A Generalized Binet's Formula for k^{th} Order Linear Recurrences

A Markov Chain Approach

by

Christopher R. H. Hanusa

Francis Su, Advisor

Advisor: ____

Second Reader: ____

(Arthur Benjamin)

April 2001

Department of Mathematics

Harvey Mudd College

Abstract

A Generalized Binet's Formula for k^{th} Order Linear Recurrences

A Markov Chain Approach

by Christopher R. H. Hanusa

April 2001

The Fibonacci Numbers are one of the most intriguing sequences in mathematics. I present generalizations of this well-known sequence. Using combinatorial proofs, I derive closed form expressions for these generalizations. Then using Markov Chains, I derive a second closed form expression for these numbers which is a generalization of Binet's formula for Fibonacci Numbers. I expand further and determine the generalization of Binet's formula for any k^{th} order linear recurrence.

Table of Contents

List of Figures

Acknowledgments

I would like to thank Professors Benjamin and Su for their support and helpful comments and thought-provoking ideas. I would like to thank Tracy van Cort for her LaTeX knowledge and spirited moral support. And for random little things, I would like to thank Anand Patil, Cam McLeman and Profs. Bernoff and Ward. Thanks everyone!

Chapter 1

Introduction

The inspiration for this thesis came two years ago when I was in a colloquium listening to Jennifer Quinn speak about tilings and how they relate to Fibonacci Numbers. Since then, combinatorial interpretations of tilings have inspired me to ask many questions dealing with generalizations of such sequences. For this thesis, I explain my year-long work, with Chapter 1, an introduction to the ideas presented within, Chapter 2, a combinatorial proof of a closed form for the generalized Fibonacci numbers, Chapter 3, a proof using Markov chains of a different closed form for the same numbers, Chapter 4, where I generalize to a larger spectrum – all k^{th} order linear recurrences, and Chapter 5, where I share my insights into possible future work.

1.1 Tilings

In this paper we will use many tilings involving an $n \times 1$ board tiled with squares (1×1 tiles) and dominoes (2×1 tiles), as illustrated in Figure 1.1.

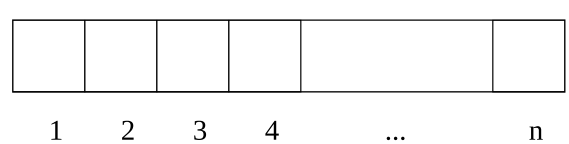

Figure 1.1: An $n \times 1$ board

Definition 1 *Let f_n denote the number of ways to tile this board with squares and dominoes.*

Theorem 1.1.1 *The recurrence formula for f_n is $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$.*

Proof: The tiling of the $n \times 1$ board can end in two possible tiles. Either it ends with a square, in which case we can tile the rest of the $(n-1) \times 1$ board in f_{n-1} ways (by definition of f_{n-1}), or it ends with a domino, in which case we can tile the rest of the board in f_{n-2} ways. Thus, the number of ways to tile an $n \times 1$ board is $f_{n-1} + f_{n-2}$. $\square$

Since a board of length 0 has one tiling (the empty tiling) and a board of length 1 has one tiling, we have $f_0 = f_1 = 1$. Hence the f_n's are the Fibonacci numbers. The standard Fibonacci sequence starts with $F_0 = 0$, $F_1 = 1$ and goes on from there; thus $f_n = F_{n+1}$. Summarizing, we have

Theorem 1.1.2 *The number of ways to tile an $n \times 1$ board with squares and dominoes is the n^{th} Fibonacci number, f_n.*

The structure of tiling the $n \times 1$ board in this way was used by Arthur Benjamin and Jennifer Quinn in [4] and [2] as well as by Brigham et al in [5] to prove many identities involving Fibonacci numbers.

1.2 Combinatorial Proofs

The general idea of a combinatorial proof is to show two quantities are equal by counting the same set of objects in two different ways, thus establishing their equality. As an example, we show how to prove a classic algebraic result of Fibonacci using a combinatorial proof. (See [4].)

Example 1 $f_{2n} = f_n^{\,2} + f_{n-1}^{\,2}$

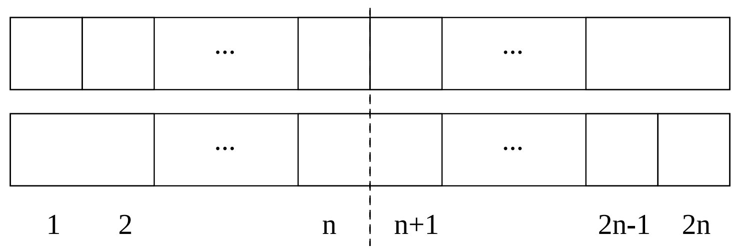

Figure 1.2: The two cases in Example 1.

Proof: In this example, we want to count the number of ways to tile a board of length $2n$.

On one hand, this is the definition of f_{2n}. In another respect, we can look at this problem in two cases – there is either a break or no break between cells n and $n+1$, as exhibited in Figure 1.2.

If there is a break, the number of ways to independently tile each half of the board is f_n, so the number of ways to tile the whole board is $f_n \cdot f_n = f_n^2$. If there is no break, it is necessary that there be a domino covering cells n and $n+1$, so there are $n-1$ cells to tile on either half, and the number of ways to do this is $f_{n-1} \cdot f_{n-1} = f_{n-1}^2$. So the total number of ways to tile the $2n \times 1$ board is $f_{2n} = f_n^2 + f_{n-1}^2$. Thus the equality is established. $\square$

In the following chapter, the reader will see similar techniques used to exhibit other identities involving generalizations of Fibonacci numbers.

1.3 Markov Chains

In Chapters 3 and 4, I use the technique of Markov chains. A Markov chain is a sequence of random variables on some state space, with a fixed probability p_{ij} of moving from state i to state j at each step. This probability does not depend on the past; only on the current state. Thus, given an object's initial state, one can determine the probability that the object is in any particular state at any time.

Example 2 *Given the matrix M*

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{1.1}$$

of transition probabilities and given that at time 0, the model starts in state 4, what is the probability that the model is in state 2 at time 7?

Demonstration: In this example, we want to calculate M^7 (representing seven time steps) and apply it to the initial vector

$$\mathbf{x}_0 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{1.2}$$

which represents our initial probabilities (we start in state 4 at time zero). Once we do this, we arrive at the probability distribution at step 7 of:

$$\mathbf{x}_7 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}^7 = \begin{bmatrix} \frac{13}{48} & \frac{5}{64} & \frac{67}{576} & \frac{53}{288} & \frac{101}{288} \end{bmatrix} \tag{1.3}$$

From this, we see that the probability that the model is in state 2 at time 7 is $5/64$. We will use this Markov Chain approach more abstractly in Chapter 4.

