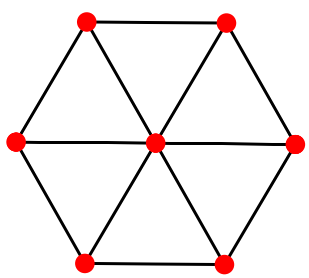
(Vertex) Colorings

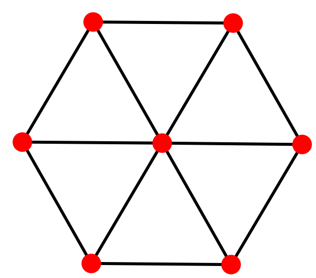
Definition. A **coloring** of a graph G (with c colors) is a function $f : V(G) \to \{1, 2, \ldots, c\}$.

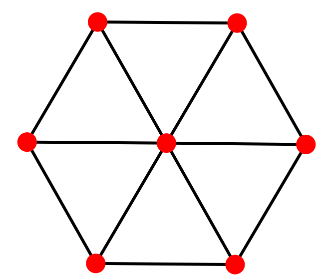
In other words, we assign colors to each of the vertices of G.

Definition. A **proper coloring** of G is a coloring of G such that no two adjacent vertices are labeled by the same color.

Example. W_6:







We can properly color W_6 with ____ colors and no fewer.

Of interest: What is the fewest colors necessary to properly color G?

The chromatic number of a graph

Definition. The minimum # of colors necessary to properly color a graph G is called the **chromatic number** of G, denoted $\chi(G)$. (chi)

Example. Find $\chi(K_n)$.

Proof. A proper coloring of K_n must use at least ____ colors, because every vertex is adjacent to every other vertex. With fewer than ____ colors, there would be two adjacent vertices colored the same. And indeed, placing a different color on each vertex is a proper coloring of K_n.

$\chi(G) = k$ is the same as:

1. There is a proper coloring of G with k colors. (Show it!)
2. There is no proper coloring of G with $k-1$ colors. (Prove it!)

Chromatic numbers and subgraphs

Lemma C: If H is a subgraph of G, then $\chi(H) \le \chi(G)$.

Proof. If $\chi(G) = k$, then ______________________________.
Let the vertices of H inherit their coloring from G.
This gives a proper coloring of H using k colors.
In turn, this implies $\chi(H) \le k$.

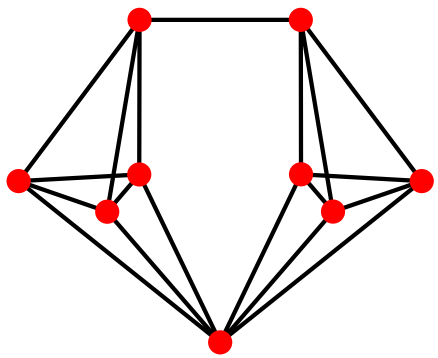
If G contains a **clique** of size k (subgraph isomorphic to K_k), then what can we say about $\chi(G)$?

Definition. The **clique number** $\omega(G)$ is the size of the largest complete graph contained in G.

Theorem. For any graph G, $\chi(G) \ge \omega(G)$.

Proof. Apply Lemma C to the subgraph of G isomorphic to $K_{\omega(G)}$.

Example. Calculate $\chi(G)$ for this graph G:



Critical graphs

How to prove $\chi(G) \geq k$?

One way: Find a (small) subgraph H of G that requires k colors.

Definition. A graph H is called **critical** if for every proper subgraph $J \subsetneq H$, then $\chi(J) < \chi(H)$.

Theorem 2.1.2: Every graph G contains a critical subgraph H such that $\chi(H) = \chi(G)$.

(Stupid) Proof. If G is critical, stop. Define $H = G$.

If not, then there exists a proper subgraph G_1 of G with ________.

If G_1 is critical, stop. Define $H = G_1$.

If not, then there exists a proper subgraph G_2 of G_1 with ________.

If G_2 is critical, stop. Define $H = G_2$.

If not, then there exists $\cdots$

Since ________, there will be some proper subgraph G_l of G_{l-1} such that G_l is critical and $\chi(G_l) = \chi(G_{l-1}) = \cdots = \chi(G)$.

Critical graphs

What do we know about critical graphs?

Thm 2.1.1: Every critical graph is connected.

Thm 2.1.3: If G is critical and $\chi(G) = 4$, then $\deg(v) \geq 3$ for all v.

Proof. Suppose not. Then there is some $v \in V(G)$ with $\deg(v) \leq 2$. Remove v from G to create H.

Similarly: If G is critical, then for all $v \in V(G)$, $\deg(v) \geq \chi(G) - 1$.

Bipartite graphs

Question. What is $\chi(C_n)$ when n is odd?

Answer.

Definition. A graph is called **bipartite** if $\chi(G) \leq 2$.

Example. $K_{m,n}$, $\square_n$, Trees

Thm 2.1.6: G is bipartite $\iff$ every cycle in G has even length.

($\Rightarrow$) Let G be bipartite. Assume that there is some cycle C of odd length contained in $G \ldots$

Proof of Theorem 2.1.6

($\Leftarrow$) Suppose that every cycle in G has even length. We want to show that G is bipartite. Consider the case when G is connected.

Plan: Construct a coloring on G and prove that it is proper.

Choose some starting vertex x and color it blue. For every other vertex y, calculate the distance from y to x and then color y:

$$\begin{cases} \text{blue} & \text{if } d(x, y) \text{ is even.} \\ \text{red} & \text{if } d(x, y) \text{ is odd.} \end{cases}$$

Question: Is this a proper coloring of G?

If not, then there are two adjacent vertices v and w of the same color.

Claim 1: Their distance to the x is the same.

Claim 2: There exists an odd cycle in G.

This contradicts our hypothesis, so a 2-coloring exists; G is bipartite.