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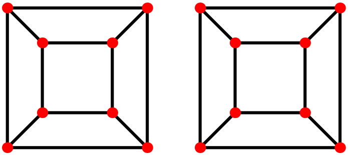

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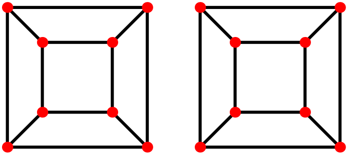

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Definition. The minimum number of colors necessary to properly edge color a graph G is called the **edge chromatic number** of G, denoted $\chi'(G) =$ "chi prime".

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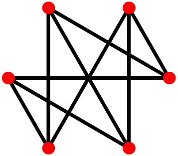
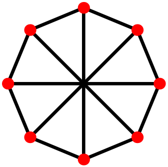
Fact: **Most** 3-regular graphs have edge chromatic number 3.





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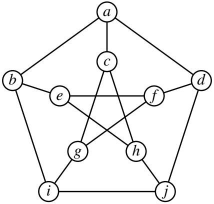

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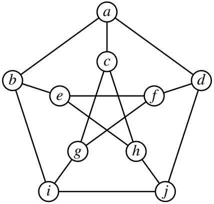

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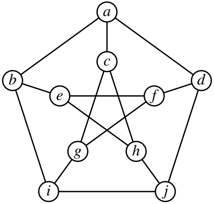

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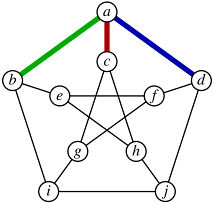

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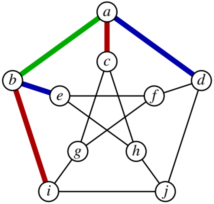

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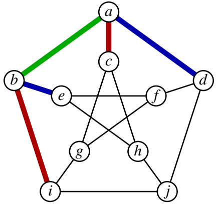

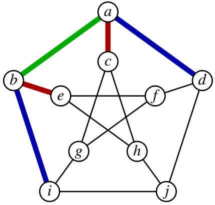

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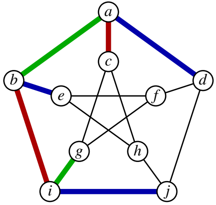

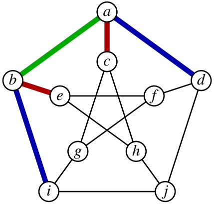

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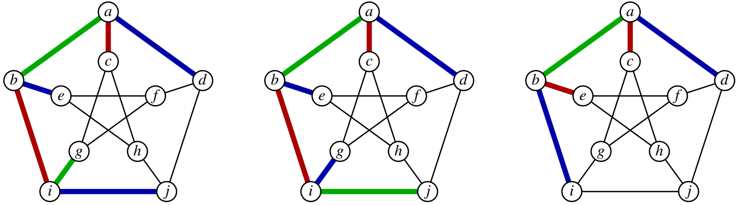

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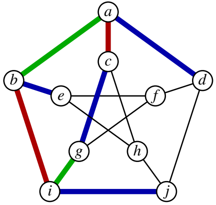

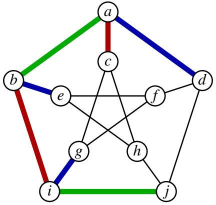

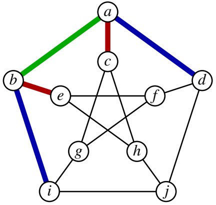

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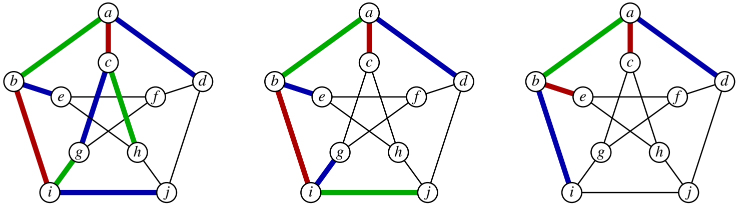

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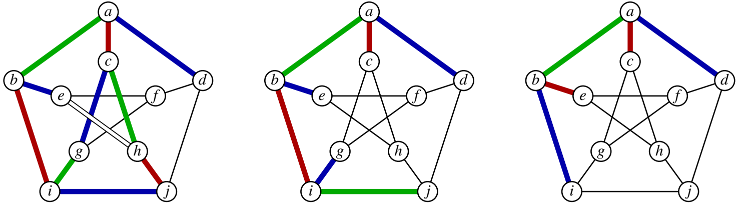

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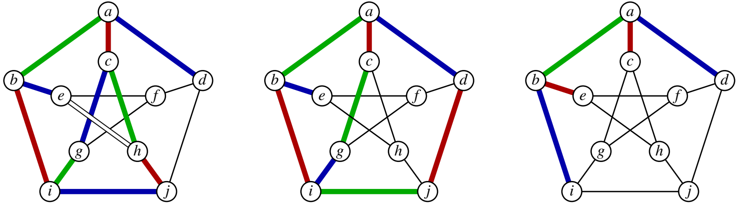

