Course Notes

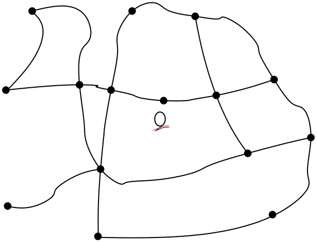
Graph Theory, Fall 2022

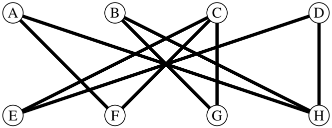
Queens College, Math 334/634

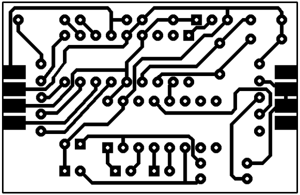
Prof. Christopher Hanusa

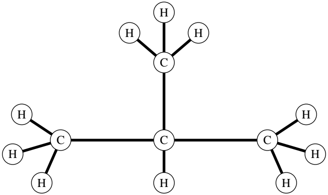
http://qc.edu/~chanusa/courses/634/22/

What is a graph?









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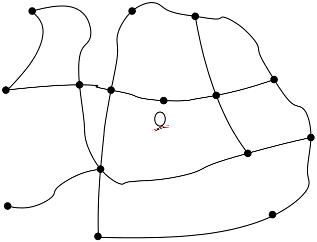

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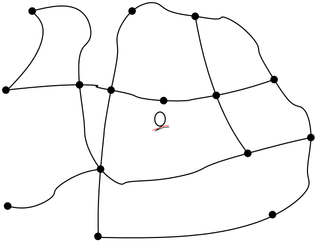

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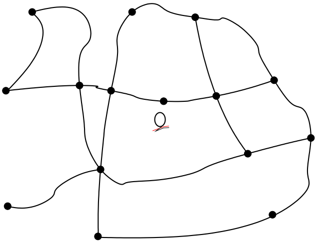

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- Represent each city or intersection as a vertex
- Roads correspond to edges.

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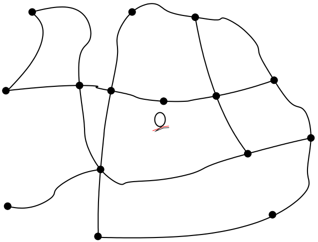

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However, a graph is an abstract concept.

- It doesn't matter whether the edge is straight or curved.
- All we care about is which vertices are connected.

Concept: Matchings

Suppose that:

Erika likes cherries and dates.
Frank likes apples and cherries.
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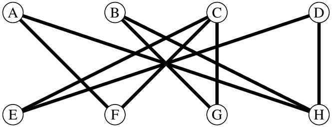

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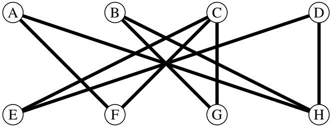

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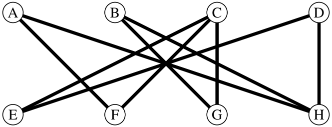

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Related topics: assignments, perfect matchings, counting questions.

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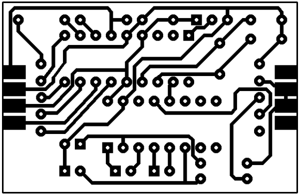

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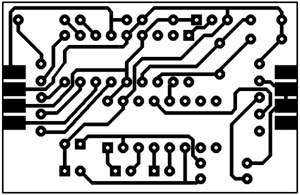

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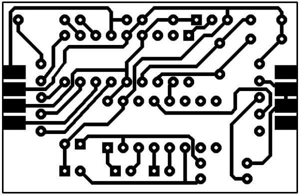

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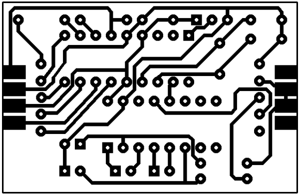

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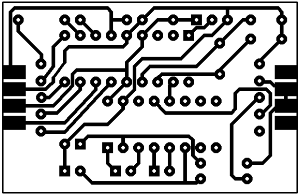

