{x}_1(c_2)$ and $\mathbf{x}_2(c_1) < \mathbf{x}_2(c_2)$. For a voter v_i, define v_i's *preference ranking* r_i to be a one-to-one function from the set of candidates to the integers $[n] = \{1, \ldots, n\}$, where if $r_i(c_1) > r_i(c_2)$ for candidates c_1 and c_2, then v_i prefers c_1 to c_2. With this notation, if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a scoring rule, then $\mathbf{x}(c) = \sum_{i=1}^{m} x_{n+1-r_i(c)}$.

Section 2 explores the minimal number of voters necessary to disorder n candidates. Section 3 explores which pairs (m, n) are disordering pairs.

2. Finding minimal disordering pairs

In order to find the minimal number of voters necessary to disorder n candidates, we first establish and prove a necessary and sufficient condition under which two candidates are disordered. Building upon this, we establish a necessary condition for a profile to be a disordering profile.

We define $R_j(c)$ as the number of voters for which candidate c is ranked at least j, that is, the number of voters v_i such that $r_i(c) \geq j$. The following lemma gives a necessary and sufficient condition under which two candidates are disordered.

Lemma 1. *In a given profile, two candidates c_1 and c_2 are disordered if and only if there exist integers j and k such that $R_j(c_1) > R_j(c_2)$ and $R_k(c_1) < R_k(c_2)$.*

Proof. We first prove that this condition is necessary. Order the rankings $r_i(c_1)$ in decreasing order and do the same for the rankings $r_i(c_2)$. Suppose that $R_k(c_1) \geq R_k(c_2)$ for all k from 1 to n. Then the j-th highest ranking of candidate c_1 is at least as high as the j-th highest ranking of candidate c_2. Since in all scoring rules $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, the x_i are monotonically nonincreasing, the contribution to the total score of the each candidate's j-th highest ranking is always at least as large for c_1 as for c_2. Therefore, $\mathbf{x}(c_1) \geq \mathbf{x}(c_2)$ for all scoring rules $\mathbf{x}$. This implies that candidate c_2 will never be preferred to candidate c_1.

In addition, this condition is sufficient to disorder two candidates. For example, if j and k are integers such that $R_j(c_1) > R_j(c_2)$ and $R_k(c_1) < R_k(c_2)$, the scoring rules $(1, \ldots, 1, \underbrace{0, \ldots, 0}_{j-1})$ and $(1, \ldots, 1, \underbrace{0, \ldots, 0}_{k-1})$ disorder candidates c_1 and c_2. □

A direct corollary of Lemma 1 is the following.

Lemma 2. *For every pair of candidates c_i and c_j in a disordering profile, there exist integers j and k such that $R_j(c_i) > R_j(c_j)$ and $R_k(c_i) < R_k(c_j)$.*

We will repeatedly use Lemma 2 in order to prove that a particular profile is not a disordering profile.

That every pair of candidates is disordered in a profile does not guarantee that the profile is a disordering. Figure 1 gives two examples of profiles in which every pair of candidates is disordered yet are not disordering profiles. In this and subsequent examples, each column will represent one voter's preferences for the candidates, ordered from highest preference at the top to lowest preference at the bottom. Figure 1(a) is an example of four candidates and four voters, where candidate c_2 can only tie for the lead (with the scoring rule $(1, 2/3, 1/3, 0)$). In order to verify that candidate c_2 cannot win outright, we must verify that this is true that under every scoring rule. Figure 2(a) breaks down the region $1 \geq x_2 \geq x_3 \geq 0$ into the subregions in which candidates c_1, c_3, and c_4 win. Figure 1(b) contains a larger example in which candidate c_3 can win for no scoring rule. This is shown by the graph in Figure 2(b) by plotting the scores of candidates c_1, c_2, and c_3 as x_2 varies.

For the following theorem and throughout this article, the computer software Maple was used to aid example generation and verification. For example, with a small enough number of voters and candidates, a computer can generate all profiles of that size and then verify whether Lemma 2 holds. Appendix C contains a sample of the types of calculations done with Maple. The complete Maple worksheet and calculations are available directly from the author and as a download from the author's website[1].

[1]http://qcpages.qc.edu/~chanusa/papers.html

$$\text{(a)}\quad \begin{array}{cccc} c_1 & c_1 & c_2 & c_3 \\ c_2 & c_4 & c_4 & c_4 \\ c_3 & c_3 & c_3 & c_2 \\ c_4 & c_2 & c_1 & c_1 \end{array} \qquad \text{(b)}\quad \begin{array}{cccccccccc} c_1 & c_1 & c_1 & c_1 & c_1 & c_2 & c_2 & c_3 & c_3 & c_3 \\ c_2 & c_2 & c_2 & c_2 & c_3 & c_3 & c_3 & c_2 & c_2 & c_2 \\ c_3 & c_3 & c_3 & c_3 & c_2 & c_1 & c_1 & c_1 & c_1 & c_1 \end{array}$$

FIGURE 1. (a) A profile with four candidates and four voters where each pair of candidates is disordered. (b) A profile with three candidates and ten voters where each pair of candidates is disordered.

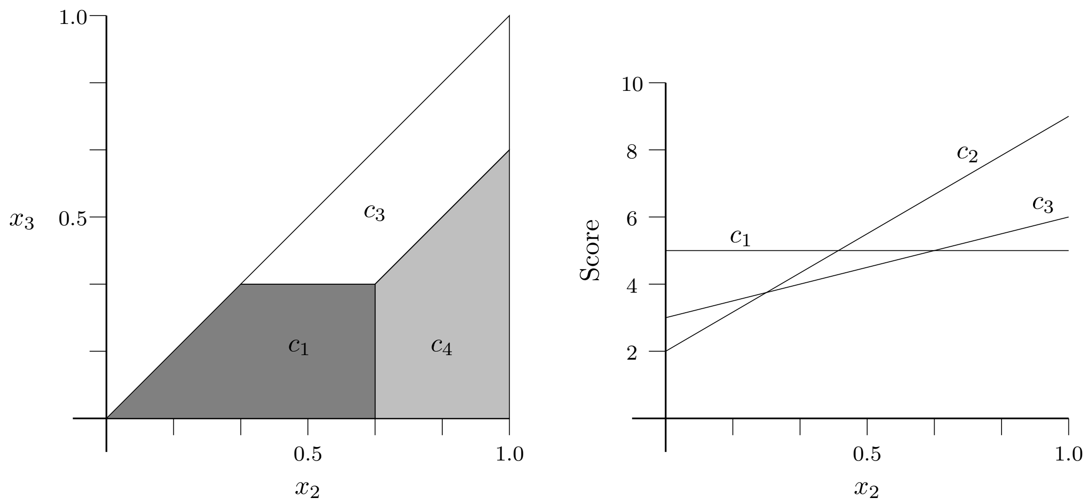

FIGURE 2. (a) The decomposition of the region $1 \geq x_2 \geq x_3 \geq 0$ into subregions for which candidates c_1, c_3, and c_4 win in the profile in Figure 1(a). (b) A graph of the scores that candidates c_1, c_2, and c_3 receive under the profile in Figure 1(b) for values of x_2 between 0 and 1.

Theorem 3. *The minimal number of voters necessary to disorder three candidates is nine. Similarly, six voters are necessary to disorder four candidates, four voters to disorder five candidates, and three voters to disorder nine candidates.*

The proof of Theorem 3 is a verification that every profile with fewer candidates is not a disordering profile. Examples of the cited minimal configurations are presented in Figure 3, along with the scoring rules which verify that each is disordering. In the vast majority of the cases, the profile in question includes two candidates who are not disordered. The few profiles which do not have two disordered candidates are treated similarly to the profile in Figure 1(a) above.

The remainder of this section verifies that (m, n) cannot be a disordering pair if $n = 2$ (Lemma 4), if $m = 2$ (Lemma 5), or if $m = 3$ and $n \leq 8$ (Lemma 6). The two remaining cases are less instructive and are included as Appendix A for completeness. Lemma 15 proves that (m, n) is not a disordering pair if $n = 3$ and $m \leq 8$, while Lemma 16 addresses the case when $n = 4$ and m is either 4 or 5.

It seems counter-intuitive at first that the number of voters needed to disorder candidates decreases with the number of candidates. The reason for this is that with more candidates come more degrees of freedom in the scoring rule, so fewer voters should be necessary. Two

(a)

									$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
c_1	c_1	c_1	c_1	c_1	c_2	c_2	c_2	c_2	1.0	1.0	1.0
c_2	c_2	c_3	c_3	c_3	c_3	c_3	c_3	c_3	0.0	0.6	1.0
c_3	c_3	c_2	c_2	c_2	c_1	c_1	c_1	c_1	0.0	0.0	0.0

(b)

						$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$
c_3	c_1	c_1	c_2	c_2	c_1	1.0	1.0	1.0	1.0
c_2	c_2	c_3	c_3	c_3	c_4	0.0	0.8	1.0	1.0
c_4	c_4	c_4	c_4	c_4	c_3	0.0	0.0	0.1	1.0
c_1	c_3	c_2	c_1	c_1	c_2	0.0	0.0	0.0	0.0

(c)

				$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$
c_1	c_1	c_5	c_4	1.0	1.0	1.0	1.0	1.0
c_2	c_3	c_2	c_2	0.0	1.0	1.0	0.9	0.9
c_3	c_5	c_3	c_5	0.0	0.0	1.0	0.9	0.9
c_4	c_4	c_4	c_3	0.0	0.0	0.1	0.9	0.0
c_5	c_2	c_1	c_1	0.0	0.0	0.0	0.0	0.0

(d)

			$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_7$	$\mathbf{x}_8$	$\mathbf{x}_9$
c_1	c_9	c_8	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_6	c_7	0.9	0.9	1.0	0.9	0.9	0.9	1.0	0.3	0.3
c_5	c_3	c_3	0.9	0.8	1.0	0.9	0.9	0.9	0.3	0.3	0.3
c_4	c_4	c_2	0.9	0.8	0.0	0.9	0.9	0.9	0.3	0.3	0.3
c_7	c_1	c_5	0.9	0.0	0.0	0.7	0.7	0.7	0.3	0.3	0.3
c_6	c_8	c_6	0.0	0.0	0.0	0.7	0.5	0.7	0.25	0.3	0.3
c_9	c_5	c_9	0.0	0.0	0.0	0.7	0.5	0.0	0.25	0.25	0.3
c_8	c_7	c_4	0.0	0.0	0.0	0.7	0.0	0.0	0.25	0.25	0.25
c_3	c_2	c_1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FIGURE 3. Disordering profiles with (a) 9 voters and 3 candidates (b) 6 voters and 4 candidates (c) 4 voters and 5 candidates (d) 3 voters and 9 candidates.

voters, however, cannot impose a total disorder on any number of candidates, nor can two candidates be disordered by any number of voters.

