Multiplying two generating functions (Convolution)

Let $A(x) = \sum_{k \geq 0} a_k x^k$ and $B(x) = \sum_{k \geq 0} b_k x^k$.

Question: What is the coefficient of x^k in $A(x)B(x)$?

When expanding the product $A(x)B(x)$ we multiply terms $a_i x^i$ in A by terms $b_j x^j$ in B. This product contributes to the coefficient of x^k in $A(x)B(x)$ only when ____________________________.

Therefore, $A(x)B(x) = \sum_{k \geq 0} \Big(\underline{\qquad\qquad\qquad} \Big) x^k$

Example.
$[x^9]\frac{x^3(1+x)^4}{(1-2x)}$

Combinatorial interpretation of the convolution:

If a_k counts all "A" objects of "size" k, and
b_k counts all "B" objects of "size" k,

Then $[x^k]\big(A(x)B(x)\big)$ counts all pairs of objects (A, B) with *total* size k.

A Halloween Multiplication

Example. In how many ways can we fill a halloween bag w/30 candies, where for each of 20 BIG candy bars, we can choose at most one, and for each of 40 different small candies, we can choose as many as we like?

Big candy g.f.: $B(x) = (1+x)^{20} = \sum_{k=0}^{\infty} \binom{20}{k} x^k.$

b_k counts (k big candies)

Small candy g.f.: $S(x) = \dfrac{1}{(1-x)^{40}} = \sum_{k=0}^{\infty} \left(\!\binom{40}{k}\!\right) x^k.$

s_k counts (k small candies)

Total g.f.: $B(x)S(x) = \sum_{k=0}^{\infty} \left[\sum_{i=0}^{k} \binom{20}{i} \left(\!\binom{40}{k-i}\!\right) \right] x^k$

Conclusion: $[x\quad]B(x)S(x) = \sum_{i=0} \binom{20}{i} \left(\!\binom{40}{\quad - i}\!\right)$

So, $[x^k]B(x)S(x)$ counts pairs of the form ∨ w/k total candies.
(some number of big candies, some number of small candies)

Example: Rolling dice

Example. When two standard six-sided dice are rolled, what is the distribution of the sums that appear?

Solution. The generating function for one die is $D(x) =$
Therefore, the distribution of sums for rolling two dice is

Question: What does $D(1)$ count?
Answer:

Example. Is it possible to relabel two six-sided dice differently to give the *exact same distribution* of sums?

Solution. Find two generating functions $F(x)$ and $G(x)$ such that $F(x)G(x) = D^2(x)$ and $F(1) = G(1) = 6$. Rearrange the factors:

$$\begin{aligned} D(x)^2 &= x^2(1+x)^2\left(1-x+x^2\right)^2\left(1+x+x^2\right)^2. \\ &= \left[x(1+x)\left(1+x+x^2\right)\right] \cdot \left[x\left(1-x+x^2\right)^2(1+x)\left(1+x+x^2\right)\right] \\ &= \left[x+2x^2+2x^3+x^4\right] \cdot \left[x+x^3+x^4+x^5+x^6+x^8\right] \end{aligned}$$

Die F: and die G: .

Vandermonde's Identity (p. 117)

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}$$

Combinatorial proof

Generating function proof

Powers of generating functions

A special case of convolution gives the coefficients of powers of a g.f.:

$$(A(x))^2 = \sum_{k\geq 0} \left(\sum_{i=0}^{k} a_i a_{k-i} \right) x^k = \sum_{k\geq 0} \left(\sum_{i_1+i_2=k} a_{i_1} a_{i_2} \right) x^k.$$

Similarly,

$$(A(x))^n = \sum_{k\geq 0} \left(\sum_{i_1+i_2+\cdots+i_n=k} a_{i_1} a_{i_2} \cdots a_{i_n} \right) x^k.$$

Conclusion:

$[x^k](A(x))^n$ counts *sequences* of objects $(A_1, A_2, \ldots, A_n)$, all of type A, with a total size (summed over all objects) of k.

Example. What is the generating function for the number of points that a basketball team can score if they hit a sequence of 10 baskets?

In how many ways can they score 20 points in those 10 baskets?

Compositions

Question: In how many ways can we write a positive integer n as a sum of positive integers?