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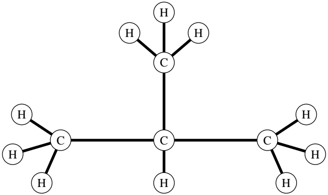
Related topics: Traveling Salesman, computer algorithms, optimization

Chemis-Tree

Graphs are used in Chemistry to draw molecules. (isobutane)



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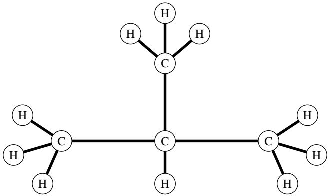

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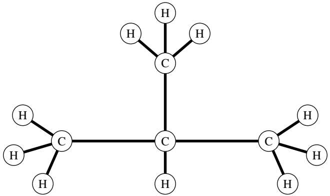

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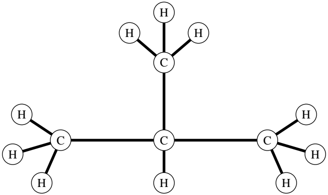
We will work to understand
some of their properties.



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- **NEW! Cross-listing of MATH 334 and MATH 634**
  - Same in-class content
  - Undergraduates can choose 334 vs 634.
  - 634: Assessment expectations higher.
  - 634: Project expectations higher. (More later.)
  - Both count toward major. Only 634 counts toward Masters.

To do well in this class:

- **Come to class prepared.**
  - Print out and read over course notes.
  - Read sections before class.
- **Form good study groups.**
  - Discuss homework and classwork.
  - Bounce proof ideas around.
  - You will depend on this group.
- **Put in the time.**
  - Three credits = (at least) nine hours / week out of class.
  - Homework stresses key concepts from class; learning takes time.
- **Stay in contact.**
  - If you are confused, ask questions (in class and out).
  - Don't fall behind in coursework or project.
  - I need to understand your concerns.

Getting to knoooooow you

Arrange yourselves into groups.

- Introduce yourself. (your name, where you are from)
- What brought you to this class?
- Fill out **the front of** your notecard:
  - Write your name. (Stylize if you wish.)
  - Write some words about how I might remember you & your name.
  - *Draw* something (anything!) in the remaining space.
- Exchange contact information. (phone / email / other)
- *Small talk suggestion:* What's been keeping you busy?

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Example. Let $G = (V, E)$, where
$V = \{v_1, v_2, v_3, v_4\}$,
$E = \{e_1, e_2, e_3, e_4, e_5\}$, and
$e_1 = \{v_1, v_2\}$, $e_2 = \{v_2, v_3\}$,
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Notation: # vertices $= |V| =$ __ $=$ __. # edges $= |E| =$ __ $=$ __.

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We say v_1 is **adjacent** to v_2 if there is an edge between v_1 and v_2. We also say v_1 and v_2 are **neighbors**.

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When multiple edges are allowed (but not loops): called **multigraphs**. When loops (& mult. edge) are allowed: called **pseudographs**.

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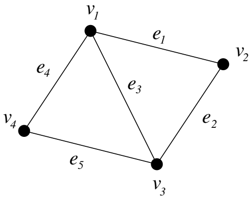

In our example,

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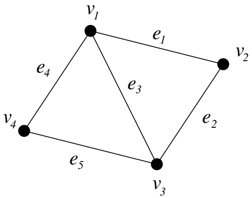

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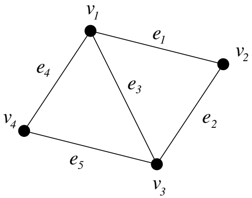

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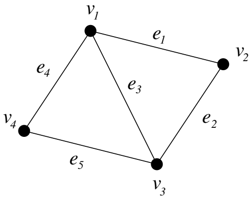

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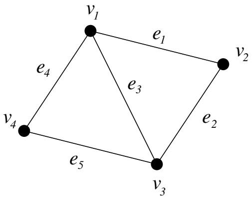

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Q. How many edges in G?