Lemma 4. *Two candidates cannot be disordered by any number of voters.*

Proof. The normalized scoring rule with two candidates is always $(1, 0)$, so no matter how many voters there are, the candidates will always be tied or one candidate will place higher than the other. □

Lemma 5. *Two voters cannot impose a disorder on any number of candidates.*

Proof. Let v_1 and v_2 be our two voters with preference rankings r_1 and r_2. Let c_1 be the first preference of voter v_1 and c_2 be the first preference of voter v_2. If $c_1 = c_2$, then no other candidate can ever be preferred to this candidate. If $c_1 \neq c_2$, then consider the values $r_1(c_2)$ and $r_2(c_1)$. If these values are equal, c_1 and c_2 will always be weighted equally and never be preferred one to the other. On the other hand, if one value is larger than the other, for example $r_2(c_1) > r_1(c_2)$, then c_2 will never be preferred to c_1. □

c_4	c_5	c_1
c_2	c_6	c_7
c_3	c_3	c_8
x	x	c_2
y	c_1	y
z	z	z
z	z	z
c_1	c_2	c_3

FIGURE 4. Visual aid for the proof of the impossibility of a $(3, 8)$ disordering profile.

Lemma 6. *There does not exist a $(3, n)$ disordering pair for $n \leq 8$.*

Proof. Assume we can construct a set of three voters' preferences that create a disordering profile with fewer than 9 candidates. Let c_1, c_2, and c_3 be the three candidates that are ranked last by the three voters. They must be distinct; otherwise, the candidate that is ranked last at least twice will have at most one non-last ranking, and can never win. Therefore, candidates c_1, c_2, and c_3 each have two non-last rankings, and each must be able to win outright over the others, using only these two non-last rankings. Lemma 2 implies how these six non-last rankings must be ordered in order for there to possibly be a disordering. Up to candidate relabeling, it must be the case that candidate c_1's highest ranking is strictly higher than c_2's, which in turn are both strictly higher than c_3's two highest rankings, which are strictly higher than c_2's second highest ranking, all of which must be higher than c_1's second highest ranking.

With three voters, there are at least 5 (possibly non-distinct) candidates ranked higher than or equal to the highest position of c_3. (See Figure 3(d) for a visual aid.) If $n \leq 7$, there must be a repetition in those five candidates, and c_3 could never win over this repeated candidate. Even when $n = 8$, it is impossible to arrange the rankings in a satisfactory manner. Figure 4 shows the only viable placement of the rankings of candidates c_1, c_2, and c_3, with the five distinct candidates ranked higher than c_3.

Notice that none of candidates c_4 through c_7 can be ranked in the positions marked x, or else c_2 could not win outright. So c_8 must be ranked in those positions. Neither c_4 nor c_5 may be ranked in a position marked y, or else c_1 could not win outright. If they are both c_6, then c_7 cannot win outright. If the positions marked y are c_6 and c_7, only one ranking remains for each, so one could never win outright, as in the proof of Lemma 5. So $(3, 8)$ is not a disordering pair. □

3. FINDING ALL DISORDERING PAIRS

It would be nice to be able to say that if (m, n) is a disordering pair, then both $(m+1, n)$ and $(m, n+1)$ are disordering pairs. But we are not even assured that by adding an additional voter to a disordering profile that the resulting augmented profile is also a disordering profile. The problem of finding the disordering pairs (m, n) is solved by means of the following theorem.

Theorem 7. *A collection of m voters can disorder n candidates whenever $m \geq 3$ and $n \geq 3$, except in the cases presented in Section 2. Those are when $m = 3$ and $n \leq 8$, when $n = 3$ and $m \leq 8$, and when $n = 4$ and $m \leq 5$.*

The remainder of this section will give rules to help prove this theorem. We start with a definition about when it is possible to add a candidate in a simple manner to a set of disordering voter preferences.

Definition. Let (m, n) be a disordering pair. We will say that (m, n) is a *splittable* disordering pair if some m-voter, n-candidate disordering profile supplemented by an $(n+1)$-st candidate ranked the same by all voters induces a disordering on the $n+1$ candidates. We will also refer to such a disordering profile as a *splittable* disordering profile.

For example, the disordering profile in Figure 3(a) is splittable since adding a row of c_4's between the second and third preferences yields a disordering of $\{c_1, c_2, c_3, c_4\}$. (We justify this assertion in the proof of Lemma 13.) When (m, n) is a splittable disordering, not only does this trivially imply that $(m, n+1)$ is disordering, but it also gives us the framework to create infinitely many disorderings through an introduction of more candidates.

Theorem 8. *If (m, n) is a splittable disordering pair, and (m, n') is a disordering pair, then $(m, n+n')$ is a disordering pair.*

We will think of the $(n+1)$-st candidate that we insert into the disordering profile with m voters and n candidates as the placeholder for the n' candidates we wish to insert. In our proof, we must first show there is enough "space" in which to insert all n' candidates.

Proof. Let (m, n) be a splittable disordering pair. Then there exists a disordering profile of m voter preferences for candidates c_1 through c_n with a way to insert an $(n+1)$-st candidate c_{n+1} between each voter's $(k-1)$-st and k-th preferences (for some k) that makes this augmented set of voter preferences a disordering profile. This implies that there exist (normalized) scoring rules $\mathbf{x}_i$ for i from 1 to $n+1$ such that candidate c_i wins outright, i.e. $\mathbf{x}_i(c_i) > \mathbf{x}_i(c_j)$ for all $j \neq i$.

Consider what "winning outright" means with respect to the geometry of the situation. The vectors $\mathbf{x}_i = (1, x_2, x_3, \ldots, x_n, 0)$ are elements of the subset S of $\mathbb{R}^{n+1}$ where $1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq 0$. Assign to every vector $\mathbf{x} \in S$ the candidates that win there. Since $\mathbf{x}(c)$ is an affine function on variables x_2 through x_n, the region associated to each candidate c is an intersection of halfspaces, the boundary of which arises from equations of the form $\mathbf{x}(c) = \mathbf{x}(c')$. For a candidate c to be able to win outright, the intersection of these halfspaces must have a non-empty interior, otherwise other candidates would tie with c at every point where c was to win.

Under this framework, the scoring rule $\mathbf{x}_{n+1}$ for which c_{n+1} wins outright will be located in the interior of the region associated to c_{n+1}, an open set. Let $\mathbf{x}_{n+1} = (x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{n+1})$. Since $\mathbf{x}_{n+1}$ is in the interior of the region on which c_{n+1} wins, there is some closed interval $[z_0, z_1]$ of positive width on which c_{n+1} wins outright for all vectors $\mathbf{x} = (x_1, \ldots, x_{k-1}, z, x_{k+1}, \ldots, x_{n+1})$, whenever $z \in [z_0, z_1]$. (That is, there is "space" around the k-th component of $\mathbf{x}_{n+1}$, in which a small perturbation does not change that c_{n+1} wins outright.)

Now since (m, n') is a disordering pair, this implies that there is a set of m voter preferences that disorder n' candidates, say c'_1 through $c'_{n'}$. Therefore there exist scoring rules $\mathbf{x}'_i$ for $1 \leq i \leq n'$ such that candidates c'_i win outright. Consider the set of voter preferences on candidates c_1 through c_n and c'_1 through $c'_{n'}$, where we insert candidates c'_1 through $c'_{n'}$ exactly where we placed the $(n+1)$-st candidate above (level k). We wish to show that this is a disordering of the $n+n'$ candidates.

For one, note that all candidates c_i for $1 \leq i \leq n$ can win outright. Since there exist scoring rules $\mathbf{x}_i = (1, x_2, \ldots, x_n, 0)$ such that c_i wins outright over candidates c_1 through c_{n+1}, the scoring rule $\widehat{\mathbf{x}}_i = (1, x_2, \ldots, x_{k-1}, x_k, x_k, \ldots, x_k, x_{k+1}, \ldots, x_n, 0)$ is such that c_i wins outright over the candidates c_1 through c_n and c'_1 through $c'_{n'}$. This is because $\widehat{\mathbf{x}}_i(c'_j) = \mathbf{x}(c_k) < \mathbf{x}_i(c_i) = \widehat{\mathbf{x}}_i(c_i)$.

Also note that all candidates c'_i can win for $1 \leq i \leq n'$. Linearly rescale the vectors $\mathbf{x}'_i = (1, x'_2, \ldots, x'_{n'-1}, 0)$ to $\mathbf{y}'_i = (z_1, y'_2, y'_3, \ldots, y'_{n'-1}, z_0)$, where z_0 and z_1 are from above. As mentioned in the introduction, a linear scaling of a scoring rule does not change the outcome of the election. Since candidate c_{n+1} wins outright over c_1 through c_n for all vectors $\mathbf{x} = (x_1, \ldots, x_{k-1}, z, x_{k+1}, \ldots, x_{n+1})$ whenever $z \in [z_0, z_1]$, then candidate c'_i will win outright over all candidates c_1 through c_n and c'_1 through $c'_{n'}$ for the scoring rule $\widehat{\mathbf{x}}'_i = (x_1, \ldots, x_{k-1}, z_1, y'_2, \ldots, y'_{n'-1}, z_0, x_{k+1}, \ldots, x_{n+1})$. Therefore $(m, n+n')$ is a disordering pair. $\square$

Corollary 9. *If (m, n) is a splittable disordering pair, and there is some sequence of n consecutive disordering pairs $(m, N), (m, N+1), \ldots, (m, N+n-1)$, then for all $n' > N$, (m, n') is a disordering pair.*

We can also generate disorderings by increasing the number of voters.

Lemma 10. *If (m, n) is a disordering pair, then (km, n) is a disordering pair for all integers $k \geq 2$.*

Proof. The introduction of clones of every voter will not change the disordering of the n candidates. $\square$

Lemma 11. *If (m, n) is a disordering pair, then $(m+n, n)$ is also a disordering pair.*

Proof. Given a disordering profile of m voters and n candidates, we can add n voters with the cyclic preferences on the candidates. For example, if $n = 3$, then let voter v_{m+1} prefer c_1 to c_2 to c_3, let voter v_{m+2} prefer c_2 to c_3 to c_1 and let voter v_{m+3} prefer c_3 to c_1 to c_2. The introduction of these new voters changes each candidate's score by the same constant amount. $\square$

Corollary 12. *If there is some sequence of n consecutive disordering pairs $(M, n), (M+1, n), \ldots, (M+n-1, n)$, then for all $m \geq M$, (m, n) is a disordering pair.*

In general, we may not be able to generate splittable disordering profiles from splittable disordering profiles as in the proof of Lemma 11, but it does work when $n = 3$.

