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We can color v with a color not used to color the neighbors of v, and we have a proper 6-coloring of G, contradicting the definition of G.

The Five Color Theorem

Theorem. Let G be a planar graph.
There exists a proper 5-coloring of G.

Proof. Let G be a the smallest planar graph (by number of vertices) that has no proper 5-coloring.

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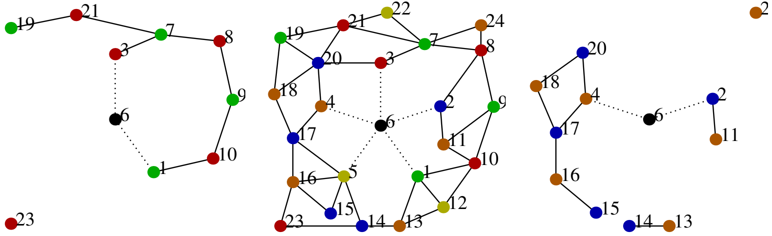
Color the vertices of $G \setminus v$ with five colors; the neighbors of v in G are colored by at most five different colors.

If they are colored with only four colors,
we can color v with a color not used to color the neighbors of v, and we have a proper 5-coloring of G, contradicting the definition of G.

The Kempe Chains Argument

Otherwise the neighbors of v are all colored differently.



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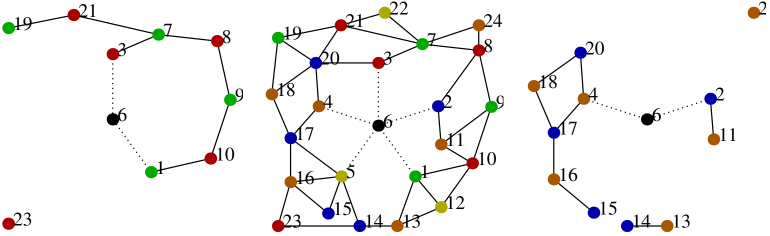

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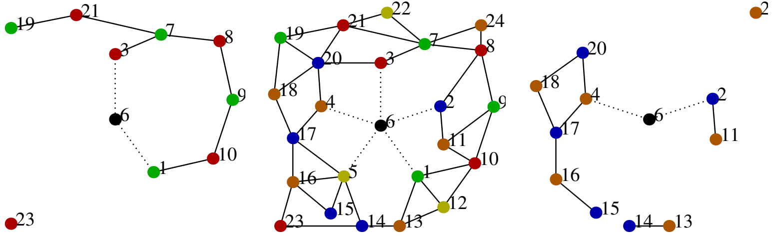

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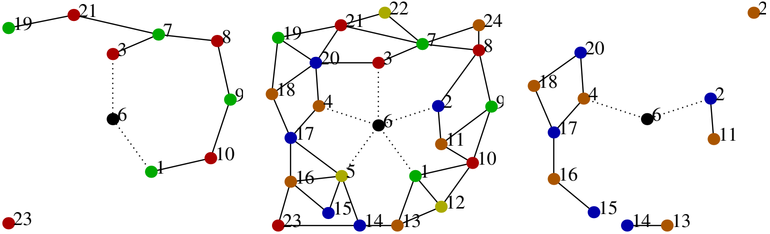

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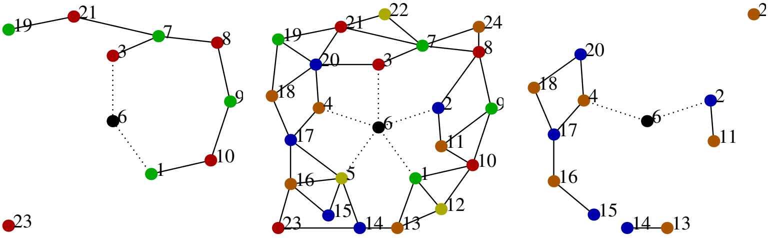

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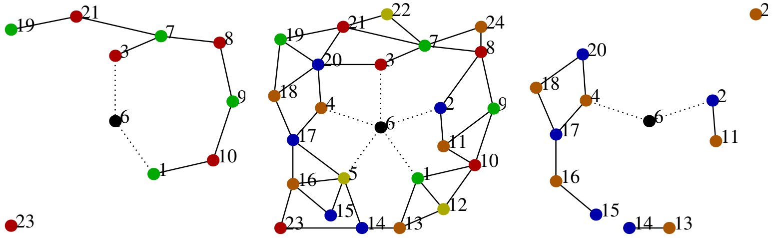

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Definition. A **Kempe chain** is a path in $G \setminus v$ between two non-consecutive neighbors of v such that the colors on the vertices of the path *alternate* between the colors on those two neighbors.

In our example, $v_3 \to v_7 \to v_8 \to v_9 \to v_{10} \to v_1$ is a Kempe chain: colors alternate between red and green & v_1 and v_3 not consecutive.