A. $m =$

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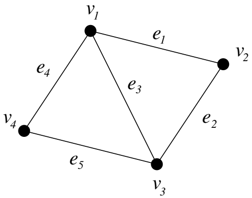

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The **degree sum** of a graph is the sum of the degrees of all vertices.

Degree sum exploration:

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A. $m =$

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Degree of a vertex

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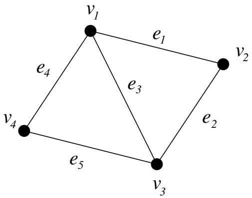

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Corollary: The degree sum of a graph is always even.

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Question. Does every sequence have a simple graph?

Answer.

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Examples: 7765333110 and 6644442

Proof of the Havel–Hakimi algorithm

Notation: Define the degree sequences to be:

$$\mathcal{S}_1 = \quad (s, \quad t_1 \quad , \quad t_2 \quad , \ldots , \quad t_s \quad , d_1 , \ldots , d_k).$$

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Question: Can this argument work in reverse?

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Game plan:

$$G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow \cdots \longrightarrow G_a$$

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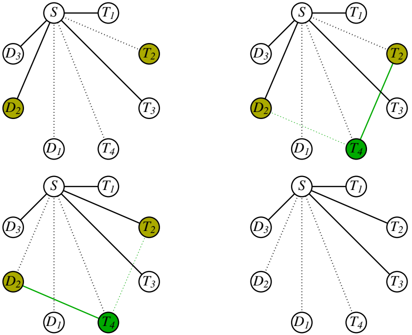
- Peel off vertex S to reveal a graph with degree sequence $\mathcal{S}_2$.

Proof of the Havel–Hakimi algorithm

Vertices $S, T_1, \ldots, T_s, D_1, \ldots, D_k$ have degrees $s, t_1, \ldots, t_s, d_1, \ldots, d_k$.



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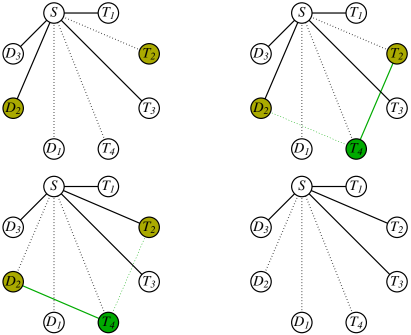
Therefore, there exists a T_i to which S is not adjacent and a D_j to which S is adjacent.



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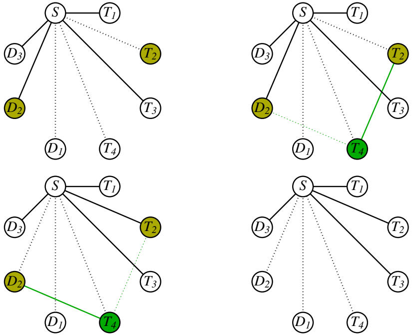

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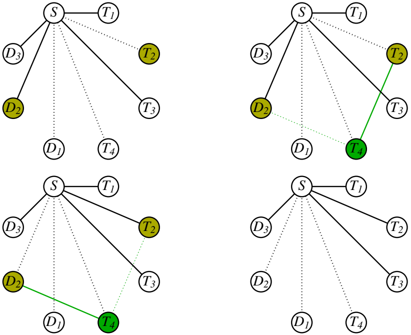

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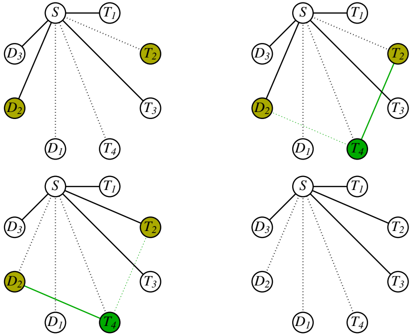

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(d) The degree sequence of the new graph is the same. (Why?) AND S is now adjacent to more T vertices. (Why?) Repeat as necessary.