The Six Color Theorem

Theorem. Let G be a planar graph.
There exists a proper 6-coloring of G.

Proof. Let G be a the smallest planar graph (by number of vertices) that has no proper 6-coloring.

By Theorem 8.1.7, there exists a vertex v in G that has degree five or less. $G \setminus v$ is a planar graph smaller than G, so it has a proper 6-coloring.

Color the vertices of $G \setminus v$ with six colors; the neighbors of v in G are colored by at most five different colors.

We can color v with a color not used to color the neighbors of v, and we have a proper 6-coloring of G, contradicting the definition of G.

The Five Color Theorem

Theorem. Let G be a planar graph.
There exists a proper 5-coloring of G.

Proof. Let G be a the smallest planar graph (by number of vertices) that has no proper 5-coloring.

By Theorem 8.1.7, there exists a vertex v in G that has degree five or less. $G \setminus v$ is a planar graph smaller than G, so it has a proper 5-coloring.

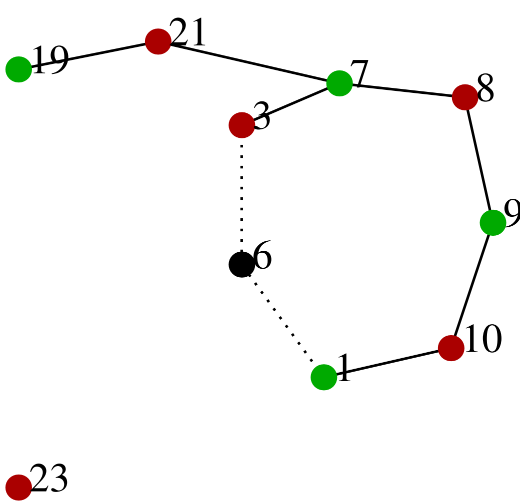
Color the vertices of $G \setminus v$ with five colors; the neighbors of v in G are colored by at most five different colors.

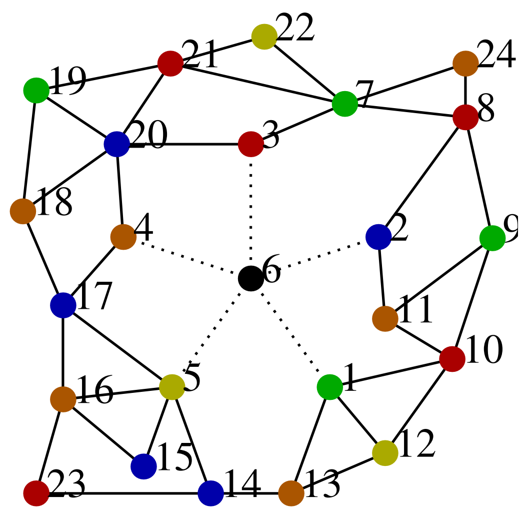
If they are colored with only four colors,
we can color v with a color not used to color the neighbors of v, and we have a proper 5-coloring of G, contradicting the definition of G.

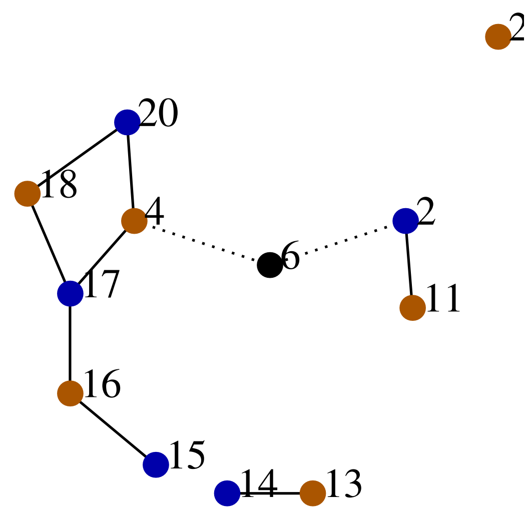
The Kempe Chains Argument

Otherwise the neighbors of v are all colored differently.
We will modify the coloring on $G \setminus v$ so **only four** colors are used.

Construct the subgraphs $H_{1,3}$ and $H_{2,4}$ of $G \setminus v$ as follows:
Let $V_{1,3}$ be the set of vertices in $G \setminus v$ colored with colors 1 or 3.
Let $V_{2,4}$ be the set of vertices in $G \setminus v$ colored with colors 2 or 4.
Let $H_{1,3}$ be the induced subgraph of G on $V_{1,3}$. (Define $H_{2,4}$ similarly)