If order **doesn't** matter:

A **partition**: $n = p_1 + p_2 + \cdots + p_\ell$ for positive integers $p_1, p_2 \ldots, p_\ell$ satisfying $p_1 \geq p_2 \geq \cdots \geq p_\ell$.

If order **does** matter:

A **composition**: $n = i_1 + i_2 + \cdots + i_\ell$ for positive integers $i_1, i_2 \ldots, i_\ell$ with no restrictions.

Example. There are 2^{n-1} compositions of n. When $n = 4$:

$$\begin{gathered} 4 \\ 3+1 \\ 2+2 \\ 2+1+1 \\ 1+1+1+1 \end{gathered}$$

Compositions of Generating Functions

Question: Let $F(x) = \sum_{n\geq 0} f_n x^n$ and $G(x) = \sum_{n\geq 0} g_n x^n$.
What can we learn about the composition $H(x) = F\big(G(x)\big)$?

Investigate $F(x) = 1/(1-x)$.

$$H(x) = F(G(x)) = \frac{1}{1-G(x)} = \qquad\qquad .$$

- This is an infinite sum of (likely infinite) power series. Is this OK?
- The constant term h_0 of $H(x)$ only makes sense if ______ .
- This implies that x^n divides $G(x)^n$.
  Hence, there are at most $n-1$ summands which contain x^{n-1}.
  We conclude that the infinite sum makes sense.

For a general composition with $g_0 = 0$,

$$F(G(x)) = \sum_{n\geq 0} f_n G(x)^n = f_0 + f_1 G(x) + f_2 G(x)^2 + f_3 G(x)^3 + \cdots .$$

Compositions. of. Generating Functions.

Interpreting $\dfrac{1}{1-G(x)} = 1 + G(x)^1 + G(x)^2 + G(x)^3 + \cdots$:

Recall: The generating function $G(x)^n$ counts sequences of length n of objects $(G_1, G_2, \ldots, G_n)$, each of type G, and the coefficient $[x^k](G(x)^n)$ counts those n-sequences that have total size equal to k.

Conclusion: As long as $g_0 = 0$, then $1 + G(x)^1 + G(x)^2 + G(x)^3 + \cdots$ counts sequences of **any length** of objects of type G, and the coefficient $[x^k]\frac{1}{1-G(x)}$ counts those that have total size equal to k.

Alternatively: Interpret $[x^k]\frac{1}{1-G(x)}$ thinking of k as this *total size*. First, find **all ways** to break down k into integers $i_1 + \cdots + i_\ell = k$. Then create **all sequences** of objects of type G in which object j has size i_j.

Think: A composition of generating functions equals

a composition. of. generating. functions.

An Example, Compositions

Example. How many compositions of k are there?

Solution. A composition of k corresponds to a sequence $(i_1, \ldots, i_\ell)$ of positive integers (of any length) that sums to k.

The objects in the sequence are positive integers; we need the g.f. that counts how many positive integers there are with "size i".

What does size correspond to?

How many have value i? Exactly one: the number i.

So the generating function for our objects is
$G(x) = 0 + 1x^1 + 1x^2 + 1x^3 + 1x^4 + \cdots = $ ____________________.

We conclude that the generating function for compositions is
$H(x) = \frac{1}{1-G(x)} =$

So the number of compositions of n is

A Composition Example

Example. How many ways are there to take a line of k soldiers, divide the line into non-empty platoons, and from each platoon choose one soldier in that platoon to be a leader?

Solution. A soldier assignment corresponds to a sequence of platoons of size $(i_1, \ldots, i_\ell)$.

Given i soldiers in a platoon, in how many ways can we assign the platoon a leader? __________

Therefore $G(x) =$

And the generating function for such a military breakdown is

$$H(x) = \frac{1}{1 - G(x)} = \frac{1 - 2x + x^2}{1 - 3x + x^2}$$

Domino Tilings

Example. How many square-domino tilings are there of a $1 \times n$ board?

Solution. A tiling corresponds to a sequence $(i_1, \ldots, i_\ell)$, where i_j _________________.

So $G(x) =$ _______________, and therefore $H(x) =$ _____________.

Another way to see this:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}
$G(x)^0 =$													
$G(x)^1 =$													
$G(x)^2 =$													
$G(x)^3 =$													
$G(x)^4 =$													
$G(x)^5 =$													
$G(x)^6 =$													
$1/(1 - G(x)) =$													