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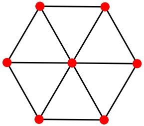
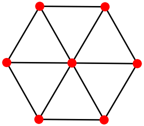
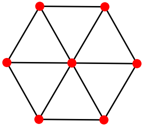

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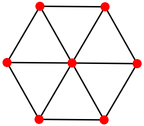

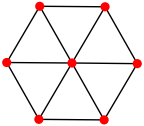

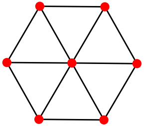

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Of interest: What is the fewest colors necessary to properly color G?

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1. There is a proper coloring of G with k colors. (Show it!)
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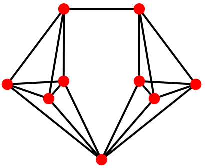
Example. Calculate $\chi(G)$ for this graph G:



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Since _________, there will be some proper subgraph G_l of G_{l-1} such that G_l is critical and $\chi(G_l) = \chi(G_{l-1}) = \cdots = \chi(G)$.

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Similarly: If G is critical, then for all $v \in V(G)$, $\deg(v) \geq \chi(G) - 1$.

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$(\Rightarrow)$ Let G be bipartite. Assume that there is some cycle C of odd length contained in G . . .

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This contradicts our hypothesis, so a 2-coloring exists; G is bipartite.