Chapter 2

Fibonacci Generalizations

In this chapter, we see how to use tiling methods to generalize Fibonacci Numbers.

2.1 Tiling Generalization

To generalize the Fibonacci Numbers, we choose to tile an $n \times 1$ board with squares and k-ominoes.

Definition 2 *A k-omino is a tile that covers k cells.*

Just like with Definition 1, we can now define a series of numbers $\mathcal{F}_{k,n}$ that depends on this constant k.

Definition 3 *Let $\mathcal{F}_{k,n}$ denote the number of ways to tile an $n \times 1$ board with squares and k-ominoes.*

Theorem 2.1.1 *The sequence $\mathcal{F}_{k,n}$ can be defined recursively*

$$\mathcal{F}_{k,n} = \mathcal{F}_{k,n-1} + \mathcal{F}_{k,n-k} \tag{2.1}$$

with starting conditions

$$\mathcal{F}_{k,0} = \mathcal{F}_{k,1} = \cdots = \mathcal{F}_{k,k-1} = 1. \tag{2.2}$$

The proof to this is analogous to the proof of Theorem 1.1.1. The reader will note that the specific case when $k = 2$ is exactly the Fibonacci sequence as exhibited in Theorem 1.1.2.

2.2 A Closed Form for $\mathcal{F}_{k,n}$

We have a recurrence relation for $\mathcal{F}_{k,n}$, but it would be nice to have a closed form. Here, from first principles of tilings, we derive such a closed form for $\mathcal{F}_{k+1,n}$. (The $k+1$ is chosen for convenience, so the end result is easier to state.)

Lemma 2.2.1 *The number of ways to tile an $n \times 1$ board using i $(k+1)$-ominoes is $\binom{n-ik}{i}$*

Proof: If you are tiling an $n \times 1$ board using i $(k+1)$-ominoes, then there are $n - i(k+1)$ cells to place squares, and i k-ominoes to place, so in all, there is a total of $n - ik$ tiles to place, where i of them are k-ominoes. Since the squares are indistinguishable from each other, as are the k-ominoes, we have established our identity. $\square$

Using this Lemma, we can establish a closed form for $\mathcal{F}_{k+1,n}$, by summing over all values of i; the number of ways to tile an $n \times 1$ board with squares and $(k+1)$-ominoes is

$$\mathcal{F}_{k+1,n} = \sum_{i=0}^{\infty} \binom{n-ik}{i}. \tag{2.3}$$

This theorem was proved by induction in the 1960's by Hoggatt [14], but this result is exciting because it was done combinatorially.

2.3 Specific Cases

To show that this is a reasonable generalization, we shall now cite some specific cases of formula (2.3), letting $k = 1$ and $k = 0$.

If we let $k = 1$ then formula (2.3) becomes

$$\mathcal{F}_{2,n} = F_n = \sum_{i=0}^{\infty} \binom{n-i}{i}. \tag{2.4}$$

This is an algebraic result of the Fibonacci numbers that was recently proved combinatorially in [4].

If we let $k = 0$, consider what this means - we are tiling with squares and 1-ominoes. So we are tiling with squares and alternate types of squares (we'll give them a different color) squares. The recurrence is

$$\mathcal{F}_{1,n} = \mathcal{F}_{1,n-1} + \mathcal{F}_{1,n-1} = 2\mathcal{F}_{1,n-1}, \tag{2.5}$$

which is the recurrence for the powers of 2, which makes sense; at each cell we have a choice of two colors, so the number of ways to cover n cells is 2^n. By letting $k = 0$ in Equation (2.3), we arrive at the identity

$$2^n = \sum_{i=0}^{\infty} \binom{n}{i}, \tag{2.6}$$

which is the well-known fact that the sum of the elements of the n^{th} row of Pascal's triangle is equal to 2^n. In fact, Equation (2.4) is the sum of the diagonals of Pascal's triangle and Equation (2.3) is the sum of lines of slope k of Pascal's triangle [6],[7].

In these specific cases, the simple elegance of the closed form is striking.

2.4 A Tiling Interpretation of General Linear Recurrences

When we deal with more types of tiles than just squares and dominoes, up to length k, we will be dealing with a k^{th} order linear recurrence.

Definition 4 *A* k^{th} order linear recurrence *is a sequence where each term in the sequence is a linear combination of the k previous terms.*

The symbolic representation of a k^{th} order linear recurrence is

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, \tag{2.7}$$

where the c_i are constant coefficients. For combinatorial purposes, the c_i's must be non-negative integers. To define a specific series, we also need initial conditions, let them be a_0, a_2, ... , a_{k-1}. Given these coefficients and initial conditions, our linear recurrence is completely defined. Since we have $2k$ pieces of information to take into account, it makes sense that we should have $2k$ types of tiles. First, to take into account the coefficients, we remember the previous section, when we tile a board with two types of squares. In that case, we used two different colors. We can so a similar thing here – we now look at the tiling of an $n \times 1$ board using c_1 colors of squares, c_2 colors of dominoes, $\ldots$, up to c_k colors of k-ominoes. To take into account the initial conditions, we must be more clever. We will let the first tile in any tiling (an i-omino) be special, giving it p_i different *phases*. (We use phases to distinguish them from colors, following the convention in [3].)

Definition 5 *Let p_i be the number of ways to tile the first tile if it is an i-omino.*

We will now calculate p_i in terms of the a_i and c_i. If a_n is to count the number of ways to tile a board of length n with colored tiles as described above, we can condition on the last tile. If $n > k$, we have $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$. Now for $n \leq k$, the last tile has length i where $1 \leq i \leq n$. If $i = n$, then our tiling consists of a single tile, for which there are p_i choices. Otherwise there are $c_i a_{n-i}$ such tilings. Thus, for $1 \leq n \leq k$, we must have that

$$a_n = p_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_{n-1} a_1 = p_n + \sum_{i=1}^{n-1} c_i a_{n-i}. \tag{2.8}$$

Thus for a proper tiling interpretation, we must define for $1 \leq n \leq k$,

$$p_n = a_n - \sum_{i=1}^{n-1} c_i a_{n-i}. \tag{2.9}$$

We can also make one further simplification: We will define a_0 to satisfy

$$a_k = c_1 a_{k-1} + c_2 a_{k-2} + \cdots + c_k a_0, \tag{2.10}$$

in which case

$$p_k = a_k - c_1 a_{k-1} - c_2 a_{k-2} - \cdots - c_{k-1} a_1 = c_k a_0, \tag{2.11}$$

and the empty board is tiled in a_0 ways.

So given the k variable p_i phases of the first tile and the k variable c_j colors of j-ominoes in the tiling, we now have the $2k$ variables we need for a tiling interpretation of a k^{th} order linear recurrence.

