Reminder: Directed Graphs

Definition. A **directed graph** (or **digraph**) is a graph $G = (V, E)$, where each edge $e = vw$ is directed from one vertex to another:

$$e : v \to w \quad \text{or} \quad e : w \to v.$$

Remark. The edge $e : v \to w$ is different from $e' : w \to v$ and a digraph including both is not considered to have multiple edges.

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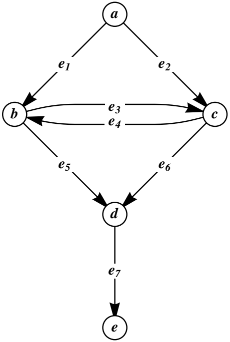

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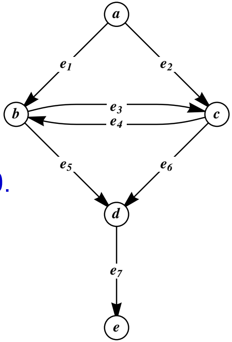

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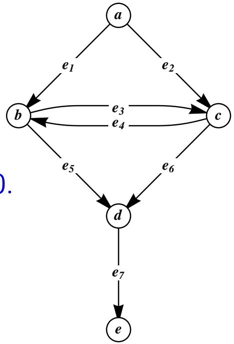

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Important. Any **path** or **cycle** in a digraph must respect the direction on each edge.

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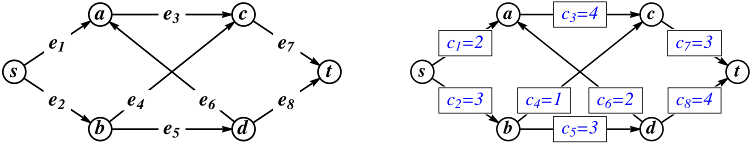

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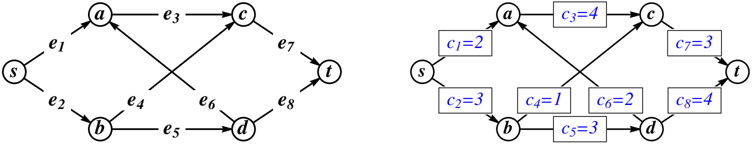

Idea. Graph networks represent real-world networks such as traffic, water, communication, etc.

Goal: Send as much "stuff" from s to t while respecting capacities.

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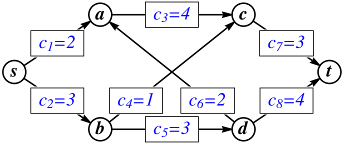

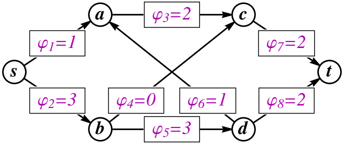

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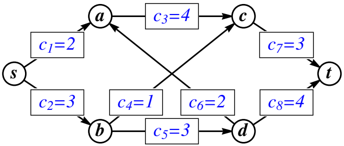

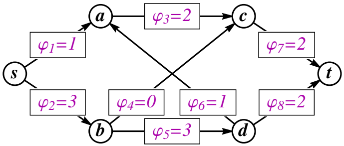

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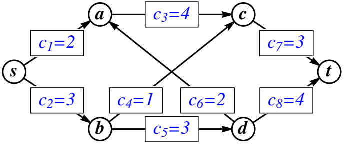

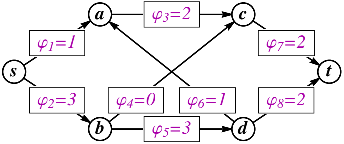

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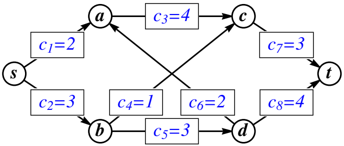

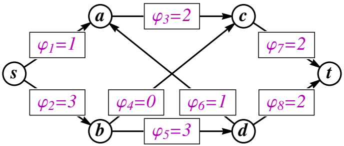

Definition. When $\varphi_e = c_e$, we say that e is **saturated**, or **at capacity**.

Maximum Flow

Theorem. Given a flow $\vec{\varphi}$ on a network G, the net flow out of s is equal to the net flow into t. Symbolically, $\sum_{e \text{ out of } s} \varphi_e = \sum_{e \text{ into } t} \varphi_e$.

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Proof. Create a new network G' by adding to G an edge $e_\infty : t \to s$ with infinite capacity, and place flow

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In G', flow is now conserved at every vertex except possibly t.
By Kirchhoff's Global Current Law (Theorem 6.2.2),
flow must be conserved at t as well.