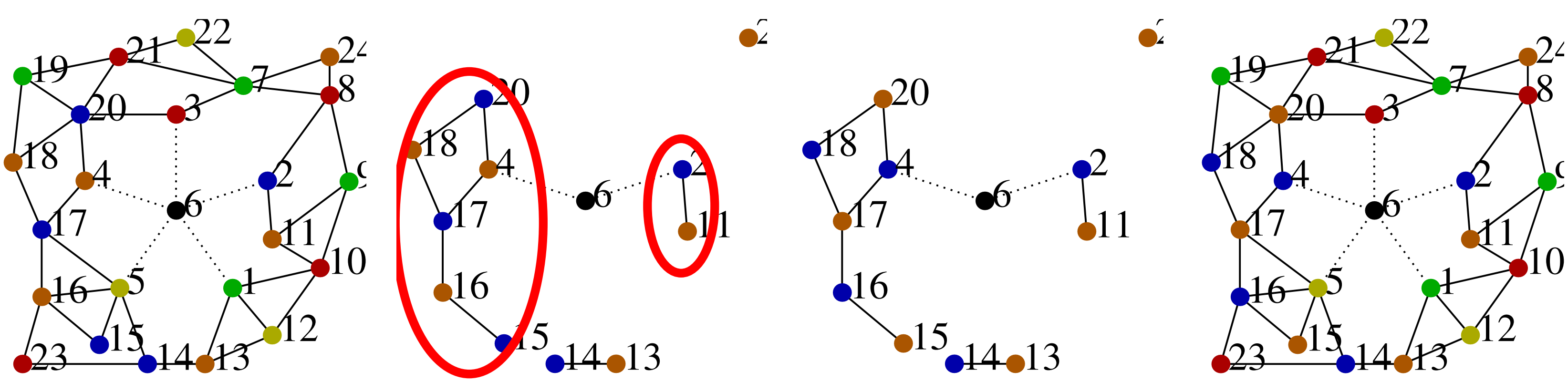
The Kempe Chains Argument

Definition. A **Kempe chain** is a path in $G \setminus v$ between two non-consecutive neighbors of v such that the colors on the vertices of the path *alternate* between the colors on those two neighbors.

In our example, $v_3 \to v_7 \to v_8 \to v_9 \to v_{10} \to v_1$ is a Kempe chain: colors alternate between red and green & v_1 and v_3 not consecutive.

For any two non-consecutive neighbors of v, (such as: v_2 and v_4.)
We ask: Are v_2 and v_4 in the **same component** of $H_{2,4}$?

- If they are, there is a Kempe chain between v_2 and v_4.
- If not, we can swap colors 2 and 4 in **one** component C of $H_{2,4}$.



The Kempe Chains Argument

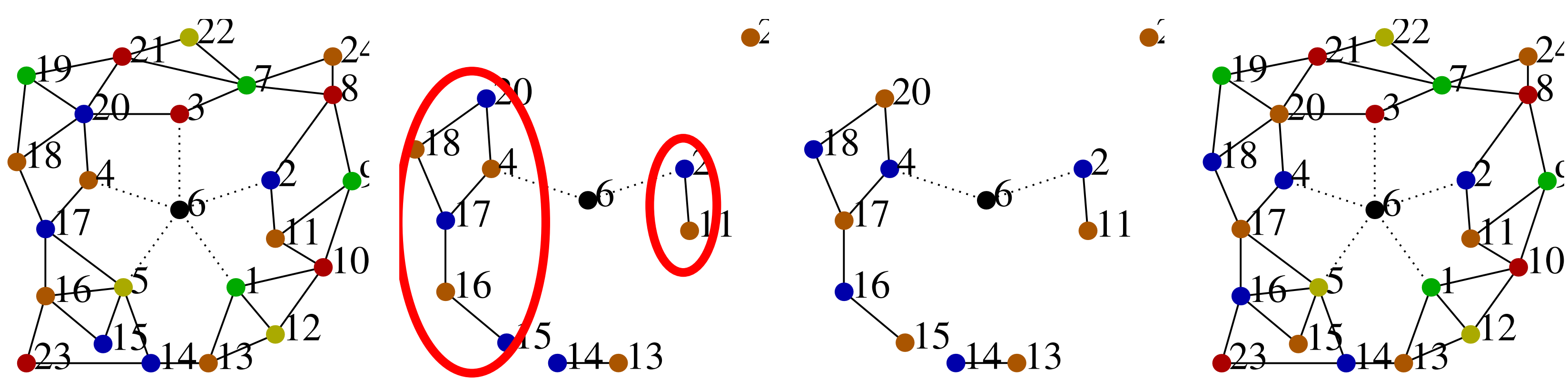
Claim. Swapping colors in $\mathcal{C}$ **is still** a proper coloring of $G \setminus v$.

Proof. We need to check that this recoloring is still proper. The only adjacencies we have to check are within $\mathcal{C}$ and with neighbors of $\mathcal{C}$.

$\mathcal{C}$ is a bipartite graph with vertices of color 2 and 4.

Swapping colors does not change this. Adjacent vertices in the newly colored $\mathcal{C}$ will be colored differently.

By construction, neighboring vertices in $G \setminus \mathcal{C}$ are not colored 2 or 4, so they do not present any conflicts before AND after recoloring. $\square$

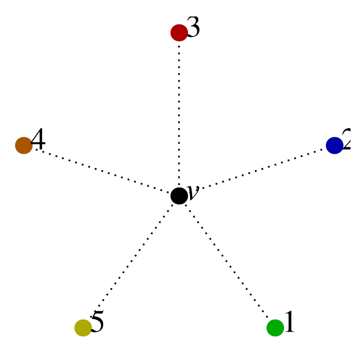


The Kempe Chains Argument

So **either** there is a Kempe chain between v_2 and v_4 **or** we can swap colors so that v's neighbors are colored only using four colors.

Similarly, **either** there is a Kempe chain between v_1 and v_3 **or** we can swap colors to color v's neighbors with only four colors.

Question. Can we have both a v_1-v_3 and a v_2-v_4 Kempe chain?



There are no edge crossings in the graph drawing, so there would exist a vertex ________________________________.

This can not exist, so it must be possible to swap colors and be able to place a fifth color on v, contradicting the definition of G.