2.5 Second Order Identities

Using this newly found knowledge leads us to many identities proven at a glance where they had not been before. We illustrate here the case when we have a generalized second order linear recurrence. Given non-negative integers a, b, H_0, and H_1, the recurrence

$$H_n = aH_{n-1} + bH_{n-2} \tag{2.12}$$

for $n \geq 2$ can be given a combinatorial interpretation. Specifically, H_n counts the number of tilings of an $n \times 1$ board with squares of a different colors and dominoes of b different colors, and an initial square can be chosen in H_1 possible ways and an initial domino can be chosen in bH_0 possible ways. Now let's use this to prove some identities!

Identity 1 $b\sum_{k=0}^{n} a^{n-k}H_k = H_{n+2} - H_1a^{n+1}$

Proof: This identity comes from counting the number of tilings of an $(n+2) \times 1$ board using at least one domino. The left hand side of the equation comes from remarking that there must be a *last* domino. If the domino is to be in the last position (i.e. covering cells $n+1$ and $n+2$), there are bH_n ways to tile the board. If the domino is in the second to last position, there are a square possibilities for position $n+2$, b possibilities for the domino in cells n and $n+1$, and H_{n-1} ways to tile the rest of the board, for a total of baH_{n-1} tilings. In general, if the last domino covers cells $k+1$ and $k+2$, then there are $ba^{n-k}H_k$ ways to create such a board. Note that this expression is also valid when $k = 0$. This establishes the left side of the equation. The right side is much simpler; there are H_{n+2} total ways to tile the board, and $H_1 \cdot a^{n+1}$ ways to tile the board using only squares. $\square$

Identity 2 $a\sum_{k=1}^{n} b^{n-k}H_{2k-1} = H_{2n} - H_0 b^n$

Proof: Similar reasoning produces this identity, tiling a $2n \times 1$ board using at least one square. Here we condition on the location of the last square. Note that the last square can not be in an odd position, because then there would be an odd number of cells afterwards to tile using only dominoes. Thus if the last square is to be in cell $2k$ $(1 \leq k \leq n)$, there are H_{2k-1} ways to tile cells $1, \ldots, k$, a choices for the square in cell $2k$, and then b^{n-k} ways to tile the remaining $2n-2k$ cells with $n-k$ dominoes. Summing over all k yields the left side of the identity. The right side is shown by noting that there are H_{2n} total ways to tile the board, and $bH_0 \cdot b^{n-1}$ ways to tile the board using only dominoes. $\square$

Similar reasoning produces: $a\sum_{k=1}^{n} b^{n-k}H_{2k} = H_{2n-1} - H_1 b^n$.

Identity 3 $a\sum_{i=1}^{2n} b^{2n-i}H_{i-1}H_i = H_{2n}^2 - b^{2n}H_0^2$

Proof: Looking at the terms in the series on the left, we see that this identity is different from before, in that we must use a pair of tilings, in order to achieve an $H_{i-1}H_i$ term. We accomplish this by taking an ordered pair of $2n$-tilings, (A, B) where A or B contains at least one square. The left hand side is seen by defining the parameter k_X to be the first cell of tiling X covered by a square. If X is all dominoes, we set k_X to infinity. Since there is at least one square, the minimum of k_A and k_B must be finite and odd. Let $k = \min\{k_A, k_B + 1\}$. When k is odd, A and B have dominoes covering cells 1 through $k-1$ and A has a square at cell k. In this way, the number of tilings (A,B) with odd k is the number of ways to tile A times the number of ways to tile B, i.e. $ab^{(k-1)/2}H_{2n-k} \cdot b^{(k-1)/2}H_{2n-k+1} = ab^{k-1}H_{2n-k}H_{2n-k+1}$. When k is even, A has dominoes covering cells 1 through k and B has dominoes covering cells 1 through $k-2$ and a square at cell $k-1$, so the number of tiling pairs is $b^{k/2}H_{2n-k} \cdot ab^{(k-2)/2}H_{2n-k+1} = ab^{k-1}H_{2n-k}H_{2n-k+1}$. In this way, the left hand side of the identity is satisfied. The right hand side is given when we look at the number of pairs of $2n$-boards without squares, in other words, the total number of $2n$-board pairs ($[H_{2n}]^2$) minus the number of only domino-tiled boards ($[H_0 b^n]^2$). $\square$

A similar technique yields: $a\sum_{i=2}^{2n+1} b^{2n+1-i}H_{i-1}H_i = H_{2n+1}^2 - b^{2n}H_1^2$. This proof can also be done using tail-swapping, a technique presented in [4].

The proofs of these identities were modeled after similar identities for a generalized Fibonacci sequence $a = b = 1$ proved by Benjamin, Quinn, and Su in [3]. These identities are immediately obvious from our tiling interpretation of a second order linear recurrence, which evokes the practicality of this interpretation of such systems.

Chapter 3

Binet-Like Formulas

In this chapter we discuss a generalization of Binet's Formula for Fibonacci Numbers.

3.1 Binet's Formula for Fibonacci Numbers

In [1], Benjamin, Levin, Mahlburg, and Quinn give a novel proof of Binet's formula for Fibonacci numbers, which states that

$$F_n = \mathcal{F}_{2,n-1} = \frac{1}{\sqrt{5}}\left[\phi^n - \left(\frac{-1}{\phi}\right)^n\right]. \tag{3.1}$$

where $\phi = \frac{1+\sqrt{5}}{2}$. The general idea they use is to randomly tile an infinitely long board, placing a square with probability $1/\phi$ and placing a domino with probability $1/\phi^2$, and defining a new relation based on the probability t hat a new tile starts at the n^{th} cell.

Definition 6 *We define a tiling to be* breakable at cell n *if a new tile starts at the* n^{th} *cell.*

In our random tiling of the infinite board, we let q_n be the probability that a tiling will be breakable at cell n.

3.2 Generalization: Binet's Formula for 3-omino Tilings

Benjamin et al. [1] give several proofs of Equation (3.1), including an approach using Markov chains. We apply the Markov chain approach to establish a generalized Binet's formula for $\mathcal{F}_{k,n}$. We first show how to emulate their method for the

3-omino case, and then explain how it generalizes to the k-omino case. This will give a closed form for the $\mathcal{F}_{k,n}$ numbers.

The reason why Benjamin et al. assigned the probabilities $1/\phi$ and $1/\phi^2$ for placing squares and dominoes was to ensure that up to cell n, the probability of any tiling is the same, $1/\phi^n$, and that the probability of choosing a tile at any step is $1/\phi + 1/\phi^2 = 1$. I suggest that ϕ (along with its counterpart $-1/\phi$) is fundamental in Fibonacci theory *because* $1/\phi + 1/\phi^2 = 1$, in the same way that we shall see that the k roots of $1/\chi + 1/\chi^k = 1$ are fundamental to the series $\mathcal{F}_{k,n}$.