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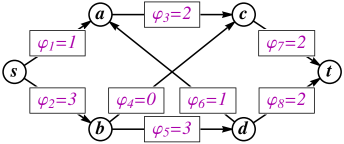
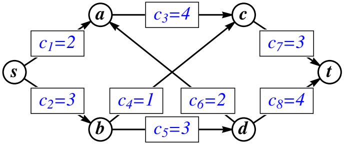
Definition. The **throughput** or **value** of a flow $\vec{\varphi}$ is $\sum_{e \text{ out of } s} \varphi_e$, denoted $|\vec{\varphi}|$.

Idea: The throughput is the amount of "stuff" flowing through G.

In our example, $|\vec{\varphi}| =$____________________.





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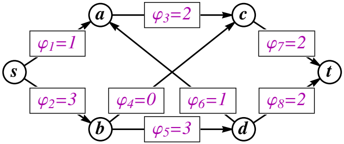

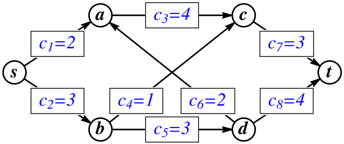

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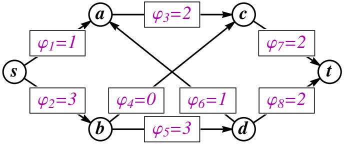

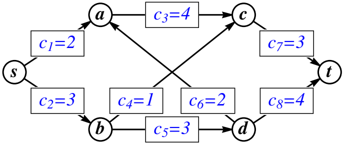

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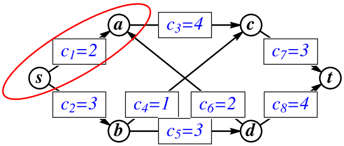
Definition. Let G be a network. Let X be a set of vertices containing s and not containing t. An st-**cut** $[X, X^c]$ is the **set of edges** between a vertex in X and a vertex in X^c (in either direction).



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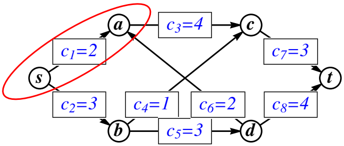

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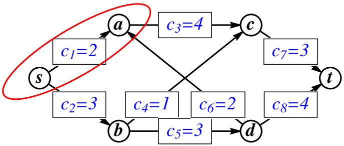

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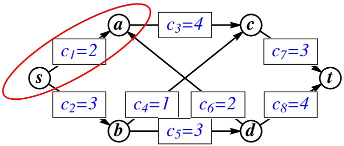

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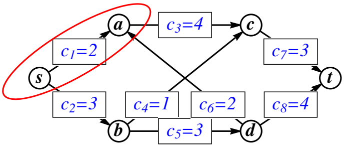

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⋆ Do **not** subtract the capacities of the edges going the other way. ⋆

Max Flow / Min Cut

Goal: For a given network, find the st-cut with the smallest capacity.

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So, IF there exists a flow $\vec{\varphi}$ and st-cut $[X^*, X^{*c}]$ where equality holds, then $\vec{\varphi}$ is a maximum flow and $[X^*, X^{*c}]$ is a minimum cut.

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Theorem. (Ford, Fulkerson, 1955) In any network G, the value of any maximum flow is equal to the capacity of any minimum cut.

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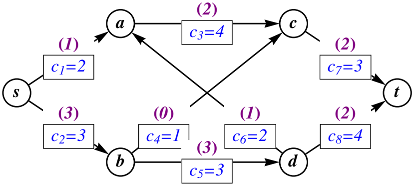

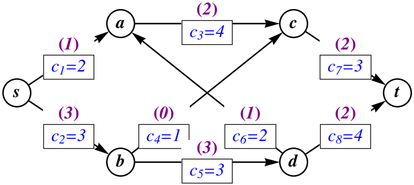

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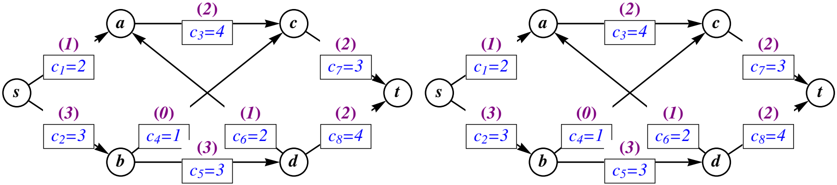

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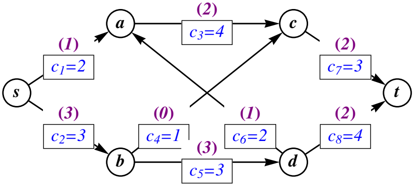

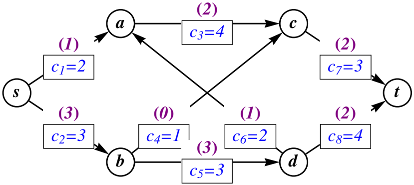

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So: Use a *companion graph* to keep track of augmentable edges/paths.