Lemma 13. *There exist splittable disordering profiles with m voters and 3 candidates for $m \geq 9$.*

Proof. There exist splittable disordering profiles with 9, 10, and 11 voters for 3 candidates, as shown in Figures 5(a–c), justified below. We claim that by adding k triples of cyclic preferences to each results in a splittable disordering profile with 3 candidates and $m + 3k$ voters for $m \in \{9, 10, 11\}$. Lemma 11 implies that profiles are indeed disordering profiles. We need only supply scoring rules that show that upon insertion of a fourth candidate preferred next-to-last by all voters, each candidate can win.

When we have $9 + 3k$ voters for $k \geq 0$ following the preferences in Figure 5(a), the scores for candidates a through d are respectively $5 + k(1 + x_2)$, $4 + 2x_2 + k(1 + x_2)$, $7x_2 + k(1 + x_2)$, and $(9+3k)x_3$. Table 2 gives scoring rules under which each candidate wins. The calculations

(a)

c_1	c_1	c_1	c_1	c_1	c_2	c_2	c_2	c_2
c_2	c_2	c_3	c_3	c_3	c_3	c_3	c_3	c_3
c_3	c_3	c_2	c_2	c_2	c_1	c_1	c_1	c_1

(b)

c_1	c_1	c_1	c_1	c_1	c_2	c_2	c_2	c_2	c_3
c_2	c_2	c_3	c_3	c_3	c_3	c_3	c_3	c_3	c_2
c_3	c_3	c_2	c_2	c_2	c_1	c_1	c_1	c_1	c_1

(c)

c_1	c_1	c_1	c_1	c_1	c_1	c_2	c_2	c_2	c_2	c_2
c_2	c_2	c_3	c_3	c_3	c_3	c_3	c_3	c_3	c_3	c_3
c_3	c_3	c_2	c_2	c_2	c_2	c_1	c_1	c_1	c_1	c_1

FIGURE 5. Splittable disordering profiles with 3 candidates and (a) 9 voters (b) 10 voters (c) 11 voters.

scoring rule	candidate scores	winning candidate
$(1,0,0,0)$	$(5+k, 4+k, k, 0)$	a
$(1,3/5,0,0)$	$(5+8k/5, 26/5+8k/5, 21/5+8k/5, 0)$	b
$(1,1,0,0)$	$(5+2k, 6+2k, 7+2k, 0)$	c
$(1,1,1,0)$	$(5+2k, 6+2k, 7+2k, 9+3k)$	d

TABLE 2. Scoring rules which allow each candidate to win in the $(9+3k)$-voter splittable disordering profile from the proof of Lemma 13.

for $10+3k$ and $11+3k$ voters for $k \geq 0$ following the preferences in Figures 5(b) and 5(c) are almost identical. In fact, the same scoring rules may be used. □

Proof of Theorem 7. We start with the following lemma, which exhibits ideas we reuse to complete the proof of the theorem. Throughout these proofs, many examples are given of splittable disorderings; Appendix B includes these profiles and the scoring rules that show that the examples are indeed disordering and splittable. The Maple worksheet provided on the author's website calculates these verifications automatically.

Lemma 14. *The pair (m,n) is disordering for $m \geq 9$ and $n \geq 3$.*

Proof. Lemma 13 constructs splittable disordering pairs $(m,3)$ for $m \geq 9$. By definition, there exist disordering pairs $(m,4)$ for $9 \leq m \leq 13$, and by Corollary 12, there exist disordering pairs of type $(m,4)$ for $m \geq 9$. Figure 3(b) is a splittable disordering of type $(6,4)$. (The preferences split between rankings two and three.) This gives a disordering pair of type $(6,5)$. Figures 3(c) and 5(d) give examples of disordering pairs $(4,5)$ and $(5,5)$. (In fact, they are splittable disordering pairs, both splitting between rankings three and four.) By Lemma 10, there exist disordering pairs $(8,5)$, $(10,5)$, and $(12,5)$. By Lemma 11, there exist disordering pairs $(9,5)$, $(11,5)$, and $(13,5)$. Corollaries 12 and 9 give the desired result. □

We argue similarly that the pair (m,n) is disordering for $m \geq 6$ and $n \geq 4$; we need only show that there exist splittable disordering pairs $(6,4)$, $(7,4)$, and $(8,4)$, as well as $(6,6)$, $(7,6)$, and $(8,6)$. These are given in Figures 7 through 9 and 14 through 16. Similarly, the pair (m,n) is disordering for $m \geq 4$ and $n \geq 5$; again, we need exhibit splittable disordering pairs $(4,5)$, $(5,5)$, $(4,6)$, $(5,6)$, $(4,8)$, and $(5,8)$. Figures 10 through 13 and Figures 17 and 18 give such pairs. Lastly, we justify that the pair (m,n) is disordering for $m=3$ and $n \geq 9$—Figures 19 through 23 give splittable disordering pairs $(3,9)$, $(3,10)$, $(3,12)$, $(3,14)$, and $(3,16)$. This proves Theorem 7. □

4. Final Thoughts

While it is possible for all candidates to be able to win outright from a fixed set of voter preferences, it is not possible for all rankings of candidates to appear as the cumulative results from a fixed set of voter preferences. This topic is discussed in Saari's [3]. An inversion of the cumulative rankings is always possible, and is discussed in [1]. It would be of interest to know how many voters are necessary to create a profile with all $(n-1)(n-1)!$ positional election rankings, as guaranteed by [3].

It is interesting to remark that the pairs of m and n which are disordering pairs are symmetric except for $(5, 4)$ and $(4, 5)$. This appears to be a coincidence.

I conclude with the following questions. Is there a simple algorithm that takes a disordering set of voter preferences and is able to perform some manipulations in order to generate an augmented disordering set of voter preferences with one additional voter or one additional candidate? If not, is there an algorithm to generate a set of m voter preferences of n candidates in an orderly manner, without passing through the constructions above? We might also allow voters to have non-equal power; we see through the example of Figure 3(a) that three voters, with 2, 3, and 4 voters respectively can disorder three candidates. Which pairs (m, n) allow for a weighted disordering? And which pairs (m, n) can be realized with a weighted disordering with three voters with m_1, m_2, and m_3 votes, where $m_1+m_2+m_3 = m$?

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Appendix A. Proof of Theorem 3

In this section, we complete the proof of Theorem 3.

Lemma 15. *There does not exist an $(m, 3)$ disordering for $m \leq 8$.*

Proof. By Lemmas 5 and 6, we need only concern ourselves with the case that there are three candidates c_1, c_2, and c_3, and that there are between four and eight voters. First, notice that if two candidates (say c_1 and c_2) are ranked first by the same number of voters (say k voters), it is not possible for both candidates to win outright. This is because under the scoring rule $(1, x, 0)$, the score for candidate c_1 will be of the form $k + l_1x$ and the score for c_2 will be of the form $k + l_2x$. Either $l_1 = l_2$, in which case both candidates always tie, or l_1 and l_2 are different, in which case one of the candidates always scores higher than the other for all x.

Consider the possible ways in which three candidates can be ranked by m voters so that no two candidates are ranked first the same number of times. If $m = 4$, the partition of first-place votes must be $4 = 3 + 1 + 0$. That is, one candidate must receive three first-place votes, one candidate must receive one first-place vote, and the last candidate must receive zero first-place votes. If $m = 5$, we must have either $5 = 4 + 1 + 0$ or $5 = 3 + 2 + 0$. If $m = 6$, the options are either $6 = 3 + 2 + 1$, $6 = 5 + 1 + 0$, and $6 = 4 + 2 + 0$. When $m = 7$, the partition possibilites are $7 = 4 + 2 + 1$, $7 = 6 + 1 + 0$, $7 = 5 + 2 + 0$, and $7 = 4 + 3 + 0$. Lastly, if $m = 8$, the five possibilites exist, namely $8 = 5 + 2 + 1$, $8 = 4 + 3 + 1$, $8 = 7 + 1 + 0$, $8 = 6 + 2 + 0$, and $8 = 5 + 3 + 0$.

After narrowing to these possibilities of first-place votes, we need to examine the number of second-place votes needed for each candidate in order for the condition of Lemma 2 to hold. For the sake of this discussion, assume that candidate c_1 has the most first-place votes, candidate c_2 has the second most first-place votes, and candidate c_3 has the fewest first-place votes. For Lemma 2 to hold, the number of first- and second-place votes for candidate c_2 must be more than those for candidate c_1. Similarly, the number of first- and second-place votes for candidate c_3 must be more than those for candidate c_2.

Imposing these conditions eliminates all possibilities for $(m, 3)$ disordering pairs. We will verify this claim in three of the above cases; the other seven examples follow from reasoning similar to the first case presented. Consider the case of $7 = 4 + 3 + 0$. That is, 7 voters allot 4 first-place votes to c_1, 3 first-place votes to c_2, and zero first-place votes to c_3. The conditions in the previous paragraph imply that c_2 has at least five first- and second-place votes and that c_3 must receive at least six second-place votes. This is impossible because we know that at least two of the seven second-place votes are alloted to c_2.

The two nontrivial cases are $6 = 3 + 2 + 1$ and $8 = 4 + 3 + 1$. With 6 voters allotting three first-place votes to c_1, two first-place votes to c_2, and one first-place vote to c_3, we see that c_1 may have zero second-place votes, c_2 may have two second-place votes, and c_3 may have four second-place votes. While Lemma 2 holds for every pair of candidates, there is no way for candidate c_2 to win outright—under scoring rule $(1, x, 0)$, the scores for candidates c_1, c_2, and c_3 are 3, $2 + 2x$, and $1 + 4x$, respectively. Candidate c_1 wins for all x in $[0, 0.5)$, candidate c_3 wins for all x in $(0.5, 1]$, and the three candidates tie when $x = 0.5$.

When 8 voters allot four first-place votes to c_1, three first-place votes to c_2, and one first-place vote to c_3, we know that c_2 must receive at least two second-place votes, and c_3 must receive at least five second-place votes. This leaves one second-place vote unallotted. If this last second-place vote is alloted to c_1 or c_2, there is no way for candidate c_3 to win. The only other option is for c_1 to receive zero second-place votes, c_2 to receive two second-place votes, and c_3 to receive the remaining six second-place votes. The exact same setup occurs as in the previous argument, with candidate c_2 never able to win outright.

These techniques, when applied to the other examples, eliminate all possibilities for $(m, 3)$ disordering pairs when $m \leq 8$. □

Lemma 16. *Neither* $(4, 4)$ *nor* $(4, 5)$ *is a disordering pair.*

Proof. We start with four voters and four candidates. We will show that there is only one profile (up to candidate permutation) for which every candidate pair is disordered; it is exactly the profile in Figure 1(a), which is not a disordering.

c_2, c_3, and c_4 to have a non-increasing number of first-place votes. If c_1 has three or more first-place votes, then there must be another candidate with at least as many last-place votes as c_1. This candidate can never score higher than c_1 in any scoring rule. There are three cases that remain. Either c_1 and c_2 each have two first-place votes, or c_1 has two first-place votes and c_2 and c_3 each have one first-place vote, or all candidates each receive one first-place vote.