Let us study the $\mathcal{F}_{3,n}$ case. We will place a square with probability $1/\tau_1$, and place a 3-omino with probability $1/\tau_1^3$, where τ_1 is the unique real root of

$$\frac{1}{\tau} + \frac{1}{\tau^3} = 1. \tag{3.2}$$

Thus, τ satisfies the characteristic equation

$$\tau^3 - \tau^2 - 1 = 0. \tag{3.3}$$

This (positive) real τ exists because of Descartes's Rule of Signs, which says that the number of positive real roots of an equation is less than or equal to the number of sign changes in the polynomial, and equal in parity. As there is only one sign change, there must be this unique τ. Later we will use the other two (complex) roots of this equation; denote them τ_2 and τ_3.

Let q_n denote the probability that the tiling is breakable at cell n. Since there are $\mathcal{F}_{3,n-1}$ ways to tile the first $n-1$ cells, and each such tiling has probability $1/\tau_1^{n-1}$ of occurring, we see that

$$q_n = \frac{\mathcal{F}_{3,n-1}}{\tau_1^{n-1}}. \tag{3.4}$$

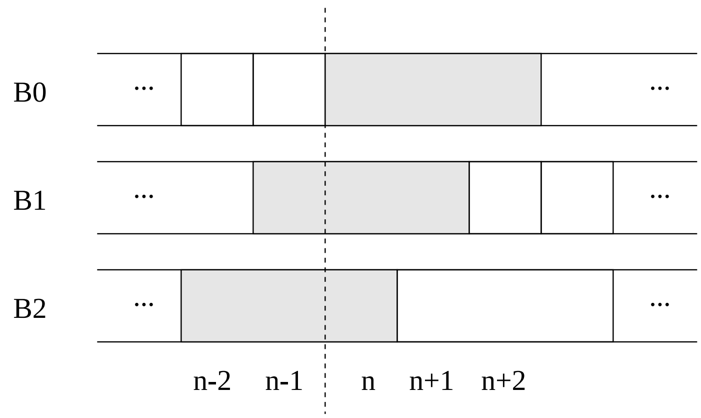

Figure 3.1: The three Markov Chain states

3.3 Markov Chain Model

We will now come up with a formula for q_n (and hence $\mathcal{F}_{k,n}$) using a Markov chain model. The process of randomly placing tiles can be described by a chain of states that moves between three states: B^0 (breakable at the current cell), B^1 (a tile beginning one cell before the current cell), and B^2 (a tile beginning two cells before the current one). (See Figure 3.3.)

The matrix of transition probabilities is as follows:

$$P = \begin{array}{c c} & \begin{array}{ccc} B^0 & B^1 & B^2 \end{array} \\ \begin{array}{c} B^0 \\ B^1 \\ B^2 \end{array} & \begin{pmatrix} \frac{1}{\tau_1} & \frac{1}{\tau_1^3} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{array} \tag{3.5}$$

where p_{ij} is the probability of going from state i to state j. Note the presence of 1's in the second and third rows. These occur because once a 3-omino is placed, there is no choice; we must continue with the 3-omino until it is completed. We begin at time (cell) 1 in the breakable state. So q_n, the probability this tiling is breakable at cell n, is the $(1, 1)$ entry of P^{n-1}. Using the diagonalization $P = Q^{-1}DQ$,

$$P^{n-1} = \begin{bmatrix} \frac{\tau_1}{3\tau_1-2} & \frac{\tau_2}{3\tau_2-2} & \frac{\tau_3}{3\tau_3-2} \\ \frac{\tau_1}{3\tau_1-2} & \frac{\tau_1^2}{\tau_2(3\tau_2-2)} & \frac{\tau_1^2}{\tau_3(3\tau_3-2)} \\ \frac{\tau_1}{3\tau_1-2} & \frac{\tau_1}{3\tau_2-2} & \frac{\tau_1}{3\tau_3-2} \end{bmatrix} \begin{bmatrix} \left(\frac{\tau_1}{\tau_1}\right)^{n-1} & 0 & 0 \\ 0 & \left(\frac{\tau_2}{\tau_1}\right)^{n-1} & 0 \\ 0 & 0 & \left(\frac{\tau_3}{\tau_1}\right)^{n-1} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{\tau_1^3} & \frac{1}{\tau_1^3} \\ 1 & \frac{1}{\tau_1^2\tau_2} & \frac{1}{\tau_1\tau_2^2} \\ 1 & \frac{1}{\tau_1^2\tau_3} & \frac{1}{\tau_1\tau_3^2} \end{bmatrix}, \tag{3.6}$$

the $(1,1)$ entry of the product simplifies to

$$\frac{1}{{\tau_1}^{n-1}}\left[\frac{\tau_1^n}{3\tau_1-2} + \frac{\tau_2^n}{3\tau_2-2} + \frac{\tau_3^n}{3\tau_3-2}\right], \tag{3.7}$$

where τ_1, τ_2, and τ_3 are the roots of the characteristic equation (3.3). It follows directly that

$$\mathcal{F}_{3,n-1} = \frac{\tau_1^n}{3\tau_1-2} + \frac{\tau_2^n}{3\tau_2-2} + \frac{\tau_3^n}{3\tau_3-2}, \tag{3.8}$$

giving a closed form expression for $\mathcal{F}_{3,n}$ only in terms of the roots of Equation (3.3).

3.4 Generalization: Binet's Formula for k-omino Tilings

To prove the general case, we use the same ideas, but we need some help from linear algebra to diagonalize the matrix we encounter.

As before, randomly tile an infinite board, using the rule: place a square with probability $1/\chi_1$, and place a k-omino with probability $1/\chi_1^k$, where χ_1 is the unique positive real root (again from Descartes's Rule of Signs) of

$$\frac{1}{\chi} + \frac{1}{\chi^k} = 1. \tag{3.9}$$

This yields the characteristic equation

$$\chi^k - \chi^{k-1} - 1 = 0. \tag{3.10}$$

Let q_n denote the probability that the tiling is breakable at cell n. Since there are $\mathcal{F}_{k,n-1}$ ways to tile the first $n-1$ cells, and $1/\chi_1^{n-1}$ is the probability of each such tiling, the total probability of tilings being breakable at cell n is

$$q_n = \frac{\mathcal{F}_{k,n-1}}{\chi_1^{n-1}}. \tag{3.11}$$

Our Markov chain will now move between k states: B^0 (breakable at the current cell), B^1 (a tile beginning one cell before the current cell), and B^2 (a tile beginning two cells before the current one) as before, and then B^3 to B^{k-1} with B^i being the state that a tile begins i cells before. Here, the matrix of transition probabilities is:

$$P = \begin{array}{c} \\ B^0 \\ B^1 \\ B^2 \\ \vdots \\ B^{k-2} \\ B^{k-1} \end{array} \begin{array}{c} \begin{array}{cccccc} B^0 & B^1 & B^2 & B^3 & \cdots & B^{k-1} \end{array} \\ \begin{pmatrix} \frac{1}{\chi_1} & \frac{1}{\chi_1^k} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ 0 & 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \end{array} \tag{3.12}$$

where p_{ij} is the probability of going from state i to state j. Note the diagonal of 1's just above the main diagonal. This occurs because once a k-omino is placed, there is no choice; we must continue with the k-omino until it is completed. We begin at time (cell) 1 in the breakable state. So q_n, the probability we are breakable at time n, is the $(1,1)$ entry of P^{n-1}. We now want to diagonalize the probability matrix P:

$$P = Q^{-1} \cdot D \cdot Q. \tag{3.13}$$

We recall from linear algebra [8] that the rows of Q and the columns of Q^{-1} consist respectively of the right and left eigenvectors corresponding to the eigenvalues on the diagonal of D. We first establish:

