# Planarity

Up until now, graphs have been completely abstract.

In Topological Graph Theory, it matters how the graphs are drawn.

- Do the edges cross?
- Are there knots in the graph structure?

*Definition.* A **drawing** of a graph $G$ is a pictorial representation of $G$ in the plane as points and curves, satisfying the following:

- The curves must be **simple**, which means no self-intersections.
- No two edges can intersect twice. (Mult. edges: Except at endpts)
- No three edges can intersect at the same point.

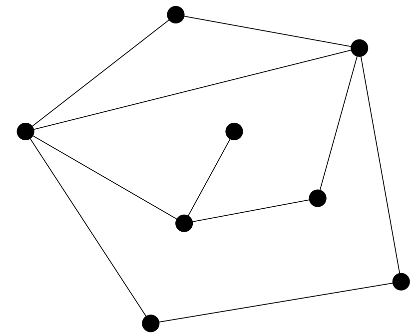
*Definition.* A **plane drawing** of a graph $G$ is a drawing of the graph in the plane with no crossings.

*Definition.* A graph $G$ is **planar** if there exists a plane drawing of $G$. Otherwise, we say $G$ is **nonplanar**.

*Example.* $K_4$ is planar because there exists a plane drawing of $K_4$.

# Vertices, Edges, and Faces

*Definition.* In a plane drawing, edges divide the plane into **regions**, or **faces**.



There will always be one face with infinite area. This is called the **outside face**.

*Notation.* Let $p = \#$ of vertices, $q = \#$ of edges, $r = \#$ of regions. Compute the following data:

| Graph | $p$ | $q$ | $r$ | |
|---|---|---|---|---|
| Tetrahedron | | | | |
| Cube | | | | |
| Octahedron | | | | |
| Dodecahedron | | | | |
| Icosahedron | | | | |

In 1750, Euler noticed that ____________ in each of these examples.

# Euler's Formula

*Theorem 8.1.1* (Euler's Formula) If $G$ is a connected planar graph, then in a plane drawing of $G$, $p - q + r = 2$.

*Proof.* (by induction on the number of cycles)

**Base Case**: If $G$ is a connected graph with no cycles, then $G$ _____ Therefore $r =$___, and we have $p - q + r = p - (p - 1) + 1 = 2$.

**Inductive Hypothesis**: Suppose that for all plane drawings with fewer than $k$ cycles, we have $p - q + r = 2$.
**Want to show**: In a plane drawing of a graph $G$ with $k$ cycles, $p - q + r = 2$ also holds.

Let $C$ be a cycle in $G$, and $e$ be an edge of $C$. Edge $e$ is adjacent to:

___________________________________________

Now remove $e$: Define $H = G \setminus e$. Now $H$ has fewer cycles than $G$, and one fewer region. The inductive hypothesis holds for $H$, giving:

# Maximal Planar Graphs

A graph with "too many" edges cannot be planar.

*Goal:* Find a numerical characterization of "too many"

*Definition.* A planar graph is called **maximal planar** if adding an edge between any two non-adjacent vertices results in a non-planar graph.

*Examples:* Octahedron $K_5 \setminus e$

What do we notice about these graphs?

# Numerical Conditions on Planar Graphs

*Claim.* Every face of a maximal planar graph is a ____________!

*Proof.* Otherwise,

*Theorem 8.1.2.* If $G$ is maximal planar and $p \geq 3$, then $q = 3p - 6$.

*Proof.* Consider any plane drawing of $G$.
Let $p = \#$ of vertices, $q = \#$ of edges, and $r = \#$ of regions.

We count the number of face-edge incidences in two ways:

From a face-centric POV, the number of face-edge incidences is

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Substituting into Euler's formula: $p - q + (2q/3) = 2$, so

*Question.* Do we need $p \geq 3$?

# Numerical Conditions on Planar Graphs

*Corollary 8.1.3.* Every planar graph with $p \geq 3$ vertices has at most $3p - 6$ edges.

- Start with any planar graph $G$ with $p$ vertices and $q$ edges.
- Add edges to $G$ until it is maximal planar. (with $Q \geq q$ edges.)
- This resulting graph satisfies $Q = 3p - 6$; hence $q \leq 3p - 6$.

*Theorem 8.1.4.* The graph $K_5$ is not planar.

*Proof.*

*Theorem 8.1.7.* Every planar graph has a vertex with degree $\leq 5$.

*Proof.*

# Numerical Conditions on Planar Graphs

Recall: The **girth** $g(G)$ of a graph $G$ is the smallest cycle size.

*Theorem 8.1.5.*$^{\star}$ If $G$ is planar with girth $\geq 4$, then $q \leq 2p - 4$.

*Proof.* Modify the above proof—instead of $3r = 2q$, we know $4r \leq 2q$. This implies that

$$2 = p - q + r \leq p - q + \frac{2q}{4} = p - \frac{q}{2}.$$

Therefore, $q \leq 2p - 4$.

*Theorem 8.1.5.* If $G$ is planar and bipartite, then $q \leq 2p - 4$.

*Theorem 8.1.6.* $K_{3,3}$ is not planar.