If c_1 and c_2 each have two first-place votes and are to be disordered, Lemma 1 implies that the candidates' two other rankings are assigned as follows. Up to permutation, c_1 is given a second-place and fourth-place ranking and c_2 is given two third-place rankings. However, in this case, there are three unassigned last-place rankings for candidates c_3 and c_4, so the candidate assigned at least two last-place rankings can never win outright against c_2.

$$
\begin{array}{ccccc}
c_1 & c_1 & c_2 & c_2 & c_3 \\
c_4 & c_3 & z & z & c_1 \\
c_2 & c_2 & z & z & c_4 \\
c_3 & c_4 & c_1 & c_1 & c_2
\end{array}
\qquad
\begin{array}{ccccc}
c_1 & c_1 & c_2 & c_2 & c_3 \\
z & z & z & c_1 & c_4 \\
z & c_2 & z & z & c_2 \\
c_2 & z & c_1 & z & c_1
\end{array}
\qquad
\begin{array}{ccccc}
c_1 & c_1 & c_2 & c_2 & c_3 \\
z & c_4 & z & z & c_1 \\
z & c_2 & z & z & c_2 \\
c_2 & c_3 & c_1 & c_1 & c_4
\end{array}
$$

Figure 6. Visual aid for the proof of the impossibility of a $(4,5)$ disordering profile.

If c_1 has two first-place votes and c_2 and c_3 each have one first-place vote, then unless c_1's other two rankings are last-place votes, c_1 will not be disordered with at least one other candidate. This leaves two remaining last-place votes. Neither c_2, c_3, nor c_4 can be allocated both last-place votes, or else that candidate would always lose against c_1. Also, if c_2 and c_3 each have one last-place vote, then it is impossible that c_2 and c_3 be disordered by placement of their remaining votes. Therefore, assume that c_2 and c_4 are allocated one last-place vote each. The three remaining rankings for c_4 must occur higher than c_2's third-highest ranking, which itself cannot be lower than third-place. Therefore, c_4 has three second-place rankings. So that c_2 and c_3 are disordered, c_2 must occupy the fourth second-place ranking. This gives us the exact profile in Figure 1(a), which is not a disordering profile.

If all candidates have one first-place ranking, then notice that the only way to disorder any pair of candidates is if one of the candidates has at least two third-place rankings. This implies that out of all six pairs of candidates, at least two candidates have two third-place rankings, so exactly two must have exactly two third-place rankings. However, these two candidates could not possibly be disordered, which eliminates this case and proves that no disordering profiles exist with four candidates and four voters.

In the case with five voters and four candidates, we again start by considering the number of first-place votes that a candidate may receive. We continue to use our convention with respect to the number of first-place votes a candidate receives. If c_1 has three or more first-place votes, then if another candidate has as many last-place votes as c_1, that candidate would never win outright against c_1. Therefore c_1 must have exactly three first-place votes, two last-place votes and c_2, c_3, and c_4 would each have one last-place vote. If c_2 has two first-place votes, then c_3 and c_4 cannot be disordered. On the other hand, if c_2 and c_3 each have one first-place vote, then c_2 and c_3 cannot be disordered.

We are then restricted to the case where c_1 has two first-place votes. We again break into two sub-cases, where either c_2 also has two first-place votes and c_3 receives the remaining first-place vote, or where c_2, c_3, and c_4 each receive one first-place vote. In the former subcase, c_1 and c_2 each have three rankings remaining. If there is any chance of c_1 and c_2 being disordered, then (up to permutation) c_1 has at least one second-place ranking, c_2 has at least two third-place rankings, leaving at least two last-place rankings for c_3 and c_4. If either c_3 or c_4 has two last-place rankings, then this candidate will never win outright against candidate c_1. The only profiles that remain to investigate are those in Figure 6.

In each profile, if c_3 and c_4 are to be disordered, there must be more second-place rankings for c_4 than c_3; however, this implies that c_2 and c_3 are not disordered, which eliminates these possibilities.

The final subcase is when c_1 has two first-place votes, and candidates c_2, c_3, and c_4 each have one first-place vote. Our argument now focuses on the last-pace votes each candidate receives. If any of c_2, c_3, or c_4 has three last-place votes, then they cannot win outright against c_1. If c_1 has three last-place votes, then the other two last-place votes are either for the same candidate, in which case the other two candidates cannot be disordered, or for two different candidates, in which case those two candidates cannot be disordered. If c_1 has one last-place vote, then either two other candidates each have two last-place votes, in which case they cannot be disordered, or one candidate has two last-place votes and the remaining two candidates each have one last-place votes. These latter two candidates cannot be disordered with their remaining three votes.

We have eliminated all cases except for the case that c_1 has two last-place votes. If another candidate (say c_2) has two last-place votes, then one of c_3 or c_4 has at least two second-place votes, and would always win against c_2. This leaves the last-place votes to be allocated as c_1 with two and c_2, c_3, and c_4 each with one (just as with the first-place votes). However with so much symmetry, it is not possible for any of the pairs c_2 and c_3, or c_2 and c_4, or c_3 and c_4 to be disordering. This eliminates all profiles with four candidates and five voters as possible disordering profiles and completes the proof of Theorem 3. □

Appendix B. Disordering Profiles and Scoring Rules

In this appendix are examples of splittable disordering profiles for various m and n. For completeness, I also include scoring rules which prove that the profiles are indeed disordering and splittable. The scoring rule labeled $\mathbf{x}_i$ is the rule under which candidate c_i wins outright of the n candidates. The grey row is the insertion of candidate c_{n+1}, and the scoring rule labeled $\mathbf{x}'_i$ is the rule under which candidate c_i wins outright of the $n+1$ candidates in the modified profile.

						$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}'_1$	$\mathbf{x}'_2$	$\mathbf{x}'_3$	$\mathbf{x}'_4$	$\mathbf{x}'_5$
c_3	c_1	c_1	c_2	c_2	c_1	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_2	c_3	c_3	c_3	c_4	0.0	0.8	1.0	1.0	0.0	0.8	1.0	0.8	1.0
c_5	c_5	c_5	c_5	c_5	c_5	–	–	–	–	0.0	0.0	0.7	0.7	1.0
c_4	c_4	c_4	c_4	c_4	c_3	0.0	0.0	0.1	1.0	0.0	0.0	0.7	0.7	1.0
c_1	c_3	c_2	c_1	c_1	c_2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Figure 7. A splittable disordering profile with four candidates and six voters, along with the scoring rules under which each candidate wins. The fifth candidate is inserted in position 3.

							$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}'_1$	$\mathbf{x}'_2$	$\mathbf{x}'_3$	$\mathbf{x}'_4$	$\mathbf{x}'_5$
c_1	c_1	c_1	c_2	c_2	c_3	c_4	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_4	c_3	c_3	c_3	c_4	c_2	c_2	0.0	0.6	1.0	1.0	0.0	0.6	1.0	1.0	1.0
c_5	c_5	c_5	c_5	c_5	c_5	c_5	–	–	–	–	0.0	0.1	0.1	0.9	1.0
c_3	c_4	c_4	c_4	c_3	c_4	c_1	0.0	0.0	0.2	1.0	0.0	0.1	0.1	0.9	0.0
c_2	c_2	c_2	c_1	c_1	c_1	c_3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Figure 8. A splittable disordering profile with four candidates and seven voters, along with the scoring rules under which each candidate wins. The fifth candidate is inserted in position 3.

								$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}'_1$	$\mathbf{x}'_2$	$\mathbf{x}'_3$	$\mathbf{x}'_4$	$\mathbf{x}'_5$
c_1	c_1	c_1	c_1	c_2	c_2	c_2	c_3	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_2	c_3	c_3	c_3	c_3	c_3	c_4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0
c_5	c_5	c_5	c_5	c_5	c_5	c_5	c_5	–	–	–	–	0.0	0.0	0.0	0.0	1.0
c_4	c_4	c_4	c_4	c_4	c_4	c_4	c_1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
c_3	c_3	c_2	c_2	c_1	c_1	c_1	c_2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FIGURE 9. A splittable disordering profile with four candidates and eight voters, along with the scoring rules under which each candidate wins. The fifth candidate is inserted in position 3.

				$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}'_1$	$\mathbf{x}'_2$	$\mathbf{x}'_3$	$\mathbf{x}'_4$	$\mathbf{x}'_5$	$\mathbf{x}'_6$
c_1	c_1	c_5	c_4	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_3	c_2	c_2	0.0	1.0	1.0	0.9	0.9	0.0	0.0	0.0	0.0	0.0	1.0
c_3	c_5	c_3	c_5	0.0	0.0	1.0	0.9	0.9	0.0	0.0	0.0	0.0	0.0	1.0
c_6	c_6	c_6	c_6	–	–	–	–	–	0.0	0.0	0.0	0.0	0.0	1.0
c_4	c_4	c_4	c_3	0.0	0.0	0.1	0.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0
c_5	c_2	c_1	c_1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FIGURE 10. A splittable disordering profile with five candidates and four voters, along with the scoring rules under which each candidate wins. The sixth candidate is inserted in position 4.

					$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}'_1$	$\mathbf{x}'_2$	$\mathbf{x}'_3$	$\mathbf{x}'_4$	$\mathbf{x}'_5$	$\mathbf{x}'_6$
c_4	c_1	c_3	c_1	c_5	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_2	c_2	c_5	c_3	0.0	1.0	1.0	1.0	1.0	0.0	1.0	1.0	1.0	1.0	1.0
c_3	c_4	c_4	c_3	c_1	0.0	0.0	1.0	1.0	0.9	0.0	0.0	1.0	1.0	0.9	1.0
c_6	c_6	c_6	c_6	c_6	–	–	–	–	–	0.0	0.0	0.0	0.9	0.9	1.0
c_5	c_5	c_5	c_4	c_4	0.0	0.0	0.0	0.9	0.9	0.0	0.0	0.0	0.9	0.9	0.0
c_1	c_3	c_1	c_2	c_2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FIGURE 11. A splittable disordering profile with five candidates and five voters, along with the scoring rules under which each candidate wins. The sixth candidate is inserted in position 4.