Theorem 3.4.1 *The eigenvalues of the probability matrix P are χ_i/χ_1 for $1 \leq i \leq k$.*

Proof: We return to the definition of eigenvalue, where since $\lambda I - A$ is a singular matrix, it must have 0 determinant, i.e.

$$|\lambda I - A| = \begin{vmatrix} \lambda - \frac{1}{\chi_1} & -\frac{1}{\chi_1^k} & 0 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & 0 & & 0 \\ 0 & 0 & \lambda & -1 & & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & & \lambda & -1 \\ -1 & 0 & 0 & \cdots & 0 & \lambda \end{vmatrix} = 0 \tag{3.14}$$

Evaluating the determinant using expansion by minors and multiplying each side by λ^k yields,

$$\lambda^k - \frac{\lambda^{k-1}}{\chi_1} + \frac{(-1)^k(-1)^{k+1}}{\chi_1^k} = 0, \tag{3.15}$$

where the $(-1)^{k+1}$ term comes from the expansion by minors method. This gives us

$$(\lambda\chi_1)^k - (\lambda\chi_1)^{k-1} - 1 = 0. \tag{3.16}$$

By satisfying Equation (3.16), $(\lambda\chi_1)$ also satisfies the characteristic Equation (3.10), so the k eigenvalues of P are related to the k roots of Equation (3.10). How cool is that!

For each root χ_i of Equation (3.10), there is a corresponding eigenvalue λ_i satisfying $\chi_i\lambda_i = \chi_1$ So, the k eigenvalues of P are

$$\lambda_i = \chi_i/\chi_1 \tag{3.17}$$

for $1 \leq i \leq k$. □

Now that we know this, we can find column vectors $\mathbf{v}_i$ and row vectors $\mathbf{w}_i$ for each eigenvalue satisfying

$$P\mathbf{v}_i = \frac{\chi_i}{\chi_1}\mathbf{v}_i \tag{3.18}$$

$$\mathbf{w}_i P = \frac{\chi_i}{\chi_1}\mathbf{w}_i \tag{3.19}$$

We can determine $\mathbf{v}_i$:

$$\begin{bmatrix} \frac{1}{\chi_1} & \frac{1}{\chi_1^k} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ 0 & 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \\ \vdots \\ \\ v_k \end{bmatrix} = \begin{bmatrix} \lambda_i v_1 \\ \lambda_i v_2 \\ \\ \vdots \\ \\ \lambda_i v_k \end{bmatrix} \tag{3.20}$$

So,

$$\lambda_i v_2 = v_3 \lambda_i v_3 = v_4 \vdots \lambda_i v_k = v_1 \tag{3.21}$$

Setting $v_1 = 1$, we have the column vectors $\mathbf{v}_i$,

$$\mathbf{v}_i = \left[1, \left(\frac{\chi_1}{\chi_i}\right)^{k-1}, \left(\frac{\chi_1}{\chi_i}\right)^{k-2}, \ldots, \frac{\chi_1}{\chi_i}\right]^{\mathsf{T}} \tag{3.22}$$

Similarly we can find $\mathbf{w}_i$,

$$\mathbf{w}_i = \left[1, \frac{1}{\chi_1^{k-1}\chi_i}, \frac{1}{\chi_1^{k-2}\chi_i^2}, \ldots, \frac{1}{\chi_1 \chi_i^{k-1}}\right] \tag{3.23}$$

These eigenvectors determine Q and Q^{-1} up to constant factors. We calculate Q and Q^{-1} independently because this is easier than taking the inverse of a $k \times k$ matrix. Since this is the case we can form Q from the row vectors $\mathbf{w}_i$:

$$Q = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \vdots \\ \mathbf{w}_k \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{\chi_1^k} & \frac{1}{\chi_1^k} & \cdots & \frac{1}{\chi_1^k} \\ 1 & \frac{1}{\chi_1^{k-1}\chi_2} & \frac{1}{\chi_1^{k-2}\chi_2^2} & & \frac{1}{\chi_1\chi_2^{k-1}} \\ 1 & \frac{1}{\chi_1^{k-1}\chi_3} & \frac{1}{\chi_1^{k-2}\chi_3^2} & & \frac{1}{\chi_1\chi_3^{k-1}} \\ \vdots & & & \ddots & \vdots \\ 1 & \frac{1}{\chi_1^{k-1}\chi_k} & \frac{1}{\chi_1^{k-2}\chi_k^2} & \cdots & \frac{1}{\chi_1\chi_k^{k-1}} \end{bmatrix}. \tag{3.24}$$

We consolidate the constants c_i into Q^{-1}; form Q^{-1} from the column vectors $\mathbf{v}_i$:

$$Q^{-1} = \begin{bmatrix} c_1\mathbf{v}_1 & c_2\mathbf{v}_2 & c_3\mathbf{v}_3 & \cdots & c_k\mathbf{v}_k \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_k \\ c_1 & c_2\frac{\chi_1}{\chi_2}^{k-1} & c_3\frac{\chi_1}{\chi_3}^{k-1} & \cdots & c_k\frac{\chi_1}{\chi_k}^{k-1} \\ c_1 & c_2\frac{\chi_1}{\chi_2}^{k-2} & c_3\frac{\chi_1}{\chi_3}^{k-2} & \cdots & c_k\frac{\chi_1}{\chi_k}^{k-2} \\ \vdots & & & & \vdots \\ c_1 & c_2\frac{\chi_1}{\chi_2} & c_3\frac{\chi_1}{\chi_3} & \cdots & c_k\frac{\chi_1}{\chi_k} \end{bmatrix}. \tag{3.25}$$

Using $QQ^{-1} = I$, we can solve for the c_i. This yields:

$$c_i = \frac{\chi_i}{k\chi_i - k + 1} \tag{3.26}$$

So the diagonalization $P = Q^{-1}DQ$ has been determined.