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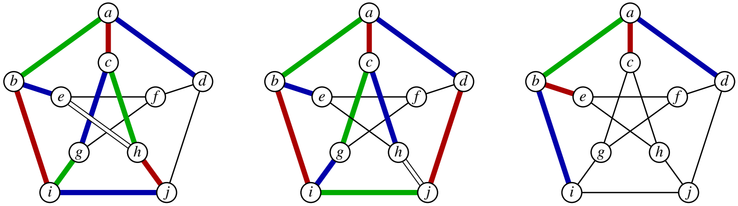

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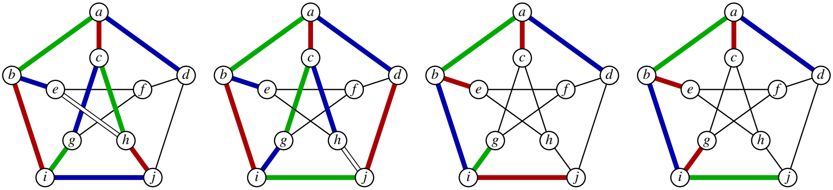

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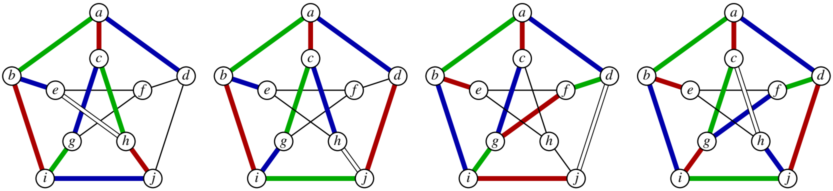

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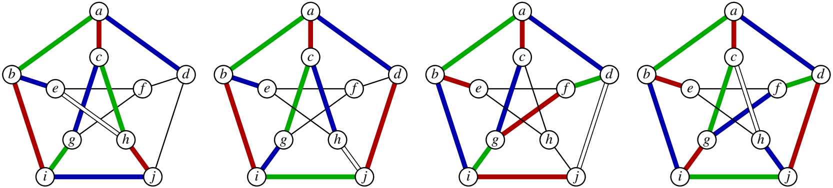

In all cases, it is not possible to edge color with 3 colors, so $\chi'(G) = 4$.

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Goal: Determine $\chi'(K_n)$ for all n.

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This is only an integer when:

So, the best we can expect is that $\begin{cases} \chi'(K_{2n}) = \\ \chi'(K_{2n-1}) = \end{cases}$

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Connect 0 with x,

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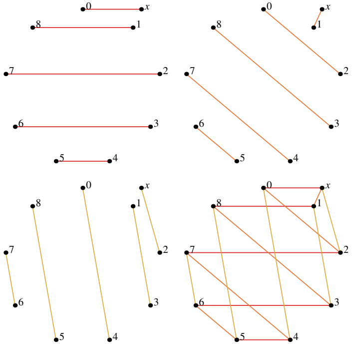

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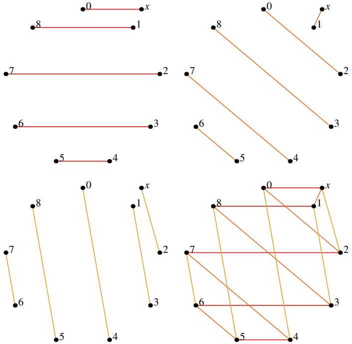

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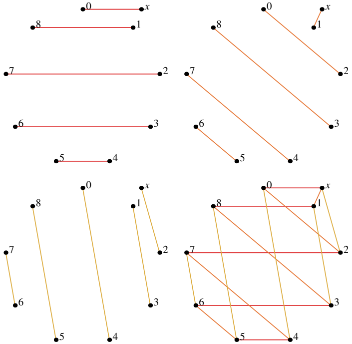

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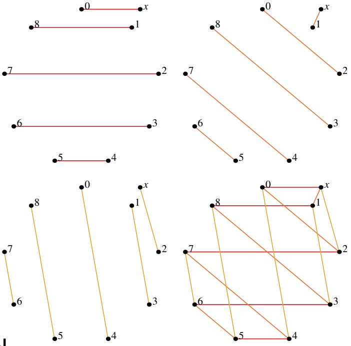

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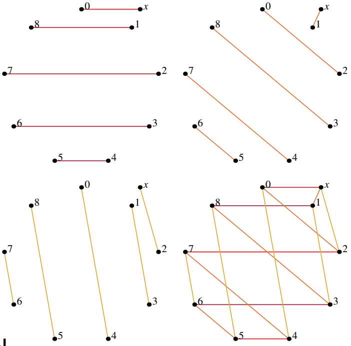

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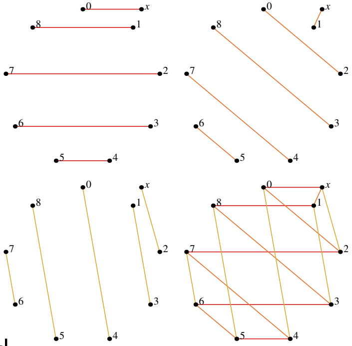

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Vertices are at circular distance 1, 3, 5, . . ., 4, 2 from each other,
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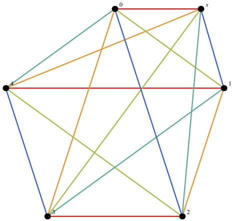
Day 1	0x	14	23
Day 2	1x	20	34
Day 3	2x	31	40
Day 4	3x	42	01
Day 5	4x	03	12



Theorem 2.2.3 proves there is such a tournament for all even numbers.