				$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}'_1$	$\mathbf{x}'_2$	$\mathbf{x}'_3$	$\mathbf{x}'_4$	$\mathbf{x}'_5$	$\mathbf{x}'_6$	$\mathbf{x}'_7$
c_6	c_1	c_3	c_1	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_5	c_5	c_2	0.0	1.0	1.0	1.0	1.0	0.6	0.0	1.0	1.0	1.0	1.0	0.6	1.0
c_3	c_3	c_4	c_4	0.0	0.2	1.0	1.0	0.6	0.6	0.0	0.2	1.0	1.0	0.6	0.6	1.0
c_7	c_7	c_7	c_7	–	–	–	–	–	–	0.0	0.2	0.0	0.6	0.6	0.6	1.0
c_4	c_4	c_2	c_6	0.0	0.2	0.0	0.6	0.6	0.6	0.0	0.2	0.0	0.6	0.6	0.6	0.0
c_5	c_6	c_6	c_5	0.0	0.0	0.0	0.5	0.5	0.5	0.0	0.0	0.0	0.5	0.5	0.5	0.0
c_1	c_2	c_1	c_3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FIGURE 12. A splittable disordering profile with six candidates and four voters, along with the scoring rules under which each candidate wins. The seventh candidate is inserted in position 4.

					$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}'_1$	$\mathbf{x}'_2$	$\mathbf{x}'_3$	$\mathbf{x}'_4$	$\mathbf{x}'_5$	$\mathbf{x}'_6$	$\mathbf{x}'_7$
c_1	c_1	c_2	c_3	c_4	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_5	c_3	c_2	c_5	0.0	1.0	1.0	0.6	1.0	1.0	0.0	1.0	1.0	0.6	1.0	1.0	1.0
c_6	c_3	c_6	c_6	c_3	0.0	0.0	1.0	0.6	0.6	1.0	0.0	0.0	1.0	0.6	0.6	1.0	1.0
c_7	c_7	c_7	c_7	c_7	–	–	–	–	–	–	0.0	0.0	0.0	0.6	0.6	0.6	1.0
c_5	c_4	c_4	c_4	c_1	0.0	0.0	0.0	0.6	0.6	0.6	0.0	0.0	0.0	0.6	0.6	0.6	0.0
c_4	c_6	c_5	c_5	c_6	0.0	0.0	0.0	0.6	0.6	0.6	0.0	0.0	0.0	0.6	0.6	0.6	0.0
c_3	c_2	c_1	c_1	c_2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FIGURE 13. A splittable disordering profile with six candidates and five voters, along with the scoring rules under which each candidate wins. The seventh candidate is inserted in position 4.

						$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}'_1$	$\mathbf{x}'_2$	$\mathbf{x}'_3$	$\mathbf{x}'_4$	$\mathbf{x}'_5$	$\mathbf{x}'_6$	$\mathbf{x}'_7$
c_1	c_1	c_1	c_2	c_3	c_4	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_2	c_5	c_5	c_5	c_6	0.0	1.0	0.6	0.9	1.0	1.0	0.0	1.0	0.6	0.9	1.0	1.0	1.0
c_3	c_6	c_6	c_6	c_6	c_5	0.0	0.6	0.6	0.9	0.4	1.0	0.0	0.6	0.6	0.9	0.4	1.0	1.0
c_4	c_4	c_2	c_4	c_4	c_1	0.0	0.6	0.6	0.9	0.0	0.0	0.0	0.6	0.6	0.9	0.0	0.0	1.0
c_7	c_7	c_7	c_7	c_7	c_7	–	–	–	–	–	–	0.0	0.6	0.6	0.0	0.0	0.0	1.0
c_6	c_3	c_3	c_3	c_2	c_3	0.0	0.6	0.6	0.0	0.0	0.0	0.0	0.6	0.6	0.0	0.0	0.0	1.0
c_5	c_5	c_4	c_1	c_1	c_2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FIGURE 14. A splittable disordering profile with six candidates and six voters, along with the scoring rules under which each candidate wins. The seventh candidate is inserted in position 5.

							$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$
c_1	c_1	c_1	c_2	c_2	c_3	c_5	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_2	c_3	c_3	c_4	c_4	c_6	0.0	1.0	1.0	1.0	0.9	1.0
c_5	c_3	c_4	c_4	c_6	c_6	c_3	0.0	0.0	1.0	1.0	0.9	1.0
c_4	c_6	c_5	c_5	c_5	c_1	c_4	0.0	0.0	0.0	1.0	0.9	0.9
c_6	c_5	c_6	c_6	c_3	c_5	c_2	0.0	0.0	0.0	0.0	0.9	0.9
c_3	c_4	c_2	c_1	c_1	c_2	c_1	0.0	0.0	0.0	0.0	0.0	0.0

							$\mathbf{x}'_1$	$\mathbf{x}'_2$	$\mathbf{x}'_3$	$\mathbf{x}'_4$	$\mathbf{x}'_5$	$\mathbf{x}'_6$	$\mathbf{x}'_7$
c_1	c_1	c_1	c_2	c_2	c_3	c_5	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_2	c_3	c_3	c_4	c_4	c_6	0.0	1.0	1.0	1.0	0.9	1.0	1.0
c_5	c_3	c_4	c_4	c_6	c_6	c_3	0.0	0.0	1.0	1.0	0.9	1.0	1.0
c_7	c_7	c_7	c_7	c_7	c_7	c_7	0.0	0.0	0.0	0.7	0.9	0.9	1.0
c_4	c_6	c_5	c_5	c_5	c_1	c_4	0.0	0.0	0.0	0.7	0.9	0.9	0.0
c_6	c_5	c_6	c_6	c_3	c_5	c_2	0.0	0.0	0.0	0.0	0.9	0.9	0.0
c_3	c_4	c_2	c_1	c_1	c_2	c_1	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FIGURE 15. A splittable disordering profile with six candidates and seven voters, along with the scoring rules under which each candidate wins. The seventh candidate is inserted in position 4.

								$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$
c_1	c_1	c_1	c_1	c_2	c_2	c_2	c_3	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_4	c_4	c_4	c_5	c_5	c_6	c_2	0.0	1.0	0.9	1.0	1.0	1.0
c_5	c_6	c_6	c_6	c_6	c_6	c_5	c_5	0.0	0.0	0.9	0.9	0.6	1.0
c_4	c_3	c_3	c_3	c_3	c_3	c_4	c_4	0.0	0.0	0.9	0.9	0.6	0.0
c_6	c_5	c_5	c_5	c_1	c_4	c_3	c_6	0.0	0.0	0.0	0.0	0.6	0.0
c_3	c_2	c_2	c_2	c_4	c_1	c_1	c_1	0.0	0.0	0.0	0.0	0.0	0.0

								$\mathbf{x}'_1$	$\mathbf{x}'_2$	$\mathbf{x}'_3$	$\mathbf{x}'_4$	$\mathbf{x}'_5$	$\mathbf{x}'_6$	$\mathbf{x}'_7$
c_1	c_1	c_1	c_1	c_2	c_2	c_2	c_3	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_4	c_4	c_4	c_5	c_5	c_6	c_2	0.0	1.0	0.9	1.0	1.0	1.0	1.0
c_5	c_6	c_6	c_6	c_6	c_6	c_5	c_5	0.0	0.0	0.9	0.9	0.6	1.0	1.0
c_4	c_3	c_3	c_3	c_3	c_3	c_4	c_4	0.0	0.0	0.9	0.9	0.6	0.0	1.0
c_7	c_7	c_7	c_7	c_7	c_7	c_7	c_7	0.0	0.0	0.0	0.0	0.6	0.0	1.0
c_6	c_5	c_5	c_5	c_1	c_4	c_3	c_6	0.0	0.0	0.0	0.0	0.6	0.0	0.0
c_3	c_2	c_2	c_2	c_4	c_1	c_1	c_1	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FIGURE 16. A splittable disordering profile with six candidates and eight voters, along with the scoring rules under which each candidate wins. The seventh candidate is inserted in position 5.

				$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_7$	$\mathbf{x}_8$
c_1	c_1	c_2	c_3	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_4	c_2	c_5	c_6	0.0	0.8	1.0	0.9	0.7	0.7	1.0	0.7
c_3	c_7	c_3	c_8	0.0	0.5	1.0	0.62	0.6	0.6	1.0	0.7
c_5	c_8	c_8	c_2	0.0	0.0	0.0	0.62	0.6	0.6	1.0	0.7
c_7	c_4	c_7	c_7	0.0	0.0	0.0	0.62	0.55	0.6	1.0	0.4
c_6	c_6	c_6	c_4	0.0	0.0	0.0	0.55	0.55	0.6	0.0	0.4
c_8	c_5	c_4	c_5	0.0	0.0	0.0	0.5	0.55	0.0	0.0	0.4
c_2	c_3	c_1	c_1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

				$\mathbf{x}'_1$	$\mathbf{x}'_2$	$\mathbf{x}'_3$	$\mathbf{x}'_4$	$\mathbf{x}'_5$	$\mathbf{x}'_6$	$\mathbf{x}'_7$	$\mathbf{x}'_8$	$\mathbf{x}'_9$
c_1	c_1	c_2	c_3	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_4	c_2	c_5	c_6	0.0	0.8	1.0	0.9	0.7	0.7	1.0	0.7	1.0
c_3	c_7	c_3	c_8	0.0	0.5	1.0	0.62	0.6	0.6	1.0	0.7	1.0
c_5	c_8	c_8	c_2	0.0	0.0	0.0	0.62	0.6	0.6	0.9	0.7	1.0
c_9	c_9	c_9	c_9	0.0	0.0	0.0	0.62	0.55	0.6	0.9	0.4	1.0
c_7	c_4	c_7	c_7	0.0	0.0	0.0	0.62	0.55	0.6	0.9	0.4	0.0
c_6	c_6	c_6	c_4	0.0	0.0	0.0	0.55	0.55	0.6	0.0	0.4	0.0
c_8	c_5	c_4	c_5	0.0	0.0	0.0	0.5	0.55	0.0	0.0	0.4	0.0
c_2	c_3	c_1	c_1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FIGURE 17. A splittable disordering profile with eight candidates and four voters, along with the scoring rules under which each candidate wins. The ninth candidate is inserted in position 5.