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3. ★ If there IS an st-path in the flow companion graph, send as many units of flow as possible through this path. Repeat Step 2.

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Theorem. (Ford, Fulkerson, 1955) In any network G, the value of any maximum flow is equal to the capacity of any minimum cut.

Proof. Use the **Ford–Fulkerson Algorithm**, which finds a max flow.

1. Start with any flow $\vec{\varphi}$ on G.
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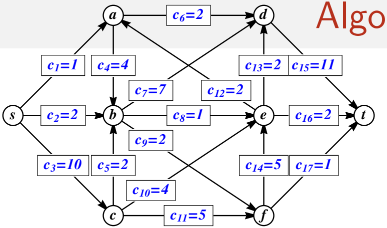
$\longrightarrow$ Upon STOP, the current flow is a maximum flow. $\longleftarrow$

In addition, let X be the set of vertices **reachable from s** in the flow companion graph. Then $[X, X^c]$ is a minimum st-cut.

A Ford–Fulkerson Algorithm Example



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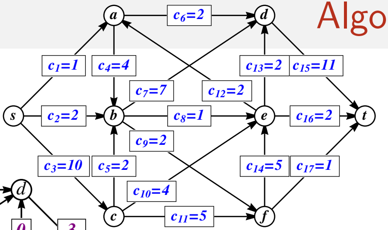

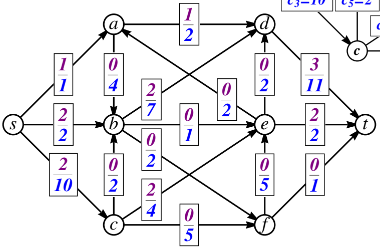

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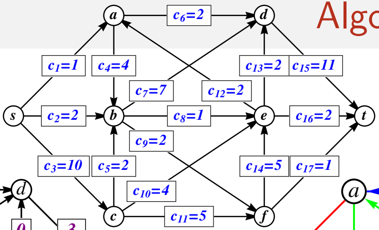

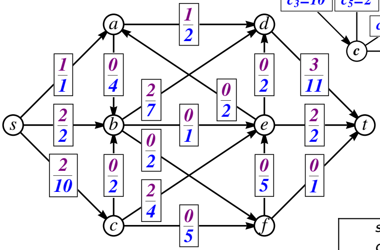

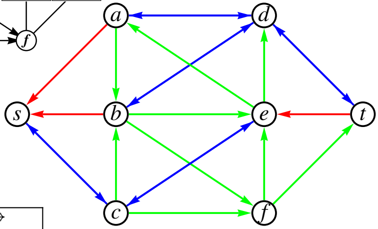

$s \to c \to b \to d \to t$: 2 units
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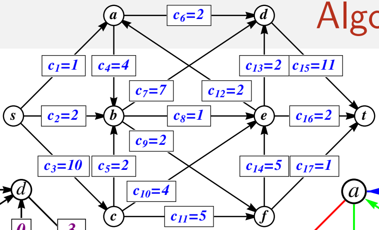

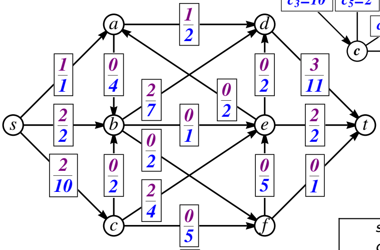

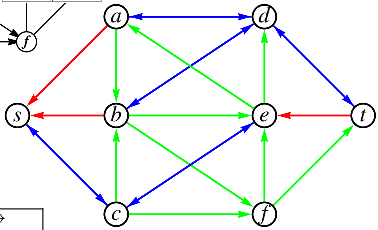

$s \to c \to b \to$ $d \to t$: 2 units
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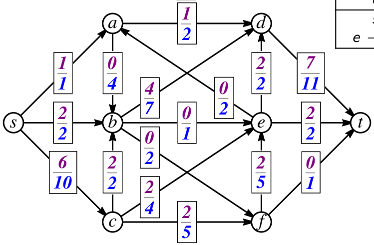

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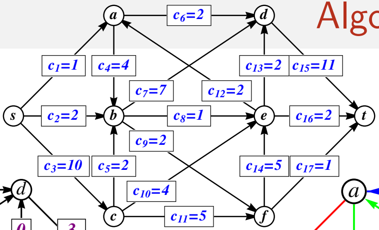