The $(1,1)$ entry of the product simplifies to

$$q_n = \frac{1}{{\chi_1}^{n-1}} \left[\frac{\chi_1^n}{k\chi_1 - k + 1} + \frac{\chi_2^n}{k\chi_2 - k + 1} + \cdots + \frac{\chi_k^n}{k\chi_k - k + 1} \right], \tag{3.27}$$

where $\chi_2, \chi_3, \ldots, \chi_k$ are the other $k-1$ roots of our characteristic equation. Hence,

$$\mathcal{F}_{k,n-1} = \frac{\chi_1^n}{k\chi_1 - k + 1} + \frac{\chi_2^n}{k\chi_2 - k + 1} + \cdots + \frac{\chi_k^n}{k\chi_k - k + 1} \tag{3.28}$$

giving a closed form expression for $\mathcal{F}_{k,n}$ in terms of the roots of Equation (3.10).

3.5 A Quick Check

Here we note that the general form reduces to Binet's Formula for $k = 2$ as follows.

The characteristic polynomial in this case is

$$\chi^2 - \chi - 1 = 0. \tag{3.29}$$

The two roots of this equation are ϕ and $-1/\phi$. So Equation (3.28) says the closed form expression for $F_n = \mathcal{F}_{2,n-1}$ is

$$F_n = \frac{\phi^n}{2\phi - 1} + \frac{(-1/\phi)^n}{2(-1/\phi) - 1} \tag{3.30}$$

which reduces nicely to Binet's formula for the Fibonacci numbers as presented at the beginning of this chapter.

But that's not all! In the next chapter, I explain how to generalize this formula even further – any k^{th} order linear recurrence.

Chapter 4

Generalized Markov Chain Method

Here we explore what happens to our Markov chain method if we do not restrict ourselves to tiling only with squares and k-ominoes, but allow a generalized k^{th} order linear recurrence.

4.1 Easy k^{th} Order Linear Recurrences

In this section, we take the Initial Conditions of the linear recurrence to be that which we would get if the p_i phases are equal to the c_i types of tiles, as explained in Section 2.4. The first two terms, for example are $a_1 = c_1$ and $a_2 = c_1^2 + c_2$. Later we will generalize further.

From Equation (2.7), we have the determining characteristic equation

$$x^k - c_1 x^{k-1} - c_2 x^{n-2} - \cdots - c_{k-1} x - c_k = 0, \tag{4.1}$$

which we can use as the basis of our Markov Chain Method. We let μ_1 be the positive real root of Equation (4.1) that exists from Descartes's Rule of Signs, and let $\mu_2, \mu_3, \cdots, \mu_k$ be the other roots. [Note that in Chapter 3, only c_k and c_1 were 1, all the rest were 0.] We let the probability that we place an i-omino be $\frac{c_i}{\mu_1^i}$. We now note that the sum of the probabilities to place a tile is 1 since the sum is the same as equation

$$c_1 \mu_1^{k-1} + c_2 \mu_1^{n-2} + \cdots + c_{k-1}\mu_1 + c_k = \mu_1^k, \tag{4.2}$$

and by the definition of μ_1, this equality holds. This Markov chain still moves between the k states $B^0, B^1, B^2, \ldots, B^{k-1}$, where B^i is the state where the current tile ends after $n - i$ more cells, with the matrix of transition probabilities:

$$P = \Bigg(\quad B^0 \quad B^1 \quad B^2 \quad B^3 \quad \cdots \quad B^{k-1} B^0 \quad \tfrac{c_1}{\mu_1} \quad \tfrac{c_k}{\mu_1^k} \quad \tfrac{c_{k-1}}{\mu_1^{k-1}} \quad \tfrac{c_{k-2}}{\mu_1^{k-2}} \quad \cdots \quad \tfrac{c_1}{\mu_1^2} B^1 \quad 0 \quad 0 \quad 1 \quad 0 \qquad 0B \tag{4.3}$$

where p_{ij} is the probability of going from state i to state j. To find the eigenvalues λ_i of the matrix P, we take $\lambda I - P$ as singular, and we take the determinant to find:

$$|\lambda I - P| = (\lambda\mu_1)^k - c_1(\lambda\mu_1)^{k-1} - c_2(\lambda\mu_1)^{k-2} - \cdots - c_{k-1}(\lambda\mu_1) - c_k = 0. \tag{4.4}$$

We now see a correlation to Theorem 3.4.1, where we can determine the eigenvalues of P with respect to the roots of our new characteristic equation, Equation (4.1). By satisfying Equation (4.4), $(\lambda\mu_1)$ also satisfies the characteristic Equation (4.1), so the k eigenvalues of P in this general case are

$$\lambda_i = \mu_i/\mu_1 \tag{4.5}$$

for $1 \leq i \leq k$. From this information, we can diagonalize P as we did in (3.24) and (3.25). This time,

$$\mathbf{v}_i = \left[1, \left(\frac{\mu_1}{\mu_i}\right)^{k-1}, \left(\frac{\mu_1}{\mu_i}\right)^{k-2}, \ldots, \frac{\mu_1}{\mu_i}\right]^{\mathsf{T}} \tag{4.6}$$

and

$$\mathbf{w}_i = \left[1, \frac{c_k}{\mu_1^{k-1}\mu_i}, \frac{c_{k-1}\mu_i + c_k}{\mu_1^{k-2}\mu_i^2}, \ldots, \frac{c_2\mu_i^{k-2} + c_3\mu_1^{k-3} + \cdots + c_k}{\mu_1\mu_i^{k-1}}\right]. \tag{4.7}$$

Now we find Q and Q^{-1} as before:

$$Q = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \vdots \\ \mathbf{w}_k \end{bmatrix} = \begin{bmatrix} 1 & \frac{c_k}{\mu_1^k} & \frac{c_{k-1}+c_k}{\mu_1^k} & \cdots & \frac{c_2+c_3+\cdots+c_k}{\mu_1^k} \\ 1 & \frac{c_k}{\mu_1^{k-1}\mu_2} & \frac{c_{k-1}\mu_2+c_k}{\mu_1^{k-2}\mu_2^2} & & \frac{c_2\mu_2^{k-2}+c_3\mu_2^{k-3}+\cdots+c_k}{\mu_1\mu_2^{k-1}} \\ 1 & \frac{c_k}{\mu_1^{k-1}\mu_3} & \frac{c_{k-1}\mu_3+c_k}{\mu_1^{k-2}\mu_3^2} & & \frac{c_2\mu_3^{k-2}+c_3\mu_3^{k-3}+\cdots+c_k}{\mu_1\mu_3^{k-1}} \\ \vdots & & & \ddots & \vdots \\ 1 & \frac{c_k}{\mu_1^{k-1}\mu_k} & \frac{c_{k-1}\mu_k+c_k}{\mu_1^{k-2}\mu_k^2} & \cdots & \frac{c_2\mu_k^{k-2}+c_3\mu_k^{k-3}+\cdots+c_k}{\mu_1\mu_k^{k-1}} \end{bmatrix}. \tag{4.8}$$