					$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_7$	$\mathbf{x}_8$
c_1	c_1	c_4	c_2	c_3	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_5	c_2	c_6	c_5	0.0	0.8	0.7	0.4	0.9	0.7	1.0	1.0
c_3	c_3	c_8	c_8	c_8	0.0	0.5	0.6	0.4	0.62	0.6	1.0	1.0
c_7	c_7	c_7	c_4	c_7	0.0	0.5	0.6	0.4	0.62	0.6	1.0	0.5
c_6	c_6	c_5	c_7	c_6	0.0	0.0	0.55	0.35	0.62	0.6	0.0	0.5
c_8	c_4	c_6	c_5	c_4	0.0	0.0	0.55	0.35	0.55	0.6	0.0	0.5
c_5	c_8	c_3	c_3	c_2	0.0	0.0	0.55	0.0	0.5	0.0	0.0	0.0
c_4	c_2	c_1	c_1	c_1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

					$\mathbf{x}'_1$	$\mathbf{x}'_2$	$\mathbf{x}'_3$	$\mathbf{x}'_4$	$\mathbf{x}'_5$	$\mathbf{x}'_6$	$\mathbf{x}'_7$	$\mathbf{x}'_8$	$\mathbf{x}'_9$
c_1	c_1	c_4	c_2	c_3	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_5	c_2	c_6	c_5	0.0	0.8	0.7	0.4	0.9	0.7	1.0	1.0	1.0
c_3	c_3	c_8	c_8	c_8	0.0	0.5	0.6	0.4	0.62	0.6	1.0	1.0	1.0
c_7	c_7	c_7	c_4	c_7	0.0	0.5	0.6	0.4	0.62	0.6	1.0	0.5	1.0
c_9	c_9	c_9	c_9	c_9	0.0	0.0	0.55	0.35	0.62	0.6	0.0	0.5	1.0
c_6	c_6	c_5	c_7	c_6	0.0	0.0	0.55	0.35	0.62	0.6	0.0	0.5	0.0
c_8	c_4	c_6	c_5	c_4	0.0	0.0	0.55	0.35	0.55	0.6	0.0	0.5	0.0
c_5	c_8	c_3	c_3	c_2	0.0	0.0	0.55	0.0	0.5	0.0	0.0	0.0	0.0
c_4	c_2	c_1	c_1	c_1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FIGURE 18. A splittable disordering profile with eight candidates and five voters, along with the scoring rules under which each candidate wins. The ninth candidate is inserted in position 5.

			$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_7$	$\mathbf{x}_8$	$\mathbf{x}_9$
c_1	c_9	c_8	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_6	c_7	0.9	0.9	1.0	0.9	0.9	1.0	1.0	0.3	0.3
c_5	c_3	c_3	0.9	0.8	1.0	0.9	0.9	0.3	0.3	0.3	0.3
c_4	c_4	c_2	0.9	0.8	0.0	0.9	0.9	0.3	0.3	0.3	0.3
c_7	c_1	c_5	0.9	0.0	0.0	0.7	0.7	0.3	0.3	0.3	0.3
c_6	c_8	c_6	0.0	0.0	0.0	0.7	0.5	0.3	0.25	0.3	0.3
c_9	c_5	c_9	0.0	0.0	0.0	0.7	0.5	0.0	0.25	0.25	0.3
c_8	c_7	c_4	0.0	0.0	0.0	0.7	0.0	0.0	0.25	0.25	0.0
c_3	c_2	c_1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

			$\mathbf{x}'_1$	$\mathbf{x}'_2$	$\mathbf{x}'_3$	$\mathbf{x}'_4$	$\mathbf{x}'_5$	$\mathbf{x}'_6$	$\mathbf{x}'_7$	$\mathbf{x}'_8$	$\mathbf{x}'_9$	$\mathbf{x}'_{10}$
c_1	c_9	c_8	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_6	c_7	0.9	0.9	1.0	0.9	0.9	1.0	1.0	0.3	0.3	1.0
c_5	c_3	c_3	0.9	0.8	1.0	0.9	0.9	0.3	0.3	0.3	0.3	1.0
c_4	c_4	c_2	0.9	0.8	0.0	0.9	0.9	0.3	0.3	0.3	0.3	1.0
c_7	c_1	c_5	0.9	0.0	0.0	0.7	0.7	0.3	0.3	0.3	0.3	1.0
c_{10}	c_{10}	c_{10}	0.0	0.0	0.0	0.7	0.5	0.3	0.25	0.3	0.3	1.0
c_6	c_8	c_6	0.0	0.0	0.0	0.7	0.5	0.3	0.25	0.3	0.3	0.0
c_9	c_5	c_9	0.0	0.0	0.0	0.7	0.5	0.0	0.25	0.25	0.3	0.0
c_8	c_7	c_4	0.0	0.0	0.0	0.7	0.0	0.0	0.25	0.25	0.0	0.0
c_3	c_2	c_1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FIGURE 19. A splittable disordering profile with nine candidates and three voters, along with the scoring rules under which each candidate wins. The tenth candidate is inserted in position 6.

			$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_7$	$\mathbf{x}_8$	$\mathbf{x}_9$	$\mathbf{x}_{10}$
c_1	c_4	c_5	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_7	c_2	c_6	0.9	0.9	1.0	0.3	0.3	1.0	1.0	0.9	0.9	0.9
c_8	c_3	c_3	0.9	0.8	1.0	0.3	0.3	0.3	0.3	0.9	0.9	0.9
c_2	c_{10}	c_9	0.9	0.8	0.0	0.3	0.3	0.3	0.3	0.7	0.9	0.9
c_9	c_8	c_1	0.9	0.0	0.0	0.3	0.3	0.3	0.3	0.7	0.7	0.7
c_{10}	c_6	c_{10}	0.0	0.0	0.0	0.3	0.3	0.3	0.3	0.5	0.5	0.7
c_5	c_7	c_7	0.0	0.0	0.0	0.3	0.3	0.25	0.3	0.5	0.5	0.0
c_4	c_9	c_4	0.0	0.0	0.0	0.25	0.25	0.25	0.0	0.5	0.5	0.0
c_6	c_5	c_8	0.0	0.0	0.0	0.0	0.25	0.25	0.0	0.5	0.0	0.0
c_3	c_1	c_2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

			$\mathbf{x}'_1$	$\mathbf{x}'_2$	$\mathbf{x}'_3$	$\mathbf{x}'_4$	$\mathbf{x}'_5$	$\mathbf{x}'_6$	$\mathbf{x}'_7$	$\mathbf{x}'_8$	$\mathbf{x}'_9$	$\mathbf{x}'_{10}$	$\mathbf{x}'_{11}$
c_1	c_4	c_5	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_7	c_2	c_6	0.9	0.9	1.0	0.3	0.3	1.0	1.0	0.9	0.9	0.9	1.0
c_8	c_3	c_3	0.9	0.8	1.0	0.3	0.3	0.3	0.3	0.9	0.9	0.9	1.0
c_2	c_{10}	c_9	0.9	0.8	0.0	0.3	0.3	0.3	0.3	0.7	0.9	0.9	1.0
c_9	c_8	c_1	0.9	0.0	0.0	0.3	0.3	0.3	0.3	0.7	0.7	0.7	1.0
c_{11}	c_{11}	c_{11}	0.0	0.0	0.0	0.3	0.3	0.3	0.3	0.5	0.5	0.7	1.0
c_{10}	c_6	c_{10}	0.0	0.0	0.0	0.3	0.3	0.3	0.3	0.5	0.5	0.7	0.0
c_5	c_7	c_7	0.0	0.0	0.0	0.3	0.3	0.25	0.3	0.5	0.5	0.0	0.0
c_4	c_9	c_4	0.0	0.0	0.0	0.25	0.25	0.25	0.0	0.5	0.5	0.0	0.0
c_6	c_5	c_8	0.0	0.0	0.0	0.0	0.25	0.25	0.0	0.5	0.0	0.0	0.0
c_3	c_1	c_2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FIGURE 20. A splittable disordering profile with ten candidates and three voters, along with the scoring rules under which each candidate wins. The eleventh candidate is inserted in position 6.

			$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_7$	$\mathbf{x}_8$	$\mathbf{x}_9$	$\mathbf{x}_{10}$	$\mathbf{x}_{11}$	$\mathbf{x}_{12}$
c_4	c_1	c_5	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_6	c_7	0.9	0.9	1.0	0.3	0.3	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_3	c_8	c_9	0.9	0.8	1.0	0.3	0.3	0.3	0.3	1.0	1.0	1.0	1.0	1.0
c_{10}	c_3	c_{11}	0.9	0.8	1.0	0.3	0.3	0.3	0.3	0.3	0.3	1.0	1.0	1.0
c_{12}	c_{12}	c_2	0.9	0.8	0.0	0.3	0.3	0.3	0.3	0.3	0.3	1.0	1.0	1.0
c_8	c_{11}	c_1	0.9	0.0	0.0	0.3	0.3	0.3	0.3	0.3	0.3	0.8	1.0	0.7
c_9	c_{10}	c_{10}	0.0	0.0	0.0	0.3	0.3	0.3	0.3	0.25	0.25	0.8	0.8	0.7
c_{11}	c_9	c_6	0.0	0.0	0.0	0.3	0.3	0.3	0.3	0.25	0.25	0.0	0.8	0.7
c_7	c_7	c_4	0.0	0.0	0.0	0.3	0.3	0.25	0.3	0.25	0.0	0.0	0.0	0.7
c_5	c_5	c_8	0.0	0.0	0.0	0.25	0.3	0.25	0.0	0.25	0.0	0.0	0.0	0.7
c_6	c_4	c_{12}	0.0	0.0	0.0	0.25	0.0	0.25	0.0	0.0	0.0	0.0	0.0	0.7
c_1	c_2	c_3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

			$\mathbf{x}'_1$	$\mathbf{x}'_2$	$\mathbf{x}'_3$	$\mathbf{x}'_4$	$\mathbf{x}'_5$	$\mathbf{x}'_6$	$\mathbf{x}'_7$	$\mathbf{x}'_8$	$\mathbf{x}'_9$	$\mathbf{x}'_{10}$	$\mathbf{x}'_{11}$	$\mathbf{x}'_{12}$	$\mathbf{x}'_{13}$
c_4	c_1	c_5	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_6	c_7	0.3	1.0	1.0	0.3	0.3	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_3	c_8	c_9	0.3	0.3	1.0	0.3	0.3	0.3	0.3	1.0	1.0	1.0	1.0	1.0	1.0
c_{10}	c_3	c_{11}	0.3	0.3	1.0	0.3	0.3	0.3	0.3	0.3	0.3	1.0	1.0	1.0	1.0
c_{12}	c_{12}	c_2	0.3	0.3	0.0	0.3	0.3	0.3	0.3	0.3	0.3	0.8	0.8	1.0	1.0
c_{13}	c_{13}	c_{13}	0.3	0.0	0.0	0.3	0.3	0.3	0.3	0.3	0.3	0.8	0.8	0.7	1.0
c_8	c_{11}	c_1	0.3	0.0	0.0	0.3	0.3	0.3	0.3	0.3	0.3	0.8	0.8	0.7	0.0
c_9	c_{10}	c_{10}	0.0	0.0	0.0	0.3	0.3	0.3	0.3	0.25	0.25	0.8	0.65	0.7	0.0
c_{11}	c_9	c_6	0.0	0.0	0.0	0.3	0.3	0.3	0.3	0.25	0.25	0.0	0.65	0.7	0.0
c_7	c_7	c_4	0.0	0.0	0.0	0.3	0.3	0.25	0.3	0.25	0.0	0.0	0.0	0.7	0.0
c_5	c_5	c_8	0.0	0.0	0.0	0.25	0.3	0.25	0.0	0.25	0.0	0.0	0.0	0.7	0.0
c_6	c_4	c_{12}	0.0	0.0	0.0	0.25	0.0	0.25	0.0	0.0	0.0	0.0	0.0	0.7	0.0
c_1	c_2	c_3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FIGURE 21. A splittable disordering profile with twelve candidates and three voters, along with the scoring rules under which each candidate wins. The thirteenth candidate is inserted in position 6.

