Catalan Numbers

c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
1	1	2	5	14	42	132	429	1430	4862	16796

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$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

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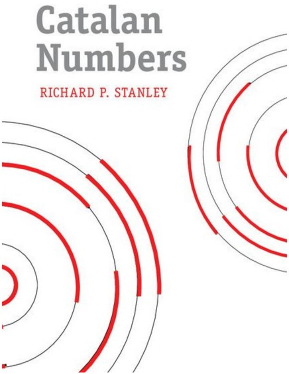

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triangulations of an $(n+2)$-gon	lattice paths from $(0,0)$ to (n,n) above $y=x$
sequences with n $+1$'s, n -1's with positive partial sums	multiplication schemes to multiply $n+1$ numbers

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4. Ways to multiply $n + 1$ numbers together two at a time.

Catalan Bijections

We claim that these objects are all counted by the Catalan numbers. So there should be bijections between the sets!

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Bijection 1:	triangulations of an $(n+2)$-gon	$\longleftrightarrow$	multiplication schemes to multiply $n+1$ numbers

Rule: Label all but one side of the $(n+2)$-gon in order. Work your way in from the outside to label the interior edges of the triangulation: When you know two sides of a triangle, the third edge is the product of the two others. Determine the mult. scheme on the last edge.

Catalan Bijections

Bijection 2: | multiplication schemes to multiply $n + 1$ #s | $\longleftrightarrow$ | seqs with n $+1$'s, n -1's with positive partial sums |

Rule: Place dots to represent multiplications. Ignore everything except the dots and right parentheses. Replace the dots by $+1$'s and the parentheses by -1's.

Catalan Bijections

Bijection 3: | seqs with n $+1$'s, n -1's with positive partial sums | $\longleftrightarrow$ | lattice paths $(0,0)$ to (n,n) above $y=x$ |
|---|---|---|

A sequence of $+$'s and $-$'s converts to a sequence of N's and E's, which is a path in the lattice.

Catalan Numbers

The underlying reason why so many combinatorial families are counted by the Catalan numbers comes back to the generating function equation that $C(x)$ satisfies:

$$C(x) = 1 + xC(x)^2.$$

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Example.

triangulations of an $(n+2)$-gon

Here, x represents one side of the polygon

Either the triangulation has a side or not.

1. No side: Empty triangulation (of *digon*): x^0.
2. Every other triangulation has one side (x contribution) and is a sequence of two other triangulations $C(x)^2$.

Catalan Numbers

Example. lattice paths $(0, 0)$ to (n, n) above $y = x$

Here, x represents an up-step down-step pair.

Either the lattice path starts with a vertical step or not.

1. No step: Empty lattice path: x^0.
2. Every other lattice path has one vertical step up from diag. and a first horizontal step returning to diag. (x contribution). "Between the V & H steps" and "after the H step" is a sequence of two lattice paths $C(x)^2$.

Therefore, $C(x) = 1 + xC(x)^2$.

A formula for the Catalan Numbers

Solve the generating function equation to find $C(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$.

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Therefore, $c_n = \frac{1}{n+1}\binom{2n}{n}$.

Expansion of $\sqrt{1-4x}$

What is the power series expansion of $\sqrt{1-4x}$?

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