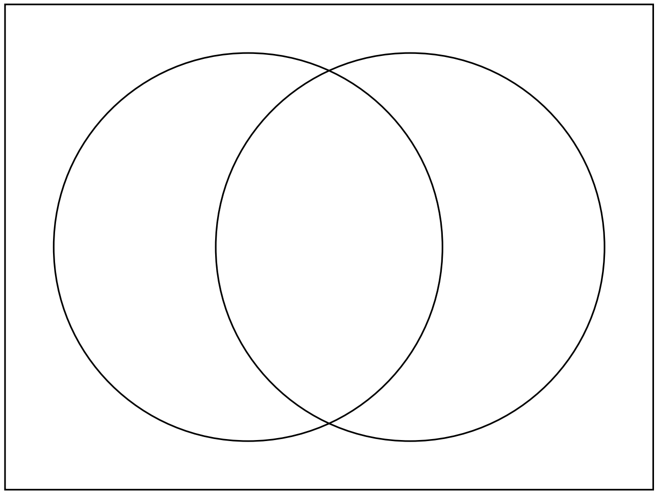
Principle of Inclusion-Exclusion

Example. Suppose that in a class of 30 students, 14 play soccer and 11 play basketball. How many students play a sport?

Solution.

Let S be the set of students who play soccer
and B be the set of students who play basketball.
Then, $|S \cup B| = |S| + |B|$__________.



Principle of Inclusion-Exclusion

When $A = A_1 \cup \cdots \cup A_k \subset \mathcal{U}$ ($\mathcal{U}$ is for universe) and the sets A_i are *pairwise disjoint*, we have $|A| = |A_1| + \cdots + |A_k|$.

When $A = A_1 \cup \cdots \cup A_k \subset \mathcal{U}$ and the A_i are **not** pairwise disjoint, we must apply the principle of inclusion-exclusion to determine $|A|$:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$|A_1 \cup \cdots \cup A_m| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| \cdots$$

It may be more convenient to apply inclusion/exclusion where the A_i are *forbidden* subsets of $\mathcal{U}$, in which case ____________________.

The Hard Part: Determining the right choice of A_i. The A_i and their intersections should be easy to count and easy to characterize.

mmm. . .PIE

Example. How many integers from 1 to 100 are divisible by 2, 3, or 5?

Solution. Let $\mathcal{U} = \{n \in \mathbb{Z}$ such that $1 \leq n \leq 100\}$.

Let $A_2 \subset \mathcal{U}$ be the multiples of 2.
Let $A_3 \subset \mathcal{U}$ be the multiples of 3.
Let $A_5 \subset \mathcal{U}$ be the multiples of 5.

We want $|A_2 \cup A_3 \cup A_5|$.

Write the interpretation of the intersections in words:

$A_2 \cap A_3$ is the set of integers
$A_2 \cap A_5$ is the set of integers
$A_3 \cap A_5$ is the set of integers
$A_2 \cap A_3 \cap A_5$ is the set of integers that are

Now calculate: $|A_2| =$ $|A_3| =$ $|A_5| =$

$|A_2 \cap A_3| =$ $|A_2 \cap A_5| =$ $|A_3 \cap A_5| =$

$|A_2 \cap A_3 \cap A_5| =$

Combinations with Repetitions

Quick review

1. How many ways are there to choose k elements out of the set $\{1 \cdot a_1, 1 \cdot a_2, \cdots, 1 \cdot a_n\}$?

2. How many ways are there to choose k elements out of the set $\{k \cdot a_1, k \cdot a_2, \cdots, k \cdot a_n\}$? $\left(\text{really } \{\infty \cdot a_1, \infty \cdot a_2, \cdots, \infty \cdot a_n\}\right)$

What we would like to calculate is:

In how many ways can we choose k elements out of an arbitrary multiset?

Now, it's as easy as PIE.

Combinations with Repetitions

Example. Determine the number of 10-combinations of the multiset $S = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}$.

Game plan: Let $\mathcal{U}$ be the set of 10-combs of $\{\infty \cdot a, \infty \cdot b, \infty \cdot c\}$. Use PIE to remove the 10-combs that violate the conditions of S

Define A_1 to be 10-combs that include at least ___ a's.
Define A_2 to be 10-combs that include at least ___ b's.
Define A_3 to be 10-combs that include at least ___ c's.

In words, $A_1 \cap A_2$ are those 10-combs that
$A_1 \cap A_3$: $A_2 \cap A_3$:
$A_1 \cap A_2 \cap A_3$

Now calculate: $|\mathcal{U}| =$ $|A_1| =$ $|A_2| = \left(\!\binom{3}{5}\!\right)$ $|A_3| = \left(\!\binom{3}{4}\!\right)$
$|A_1 \cap A_2| = 3$ $|A_1 \cap A_3| = 1$ $|A_2 \cap A_3| = 0$ $|A_1 \cap A_2 \cap A_3| = 0$

And finally: So $|\mathcal{U}| - |A_1 \cup A_2 \cup A_3| =$

Derangements

At a party, 10 partygoers check their hats. They “have a good time”, and are each handed a hat on the way out. In how many ways can the hats be returned so that no one is returned his/her own hat?

This is a derangement of ten objects.

Definition: An n-derangement is an n-permutation $\pi = p_1 p_2 \cdots p_n$ such that $p_1 \neq 1,\ p_2 \neq 2,\ \cdots,\ p_n \neq n$.

Note: A derangement is a permutation without fixed points $\pi(i) = i$.

Notation: We let D_n be the number of all n-derangements.

When you see D_n, think combinatorially: “The number of ways to return n hats to n people so no one gets his/her own hat back”

Question: Compute a formula for D_n.

Calculating the number of derangements

Solution. Let $\mathcal{U}$ be the set of all n-permutations.
We will remove all bad permutations using Inclusion / Exclusion.

Define A_i (for all $i \in [n]$) to be the set of n-perms where $p_i = i$.

In words, $A_i \cap A_j$ are n-perms where

$A_i \cap A_j \cap A_k$ are n-perms where

In general, $A_{i_1} \cap \cdots \cap A_{i_k}$ are n-perms with $p_{i_1} = i_1, \cdots, p_{i_k} = i_k$.

Now calculate: $|\mathcal{U}| =$

$|A_1| =$ $|A_2| =$

For all i and j, $|A_i \cap A_j| =$

When intersecting k sets, $|A_{i_1} \cap \cdots \cap A_{i_k}| =$

How many ways are there to intersect k sets?

We conclude:

$$D_n = |\mathcal{U}| - \sum |A_i| + \sum |A_i \cap A_j| - \sum \left|A_i \cap A_j \cap A_k\right| + \cdots$$