			$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_7$	$\mathbf{x}_8$	$\mathbf{x}_9$	$\mathbf{x}_{10}$	$\mathbf{x}_{11}$	$\mathbf{x}_{12}$	$\mathbf{x}_{13}$	$\mathbf{x}_{14}$
c_1	c_4	c_5	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_6	c_7	0.1	1.0	1.0	0.5	0.7	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.9	0.9
c_8	c_3	c_3	0.1	0.1	1.0	0.5	0.7	0.5	0.4	1.0	1.0	1.0	1.0	1.0	0.9	0.9
c_9	c_2	c_{10}	0.1	0.1	0.0	0.5	0.7	0.5	0.4	0.8	1.0	1.0	1.0	1.0	0.9	0.9
c_{11}	c_{12}	c_1	0.1	0.0	0.0	0.5	0.7	0.5	0.4	0.8	0.8	0.8	1.0	1.0	0.9	0.9
c_{13}	c_{14}	c_{14}	0.0	0.0	0.0	0.5	0.7	0.5	0.4	0.8	0.8	0.8	0.8	0.7	0.9	0.9
c_{10}	c_{13}	c_{11}	0.0	0.0	0.0	0.5	0.7	0.5	0.4	0.8	0.8	0.8	0.8	0.7	0.9	0.2
c_6	c_8	c_{13}	0.0	0.0	0.0	0.5	0.7	0.5	0.4	0.8	0.8	0.7	0.7	0.7	0.9	0.2
c_{12}	c_9	c_4	0.0	0.0	0.0	0.5	0.7	0.4	0.4	0.7	0.8	0.7	0.7	0.7	0.0	0.2
c_7	c_7	c_{12}	0.0	0.0	0.0	0.4	0.7	0.4	0.4	0.7	0.7	0.7	0.7	0.6	0.0	0.2
c_5	c_5	c_9	0.0	0.0	0.0	0.4	0.7	0.4	0.0	0.7	0.7	0.7	0.7	0.0	0.0	0.2
c_{14}	c_{11}	c_8	0.0	0.0	0.0	0.4	0.0	0.4	0.0	0.7	0.0	0.7	0.7	0.0	0.0	0.2
c_4	c_{10}	c_6	0.0	0.0	0.0	0.4	0.0	0.4	0.0	0.0	0.0	0.7	0.0	0.0	0.0	0.0
c_3	c_1	c_2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

			$\mathbf{x}'_1$	$\mathbf{x}'_2$	$\mathbf{x}'_3$	$\mathbf{x}'_4$	$\mathbf{x}'_5$	$\mathbf{x}'_6$	$\mathbf{x}'_7$	$\mathbf{x}'_8$	$\mathbf{x}'_9$	$\mathbf{x}'_{10}$	$\mathbf{x}'_{11}$	$\mathbf{x}'_{12}$	$\mathbf{x}'_{13}$	$\mathbf{x}'_{14}$	$\mathbf{x}'_{15}$
c_1	c_4	c_5	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_2	c_6	c_7	0.1	1.0	1.0	0.5	0.7	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.9	0.9	1.0
c_8	c_3	c_3	0.1	0.1	1.0	0.5	0.7	0.5	0.4	1.0	1.0	1.0	1.0	1.0	0.9	0.9	1.0
c_9	c_2	c_{10}	0.1	0.1	0.0	0.5	0.7	0.5	0.4	0.8	1.0	1.0	1.0	1.0	0.9	0.9	1.0
c_{11}	c_{12}	c_1	0.1	0.0	0.0	0.5	0.7	0.5	0.4	0.8	0.8	0.8	1.0	1.0	0.9	0.9	1.0
c_{13}	c_{14}	c_{14}	0.0	0.0	0.0	0.5	0.7	0.5	0.4	0.8	0.8	0.8	0.8	0.7	0.9	0.9	1.0
c_{10}	c_{13}	c_{11}	0.0	0.0	0.0	0.5	0.7	0.5	0.4	0.8	0.8	0.8	0.8	0.7	0.9	0.2	1.0
c_{15}	c_{15}	c_{15}	0.0	0.0	0.0	0.5	0.7	0.5	0.4	0.8	0.8	0.7	0.7	0.7	0.8	0.2	1.0
c_6	c_8	c_{13}	0.0	0.0	0.0	0.5	0.7	0.5	0.4	0.8	0.8	0.7	0.7	0.7	0.8	0.2	0.0
c_{12}	c_9	c_4	0.0	0.0	0.0	0.5	0.7	0.4	0.4	0.7	0.8	0.7	0.7	0.7	0.0	0.2	0.0
c_7	c_7	c_{12}	0.0	0.0	0.0	0.4	0.7	0.4	0.4	0.7	0.7	0.7	0.7	0.6	0.0	0.2	0.0
c_5	c_5	c_9	0.0	0.0	0.0	0.4	0.7	0.4	0.0	0.7	0.7	0.7	0.7	0.0	0.0	0.2	0.0
c_{14}	c_{11}	c_8	0.0	0.0	0.0	0.4	0.0	0.4	0.0	0.7	0.0	0.7	0.7	0.0	0.0	0.2	0.0
c_4	c_{10}	c_6	0.0	0.0	0.0	0.4	0.0	0.4	0.0	0.0	0.0	0.7	0.0	0.0	0.0	0.0	0.0
c_3	c_1	c_2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FIGURE 22. A splittable disordering profile with fourteen candidates and three voters, along with the scoring rules under which each candidate wins. The fifteenth candidate is inserted in position 8.

			$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_7$	$\mathbf{x}_8$	$\mathbf{x}_9$	$\mathbf{x}_{10}$	$\mathbf{x}_{11}$	$\mathbf{x}_{12}$	$\mathbf{x}_{13}$	$\mathbf{x}_{14}$	$\mathbf{x}_{15}$	$\mathbf{x}_{16}$
c_1	c_4	c_5	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_6	c_2	c_7	0.2	1.0	1.0	0.5	0.5	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.8	0.7	0.9	0.9
c_3	c_8	c_9	0.2	0.2	1.0	0.5	0.5	0.7	0.4	1.0	1.0	1.0	1.0	1.0	0.8	0.7	0.9	0.9
c_{10}	c_3	c_{11}	0.2	0.2	1.0	0.5	0.5	0.7	0.4	0.5	0.8	1.0	1.0	1.0	0.8	0.7	0.9	0.9
c_{13}	c_{14}	c_2	0.2	0.2	0.0	0.5	0.5	0.7	0.4	0.5	0.8	0.8	0.8	1.0	0.8	0.7	0.9	0.9
c_{15}	c_{12}	c_1	0.2	0.0	0.0	0.5	0.5	0.7	0.4	0.5	0.8	0.8	0.8	1.0	0.6	0.5	0.9	0.9
c_{16}	c_{15}	c_{13}	0.0	0.0	0.0	0.5	0.5	0.7	0.4	0.5	0.8	0.8	0.8	0.8	0.6	0.5	0.9	0.9
c_{14}	c_{16}	c_{16}	0.0	0.0	0.0	0.5	0.5	0.7	0.4	0.5	0.8	0.8	0.8	0.8	0.3	0.5	0.2	0.8
c_8	c_{11}	c_{12}	0.0	0.0	0.0	0.5	0.5	0.7	0.4	0.5	0.8	0.8	0.8	0.8	0.3	0.4	0.2	0.0
c_9	c_{10}	c_{10}	0.0	0.0	0.0	0.5	0.5	0.7	0.4	0.1	0.8	0.8	0.7	0.7	0.3	0.4	0.2	0.0
c_{12}	c_6	c_4	0.0	0.0	0.0	0.5	0.5	0.7	0.4	0.1	0.7	0.0	0.7	0.7	0.3	0.4	0.2	0.0
c_7	c_7	c_{14}	0.0	0.0	0.0	0.4	0.5	0.6	0.4	0.1	0.7	0.0	0.7	0.0	0.3	0.4	0.2	0.0
c_{11}	c_9	c_6	0.0	0.0	0.0	0.4	0.5	0.6	0.0	0.1	0.7	0.0	0.7	0.0	0.3	0.0	0.2	0.0
c_5	c_5	c_{15}	0.0	0.0	0.0	0.4	0.5	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.3	0.0	0.2	0.0
c_4	c_{13}	c_8	0.0	0.0	0.0	0.4	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.3	0.0	0.0	0.0
c_2	c_1	c_3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

			$\mathbf{x}'_1$	$\mathbf{x}'_2$	$\mathbf{x}'_3$	$\mathbf{x}'_4$	$\mathbf{x}'_5$	$\mathbf{x}'_6$	$\mathbf{x}'_7$	$\mathbf{x}'_8$	$\mathbf{x}'_9$	$\mathbf{x}'_{10}$	$\mathbf{x}'_{11}$	$\mathbf{x}'_{12}$	$\mathbf{x}'_{13}$	$\mathbf{x}'_{14}$	$\mathbf{x}'_{15}$	$\mathbf{x}'_{16}$	$\mathbf{x}'_{17}$
c_1	c_4	c_5	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
c_6	c_2	c_7	0.2	1.0	1.0	0.5	0.5	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.8	0.7	0.9	0.9	1.0
c_3	c_8	c_9	0.2	0.2	1.0	0.5	0.5	0.7	0.4	1.0	1.0	1.0	1.0	1.0	0.8	0.7	0.9	0.9	1.0
c_{10}	c_3	c_{11}	0.2	0.2	1.0	0.5	0.5	0.7	0.4	0.5	0.8	1.0	1.0	1.0	0.8	0.7	0.9	0.9	1.0
c_{13}	c_{14}	c_2	0.2	0.2	0.0	0.5	0.5	0.7	0.4	0.5	0.8	0.8	0.8	1.0	0.8	0.7	0.9	0.9	1.0
c_{15}	c_{12}	c_1	0.2	0.0	0.0	0.5	0.5	0.7	0.4	0.5	0.8	0.8	0.8	1.0	0.6	0.5	0.9	0.9	1.0
c_{16}	c_{15}	c_{13}	0.0	0.0	0.0	0.5	0.5	0.7	0.4	0.5	0.8	0.8	0.8	0.8	0.6	0.5	0.9	0.9	1.0
c_{17}	c_{17}	c_{17}	0.0	0.0	0.0	0.5	0.5	0.7	0.4	0.5	0.8	0.8	0.8	0.8	0.3	0.5	0.2	0.8	1.0
c_{14}	c_{16}	c_{16}	0.0	0.0	0.0	0.5	0.5	0.7	0.4	0.5	0.8	0.8	0.8	0.8	0.3	0.5	0.2	0.8	0.0
c_8	c_{11}	c_{12}	0.0	0.0	0.0	0.5	0.5	0.7	0.4	0.5	0.8	0.8	0.8	0.8	0.3	0.4	0.2	0.0	0.0
c_9	c_{10}	c_{10}	0.0	0.0	0.0	0.5	0.5	0.7	0.4	0.1	0.8	0.8	0.7	0.7	0.3	0.4	0.2	0.0	0.0
c_{12}	c_6	c_4	0.0	0.0	0.0	0.5	0.5	0.7	0.4	0.1	0.7	0.0	0.7	0.7	0.3	0.4	0.2	0.0	0.0
c_7	c_7	c_{14}	0.0	0.0	0.0	0.4	0.5	0.6	0.4	0.1	0.7	0.0	0.7	0.0	0.3	0.4	0.2	0.0	0.0
c_{11}	c_9	c_6	0.0	0.0	0.0	0.4	0.5	0.6	0.0	0.1	0.7	0.0	0.7	0.0	0.3	0.0	0.2	0.0	0.0
c_5	c_5	c_{15}	0.0	0.0	0.0	0.4	0.5	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.3	0.0	0.2	0.0	0.0
c_4	c_{13}	c_8	0.0	0.0	0.0	0.4	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.3	0.0	0.0	0.0	0.0
c_2	c_1	c_3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

FIGURE 23. A splittable disordering profile with sixteen candidates and three voters, along with the scoring rules under which each candidate wins. The seventeenth candidate is inserted in position 8.