$$Q^{-1} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \cdots & \mathbf{v}_k \end{bmatrix} = \begin{bmatrix} d_1 & d_2 & d_3 & \cdots & d_k \\ d_1 & cd2\frac{\mu_1}{\mu_2}^{k-1} & d_3\frac{\mu_1}{\mu_3}^{k-1} & \dots & d_k\frac{\mu_1}{\mu_k}^{k-1} \\ d_1 & d_2\frac{\mu_1}{\mu_2}^{k-2} & d_3\frac{\mu_1}{\mu_3}^{k-2} & \dots & d_k\frac{\mu_1}{\mu_k}^{k-2} \\ \vdots & & & & \vdots \\ d_1 & d_2\frac{\mu_1}{\mu_2} & d_3\frac{\mu_1}{\mu_3} & \dots & d_k\frac{\mu_1}{\mu_k} \end{bmatrix}. \quad (4.9)$$

Using $QQ^{-1} = I$, we can solve for the d_i. This yields:

$$d_i = \frac{\mu_i^k}{c_1\mu_i^{k-1} + 2c_2\mu_i^{k-2} + 3c_3\mu_i^{k-3} + \cdots + kc_k} \quad (4.10)$$

Hence our diagonalization $P = Q^{-1}DQ$ is complete. This gives us a Binet's formula for $k^{\mathbf{th}}$ order linear recurrences with simple initial conditions:

$$a_n = \sum_{i=1}^{k} d_i\mu_i^n. \quad (4.11)$$

4.2 Making Sure

We now verify this formula matches the case I presented in Chapter 3, when only c_k and c_1 were 1, all the rest were 0 and the initial conditions were $a_1 = 1, a_2 = 1, a_3 = 2$, as required. Everything looks good in our comparison except perhaps our constant factor d_i from Equation (4.10). Some algebra indeed verifies that

$$d_i = \frac{\chi_i^k}{\chi_i^{k-1} + k} = \frac{\chi_i}{k\chi_i - k + 1}. \quad (4.12)$$

And from this we see that Equation (4.11) looks good.

4.3 Inclusion of Initial Conditions

Consider the series

$$\alpha_n = c_1\alpha_{n-1} + c_2\alpha_{n-2} + \cdots + c_{k-1}\alpha_{n-k+1} + c_k\alpha_{n-k}, \quad (4.13)$$

where the c_i's are positive integers, but the first k terms, A_0, A_1, ... , A_{k-1} may be of any type. To formulate a Binet's Formula for such a wide range of recurrences,

Term	a_n
0	1
1	c_1
2	$c_1a_1 + c_2$
3	$c_1a_2 + c_2a_1 + c_3$
$\vdots$	$\vdots$
$k-1$	$c_1a_{k-2} + c_2a_{k-3} + \cdots + c_{k-2}a_1 + c_{k-1}$
k	$c_1a_{k-1} + c_2a_{k-2} + \cdots + c_{k-2}a_2 + c_{k-1}a_1$

Figure 4.1: $a_n,\ 0 \leq n \leq k$

we approach the problem, hope to establish a set of "basis series" $e_{0,n}$, $e_{1,n}$, $\ldots$, e_{k-1}, n satisfying (4.13). In this way, every series with initial conditions A_0, A_1, $\ldots$, A_{k-1} can be represented as a linear combination of these basis series. For a specific recurrence given c_i, this series is determined by the first k terms, so it is easy to verify that vectors of initial conditions determine a k dimensional vector space. Choose the initial conditions of the basis series $e_{i,n}$ to be an n-vector with a 1 in the i^{th} position and 0's elsewhere. We wish to now find the Binet's Formula for these basis series given the result in Equation (4.11). To accomplish this, we look at the zero and negative terms of this series. Because of the simple way we recursively define a_n, we can find these terms easily: $a_0 = 1$ and $a_{-1} = a_{-2} = \cdots = a_{1-k} = 0$. Now we look at the series a_n in Figure 4.1. What we can see is that to achieve a basis series $e_{0,n}$ of initial conditions $A_0 = 1$, $A_1 = A_2 = \cdots = A_{k-1} = 0$, we can take

$$e_{0,n} = a_n - c_1a_{n-1} - c_2a_{n-2} - \cdots - c_{k-1}a_{n-k+1}. \tag{4.14}$$

Similarly, we can find basis series $1 \leq i \leq k-1$,

$$e_{i,n} = a_{n-i} - c_1a_{n-i-1} - c_2a_{n-i-2} - \cdots - c_{k-i-1}a_{n-k+1}. \tag{4.15}$$

What we have done in effect is a row-echelon reduction of a $k \times k$ matrix of the initial conditions of the series $a_n, \ldots, a_{n-k+1}$. From these basis series, we can represent any $k^{\mathbf{th}}$ order linear recurrence with initial conditions A_0, A_1, . . . , A_{k-1} as

$$\alpha_n = A_0 e_{0,n} + A_1 e_{1,n} + A_2 e_{2,n} + \cdots + A_{k-1} e_{k-1,n}, \tag{4.16}$$

which is equal to

$$\alpha_n = A_0[a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_{k-1} a_{n-k+1}] + A_1[a_{n-1} - c_1 a_{n-2} - \cdots - c_{k-2} a_{n-k+1}] + A_2[a_{n-2} - \cdots - c_{k} \tag{4.17}$$

Collecting like terms gives us

$$\alpha_n = a_n[A_0] + a_{n-1}[A_1 - c_1 A_0] + a_{n-2}[A_2 - c_1 A_1 - c_2 A_0] + \cdots + a_{n-k+1}[A_{k-1} - c_1 A_{k-2} - c_2 A_{k-3} - \cdots - c_{k-1} A \tag{4.18}$$

And finally we can combine the Binet's formulas for the a_j linearly to achieve a completely generalized Binet's formula for k^{th} order linear recurrences,

$$\alpha_n = A_0 \sum_{i=1}^{k} d_i \mu_i^n + (A_1 - c_1 A_0) \sum_{i=1}^{k} d_i \mu_i^{n-1} + (A_2 - c_1 A_1 - c_2 A_0) \sum_{i=1}^{k} d_i \mu_i^{n-2} + \cdots + (A_{k-1} - c_1 A_{k-2} - c_2 A_{k-3} \tag{4.19}$$

More easily represented,

$$\alpha_n = \sum_{i=1}^{k} \frac{A_0 \mu_i^k + (A_1 - c_1 A_0)\mu_i^{k-1} + \cdots + (A_{k-1} - c_1 A_{k-2} - \cdots - c_{k-1} A_0)\mu_i}{c_1 \mu_i^{k-1} + 2c_2 \mu_i^{k-2} + 3c_3 \mu_i^{k-3} + \cdots + k c_k} \mu_i^n. \tag{4.20}$$

Comparing my results to the Tribonacci sequence as outlined in [12] confirms my solution in that specific case. YAY!