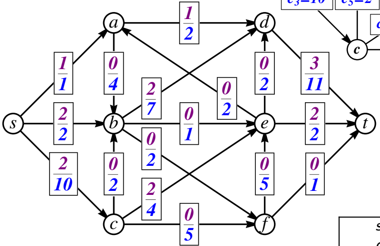

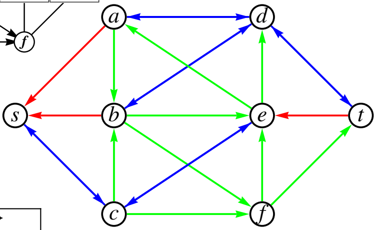

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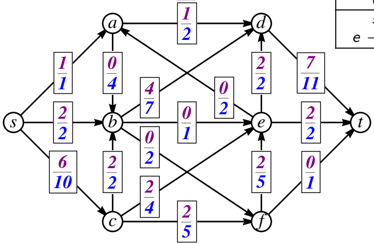

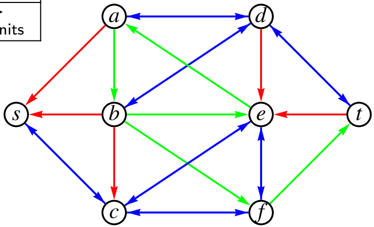

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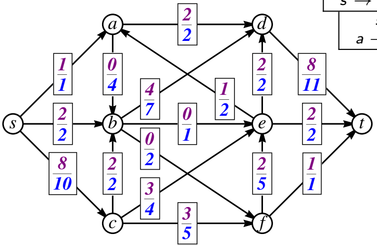

$s \to c \to f \to t$: 1 unit

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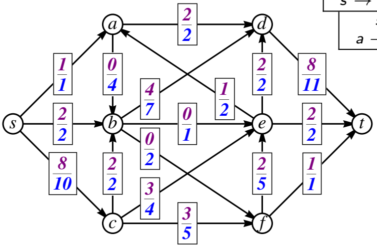

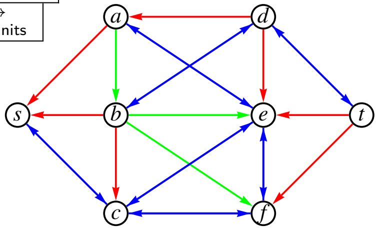

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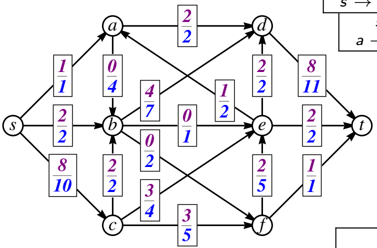

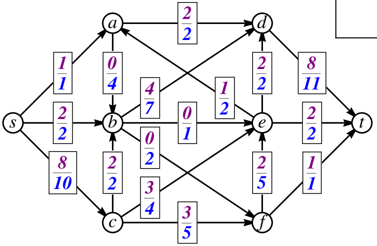

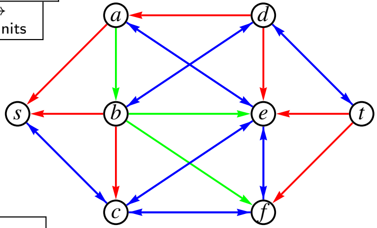

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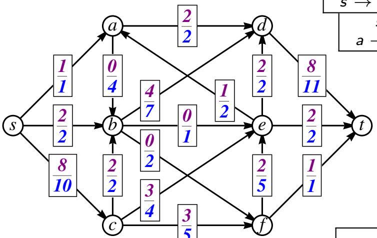

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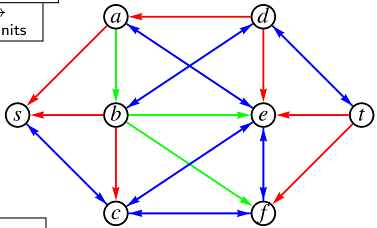

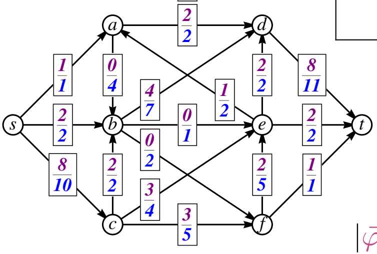

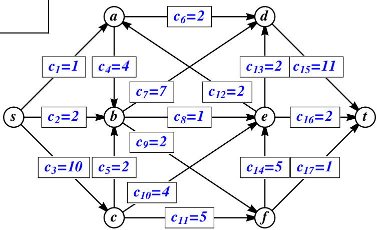

$|\vec{\varphi}| =$ ______

$X = \{$________$\}$, $[X, X^c] = \{$____________$\}$, and $|[X, X^c]| =$ ____.

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Conclusion. The flow is a max flow and the st-cut is a min cut.

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- As presented here, this algorithm may be very slow.