Appendix C. Sample Maple Calculations

We include a representative sample of the Maple code used to verify the claims in this article. The following Maple code shows that the 3-voter 9-candidate candidate preference lists given in Figures 3(d) and 19 is indeed disordering and splittable between preferences 5 and 6. The variable n represents the number of candidates and m represents the number of voters. After calculating the scoring rule generated by the preferences, we verify that n different scoring rules elect a different candidate. Following this, we insert a new candidate (c_{10}) in position 6, and generate $n+1$ new rules. Often, this is simply a slight modification of the previously defined rules. A copy of the entire Maple file is available directly from the author, or from the author's website, http://qcpages.qc.edu/˜chanusa/papers.html.

```
n,m := 9,3
for j from 1 to n do f[j]:=0; od:
for i from 1 to m do
  for j from 1 to n do
    f[op(M[j,i])]:=f[op(M[j,i])]+x[j];
od; od;
ScoringRule:=[seq(f[j],j=1..n)];
```

$$\mathit{ScoringRule} := [x_1 + x_5 + x_9, x_9 + x_2 + x_4, 2\,x_3 + x_9, x_8 + 2\,x_4, x_3 + x_7 + x_5, x_2 + 2\,x_6, x_5 + x_8 + x_2, x_8 + x_6 + x_1, 2\,x_7 + x_1]$$

```
k:=0.9:
ThisRule:=[1,k,k,k,k,0,0,0,0];
FindMax(subs({seq(x[j]=ThisRule[j],j=1..n)},ScoringRule));
k:=0.9:  l:=0.8:
ThisRule:=[1,k,l,l,0,0,0,0,0];
FindMax(subs({seq(x[j]=ThisRule[j],j=1..n)},ScoringRule));
ThisRule:=[1,1,1,0,0,0,0,0,0];
FindMax(subs({seq(x[j]=ThisRule[j],j=1..n)},ScoringRule));
k:=0.9:  l:=0.7:  p:=0.5:
ThisRule:=[1,k,k,k,l,l,l,l,0];
FindMax(subs({seq(x[j]=ThisRule[j],j=1..n)},ScoringRule));
k:=0.9:  l:=0.7:  p:=0.5:
ThisRule:=[1,k,k,k,l,p,p,0,0];
FindMax(subs({seq(x[j]=ThisRule[j],j=1..n)},ScoringRule));
k:=0.9:  l:=0.7:
ThisRule:=[1,k,k,k,l,l,0,0,0];
FindMax(subs({seq(x[j]=ThisRule[j],j=1..n)},ScoringRule));
k:=0.3:  l:=0.25:
ThisRule:=[1,1,k,k,k,l,l,l,0];
FindMax(subs({seq(x[j]=ThisRule[j],j=1..n)},ScoringRule));
k:=0.3:  l:=0.25:
ThisRule:=[1,k,k,k,k,k,l,l,0];
FindMax(subs({seq(x[j]=ThisRule[j],j=1..n)},ScoringRule));
k:=0.3:  l:=0.25:
```

```
ThisRule:=[1,k,k,k,k,k,k,l,0];
FindMax(subs({seq(x[j]=ThisRule[j],j=1..n)},ScoringRule));
```

$$ThisRule := [1, 0.9, 0.9, 0.9, 0.9, 0, 0, 0, 0]$$
$$[1.9, 1.8, 1.8, 1.8, 1.8, 0.9, 1.8, 1, 1]$$
$$c_1$$
$$ThisRule := [1, 0.9, 0.8, 0.8, 0, 0, 0, 0, 0]$$
$$[1, 1.7, 1.6, 1.6, 0.8, 0.9, 0.9, 1, 1]$$
$$c_2$$
$$ThisRule := [1, 1, 1, 0, 0, 0, 0, 0, 0]$$
$$[1, 1, 2, 0, 1, 1, 1, 1, 1]$$
$$c_3$$
$$ThisRule := [1, 0.9, 0.9, 0.9, 0.7, 0.7, 0.7, 0.7, 0]$$
$$[1.7, 1.8, 1.8, 2.5, 2.3, 2.3, 2.3, 2.4, 2.4]$$
$$c_4$$
$$ThisRule := [1, 0.9, 0.9, 0.9, 0.7, 0.5, 0.5, 0, 0]$$
$$[1.7, 1.8, 1.8, 1.8, 2.1, 1.9, 1.6, 1.5, 2.0]$$
$$c_5$$
$$ThisRule := [1, 0.9, 0.9, 0.9, 0.7, 0.7, 0, 0, 0]$$
$$[1.7, 1.8, 1.8, 1.8, 1.6, 2.3, 1.6, 1.7, 1]$$
$$c_6$$
$$ThisRule := [1, 1, 0.3, 0.3, 0.3, 0.25, 0.25, 0.25, 0]$$
$$[1.3, 1.3, 0.6, 0.85, 0.85, 1.50, 1.55, 1.50, 1.50]$$
$$c_7$$
$$ThisRule := [1, 0.3, 0.3, 0.3, 0.3, 0.3, 0.25, 0.25, 0]$$
$$[1.3, 0.6, 0.6, 0.85, 0.85, 0.9, 0.85, 1.55, 1.50]$$
$$c_8$$
$$ThisRule := [1, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.25, 0]$$
$$[1.3, 0.6, 0.6, 0.85, 0.9, 0.9, 0.85, 1.55, 1.6]$$
$$c_9$$

```
M:=InsertCandidate(M,6);
n,m := Dimension(M);
for j from 1 to n do f[j]:=0; od:
for i from 1 to m do
  for j from 1 to n do
    f[op(M[j,i])]:=f[op(M[j,i])]+x[j];
od; od;
ScoringRule:=[seq(f[j],j=1..n)];
```

[*individual candidate scoring rules input omitted*]

$$M := \begin{bmatrix} c_1 & c_9 & c_8 \\ c_2 & c_6 & c_7 \\ c_5 & c_3 & c_3 \\ c_4 & c_4 & c_2 \\ c_7 & c_1 & c_5 \\ c_{10} & c_{10} & c_{10} \\ c_6 & c_8 & c_6 \\ c_9 & c_5 & c_9 \\ c_8 & c_7 & c_4 \\ c_3 & c_2 & c_1 \end{bmatrix}$$

$$\mathit{ScoringRule} := [x_1 + x_5 + x_{10}, x_2 + x_{10} + x_4, x_{10} + 2\,x_3, 2\,x_4 + x_9, x_3 + x_8 + x_5, 2\,x_7 + x_2, x_5 + x_9 + x_2, x_9 + x_7 + x_1, 2\,x_8 + x_1, 3\,x_6]$$

$$\mathit{ThisRule} := [1, 0.9, 0.9, 0.9, 0.9, 0, 0, 0, 0, 0]$$
$$[1.9, 1.8, 1.8, 1.8, 1.8, 0.9, 1.8, 1, 1, 0]$$
$$c_1$$
$$\mathit{ThisRule} := [1, 0.9, 0.8, 0.8, 0, 0, 0, 0, 0, 0]$$
$$[1, 1.7, 1.6, 1.6, 0.8, 0.9, 0.9, 1, 1, 0]$$
$$c_2$$
$$\mathit{ThisRule} := [1, 1, 1, 0, 0, 0, 0, 0, 0, 0]$$
$$[1, 1, 2, 0, 1, 1, 1, 1, 1, 0]$$
$$c_3$$
$$\mathit{ThisRule} := [1, 0.9, 0.9, 0.9, 0.7, 0.7, 0.7, 0.7, 0.7, 0]$$
$$[1.7, 1.8, 1.8, 2.5, 2.3, 2.3, 2.3, 2.4, 2.4, 2.1]$$
$$c_4$$
$$\mathit{ThisRule} := [1, 0.9, 0.9, 0.9, 0.7, 0.5, 0.5, 0.5, 0, 0]$$
$$[1.7, 1.8, 1.8, 1.8, 2.1, 1.9, 1.6, 1.5, 2.0, 1.5]$$
$$c_5$$
$$\mathit{ThisRule} := [1, 1, 0.3, 0.3, 0.3, 0.3, 0.3, 0, 0, 0]$$
$$[1.3, 1.3, 0.6, 0.6, 0.6, 1.6, 1.3, 1.3, 1, 0.9]$$
$$c_6$$
$$\mathit{ThisRule} := [1, 1, 0.3, 0.3, 0.3, 0.3, 0.25, 0.25, 0.25, 0]$$
$$[1.3, 1.3, 0.6, 0.85, 0.85, 1.50, 1.55, 1.50, 1.50, 0.9]$$
$$c_7$$
$$\mathit{ThisRule} := [1, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.25, 0.25, 0]$$
$$[1.3, 0.6, 0.6, 0.85, 0.85, 0.9, 0.85, 1.55, 1.50, 0.9]$$
$$c_8$$
$$\mathit{ThisRule} := [1, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.25, 0, 0]$$
$$[1.3, 0.6, 0.6, 0.6, 0.85, 0.9, 0.6, 1.3, 1.50, 0.9]$$
$$c_9$$
$$\mathit{ThisRule} := [1, 1, 1, 1, 1, 1, 0, 0, 0, 0]$$
$$[2, 2, 2, 2, 2, 1, 2, 1, 1, 3]$$
$$c_{10}$$

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