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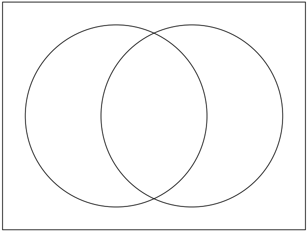
Let S be the set of students who play soccer
and B be the set of students who play basketball.
Then, $|S \cup B| = |S| + |B|$ __________.



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When $A = A_1 \cup \cdots \cup A_k \subset \mathcal{U}$ ($\mathcal{U}$ is for universe) and the sets A_i are *pairwise disjoint*, we have $|A| = |A_1| + \cdots + |A_k|$.

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The Hard Part: Determining the right choice of A_i. The A_i and their intersections should be easy to count and easy to characterize.

mmm. . .PIE

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Let $A_2 \subset \mathcal{U}$ be the multiples of 2.
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We want $|A_2 \cup A_3 \cup A_5|$.

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Combinations with Repetitions

Quick review

1. How many ways are there to choose k elements out of the set $\{1 \cdot a_1, 1 \cdot a_2, \cdots, 1 \cdot a_n\}$?

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What we would like to calculate is:

In how many ways can we choose k elements out of an arbitrary multiset?

Now, it's as easy as PIE.

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Now calculate: $|\mathcal{U}| =$ $|A_1| =$ $|A_2| = \left(\!\binom{3}{5}\!\right)$ $|A_3| = \left(\!\binom{3}{4}\!\right)$

$|A_1 \cap A_2| = 3$ $|A_1 \cap A_3| = 1$ $|A_2 \cap A_3| = 0$ $|A_1 \cap A_2 \cap A_3| = 0$

And finally: So $|\mathcal{U}| - |A_1 \cup A_2 \cup A_3| =$

Derangements

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Question: Compute a formula for D_n.

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We conclude:

$$D_n = |\mathcal{U}| - \sum |A_i| + \sum |A_i \cap A_j| - \sum \left|A_i \cap A_j \cap A_k\right| + \cdots$$