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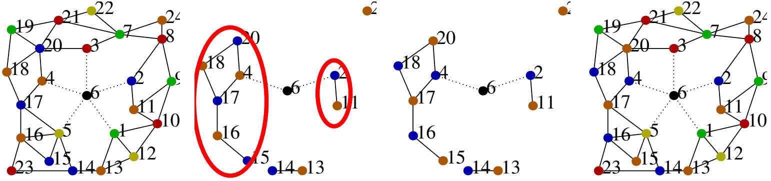
We ask: Are v_2 and v_4 in the **same component** of $H_{2,4}$?



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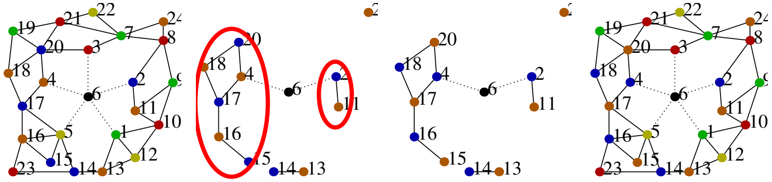

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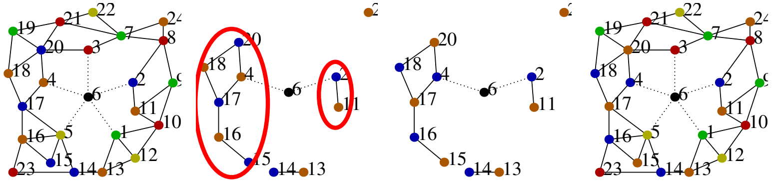
- If they are, there is a Kempe chain between v_2 and v_4.
- If not, we can swap colors 2 and 4 in **one** component $\mathcal{C}$ of $H_{2,4}$.



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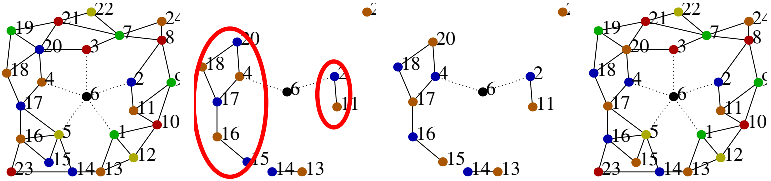
Claim. Swapping colors in $\mathcal{C}$ **is still** a proper coloring of $G \setminus v$.



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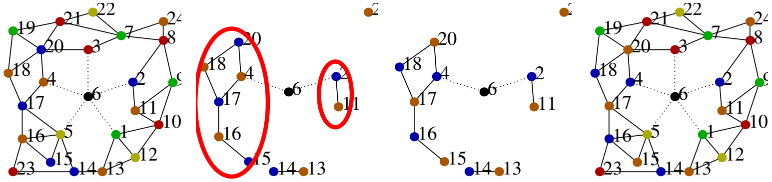

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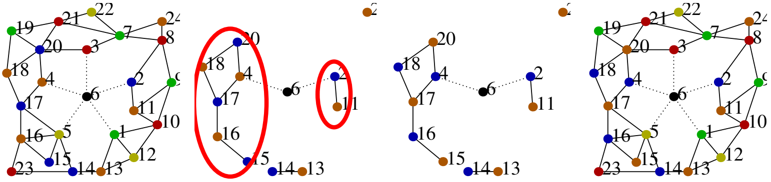

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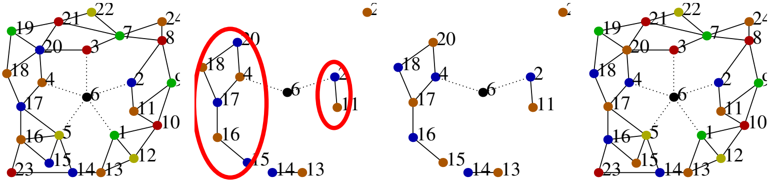
By construction, neighboring vertices in $G \setminus \mathcal{C}$ are not colored 2 or 4, so they do not present any conflicts before AND after recoloring. $\square$



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So **either** there is a Kempe chain between v_2 and v_4 **or** we can swap colors so that v's neighbors are colored only using four colors.

Similarly, **either** there is a Kempe chain between v_1 and v_3 **or** we can swap colors to color v's neighbors with only four colors.

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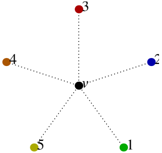

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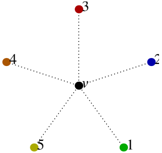

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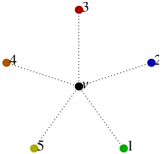

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This can not exist, so it must be possible to swap colors and be able to place a fifth color on v, contradicting the definition of G.