Chapter 5

Future Directions

In this chapter we discuss the continuation of these previous results and how they may be applied elsewhere.

5.1 p-ominoes and q-ominoes

What happens if we don't necessarily use squares as a tiling unit? What if we use two types of rectangles, such as p-ominoes and q-ominoes. In this case, we derive with a similar equation to formula (2.3).

$$\mathcal{F}^{p}_{q,n} = \sum_{\substack{i \text{ s.t.} \\ q|n-ip}} \binom{n-ip}{i} = \sum_{\substack{j \text{ s.t.} \\ p|n-jq}} \binom{n-jq}{j} \tag{5.1}$$

The first equality can be seen in that n must equal $ip + jq$ for us to be able to tile the $n \times 1$ board with p-ominoes and q-ominoes. The second equality can be seen similarly. In addition, this generalization jibes with the case when $q = 1$, because 1 always divides $n - ip$.

5.2 Whitney Numbers

In my studies so far, I have happened upon a number of identities that I have not proved (yet), and so that these are not lost forever after I retire my thesis notebook, I have included them here.

The most striking to me was an identity for the Whitney Numbers - an integral part of the square/3-omino tiling. Here's why:

When I was trying to generalize Binet's formula using pure combinatorial methods, I wanted to come up with a form for q_n as defined in Section (3.2) in terms of a sequence that had the following form:

$$q_n = a_{0,n} - a_{1,n}/\tau^3 + a_{2,n}/\tau^6 - \cdots + a_{n-1,n}(-1/\tau^3)^{n-1}, \tag{5.2}$$

trying to emulate another one of Benjamin et al.'s combinatorial methods of proving Binet's formula.

What I found was intriguing! Instead of the nice geometric series that I was hoping for, I found the Whitney numbers ($w(n,k)$) [11]. They are a set of numbers that are defined

$$w(k,n) = \sum_{i=0}^{k} \binom{n}{i} \tag{5.3}$$

I found that

$$a_{k,n} = w(k, n-k). \tag{5.4}$$

I then explored the Whitney numbers and their correlation with Pascal's triangle. I was extremely surprised to find out that I could sum variations of Pascal's triangle to come up with the Whitney numbers. Hence I came up with a new form for Whitney numbers,

$$w(k,n) = \sum_{i=0}^{\infty} \binom{k+n-2i}{i} \binom{k+n-3i}{k-i} \tag{5.5}$$

which seems like a very different combinatorial relationship than what I'm used to, so I have no idea how to prove it.

Now back to tilings. Then I wondered about the extension of these Whitney Numbers into when you tile with squares and κ-ominoes, where I now define a κ-Whitney number, $w_\kappa(n,k)$, as the coefficients of the $(-1/\chi_1^\kappa)$ expansion of q_n. Here some unexpected patterns appeared. The most interesting property of the κ-Whitney numbers is that

$$\lim_{\kappa\to\infty} w_\kappa(n,k) = \binom{n}{k} \tag{5.6}$$

so in other words, the limit of the κ-Whitney numbers is Pascal's Triangle! I would like to expand upon this further in the future.

5.3 Conclusion

I've described a number of different ideas, used combinatorics to prove new or pre-existing identities involving Generalized Fibonacci Numbers, and the identities for second-order linear recurrences, in Equation (2.3), and Identities 1, 2, and 3. I also really enjoyed using Markov Chains to prove the pretty results of Binet's Formulas, in Equations (3.28), (4.11), and (4.20), and it makes me happy to know that these have been done other ways, but I have a fresh new idea for these old theorems. If I were to suggest things that needed more looking into, I would suggest trying to find a correlation between my work and those by Spickerman and Joyner in [13] and Mouline and Rachidi in [10] and [9]. I would also be interested in a modified approach that would allow for non-integral coefficients in the recurrence. I think that the connection between Whitney Numbers and Fibonacci Generalizations is worth a look as well. I enjoyed the research that I did this year and hope that the mathematical community can benefit. Bye for now.

Bibliography

[1] A. T. Benjamin, G. M. Levin, K. Mahlburg, and J.J. Quinn. Random Approaches to Fibonacci Identities. *American Mathematical Monthly*, 107(6):511–516, 2000.

[2] A. T. Benjamin and J. J. Quinn. Fibonacci and Lucas Identities through Colored Tilings. *Fibonacci Quarterly*, 30(5):359–366, 1999.

[3] A. T. Benjamin, J.J. Quinn, and F. E. Su. Phased Tilings and Generalized Fibonacci Identities. *The Fibonacci Quarterly*, 38(3):282–288, 2000.

[4] A. T. Benjamin, F. E. Su, and J. J. Quinn. Counting on Continued Fractions. *Mathematics Magazine*, 73:98–104, 2000.

[5] R. C. Brigham, R. M. Caron, P. Z. Chinn, and R. P. Grimaldi. A tiling scheme for the Fibonacci numbers. *J. Recreational Math.*, 28(1):10–16, 1996.

[6] M. Feinberg. New Slants. *Fibonacci Quarterly*, 2(3), 1964.

[7] A Generalization of the Connection Between the Fibonacci Sequence and Pascal's Triangle. *Fibonacci Quarterly*, 1(3):21–31, 1963.

[8] D. Lay. *Linear Algebra and its Applications*. Addison-Wesley-Longman, 2^{nd} edition, 1979.

[9] M. Mouline and M. Rachidi. Suites de fibonacci généralisées et chaînes de markov. *Revista de la Real Academia de Ciencias Exatas, Fisicas y Naturales de Madrid*, 89(1–2):61–77, 1995.

[10] M. Mouline and M. Rachidi. Application of Markov Chains properties to r-Generalized Fibonacci Sequences. *Fibonacci Quarterly*, 30(1):34–38, 1999.

[11] On-line Encyclopedia of Integer Sequences. 2001. Available Online at http://www.research.att.com/ñjas/sequences/.

[12] W. R. Spickerman. Binet's Formula for the Tribonacci Sequence. *Fibonacci Quarterly*, 20(2):118–120, 1982.

[13] W. R. Spickerman and R. N. Joyner. Binet's Formula for the Recursive Sequence of Order K. *Fibonacci Quarterly*, 22(4):327–331, 1984.

[14] Jr. V. E. Hoggatt. Combinatorial Problems for Generalized Fibonacci Numbers. *Fibonacci Quarterly*, 8(4):456–462, 1970.