TANNAKIAN APPROACH TO LINEAR DIFFERENTIAL ALGEBRAIC GROUPS

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Abstract. Tannaka's Theorem states that a linear algebraic group G is determined by the category of finite dimensional G-modules and the forgetful functor. We extend this result to linear differential algebraic groups by introducing a category corresponding to their representations and show how this category determines such a group.

1. Introduction

Given a linear algebraic group G, a rational representation (or finite dimensional G-module) is a finite dimensional vector space V together with a morphism $\rho_V : G \to \mathbf{GL}(V)$. The collection of such objects forms a rigid, abelian, tensor category and Tannaka's theorem ([13, Theorem 1],[4, Theorem 2.11],[14, Theorems 2.5.3 and 2.5.7]) states that one can recover the group G as an affine variety together with the morphisms corresponding to multiplication and inverse (or equivalently, its coordinate ring and its structure as a Hopf algebra) from the knowledge of this category $\mathbf{Rep}_G$ and the forgetful functor from $\mathbf{Rep}_G$ to finite dimensional vector spaces.

In this paper, we consider linear differential algebraic groups (or, shorter, linear ∂-$\mathbf{k}$-groups). These are groups of invertible matrices with entries in a differential field $\mathbf{k}$ of characteristic 0 with derivation ∂ that are, in addition, differential varieties, that is, they are defined by the vanishing of differential polynomials. A representation of such a group is a finite dimensional vector space V over $\mathbf{k}$ together with a differential polynomial morphism from G to $\mathbf{GL}(V)$. If K is a ∂-field containing $\mathbf{k}$ then one can talk about K-points $G(K)$ of the group G. In such a way we obtain a functorial definition of a linear differential algebraic group, which we give in Section 3.2. In the preceding sections we view such a group as a Kolchin closed subset of $\mathcal{U}^n$, where $\mathcal{U}$ is a semi-universal ∂-extension of the ground ∂-field $\mathbf{k}$.

The study of these groups and their representations was initiated by Cassidy in [1, 2]. In this paper, we introduce differentiation on vector spaces (a "prolongation" functor) over ∂-fields such that representations of G correspond to this construction. We then show that for a linear differential algebraic group G, the category of objects corresponding to its representations completely determines G as a differential variety together with its morphisms for multiplication and inverse, that is,

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we show that one can recover its *differential* coordinate ring together with its Hopf algebra and *differential* structure.

The rest of the paper is organized as follows. Section 2 gives formal definitions and properties of linear differential algebraic groups. In Section 3, we introduce the category $\mathcal{V}$ and show how a representation of a linear differential algebraic group corresponds to an object of $\mathcal{V}$. In Section 4, we give various representation theoretic properties of the objects of $\mathcal{V}$ as well as some consequences (e.g., any representation can be constructed from a faithful representation using the operations of linear algebra and the prolongation functor). We then show in Section 5 how to distinguish linear algebraic groups among all differential algebraic groups using our method of differentiating representations. In Section 6, we show how to recover the group from its associated category. Although for convenience we do most of the computations in the ordinary case, everything goes through for the partial differential case. The definition of the corresponding category is given in Section 7.

We note that the categorical approach to representations of linear algebraic groups leads to the theory of Tannakian categories. This theory has found many uses and, in particular, one can develop the Galois theory of linear differential equations using this categorical approach. Recently, a theory of parameterized linear differential equations has been developed by Cassidy and Singer [3] where the Galois groups are linear differential algebraic groups. The category $\mathcal{V}$ defined in this paper was motivated by a desire to give a similar categorical development of the Galois theory of parameterized differential equations. This program is now realized in the paper [10].

2. Basic definitions

2.1. **Differential algebra.** A Δ-ring R, where $\Delta = \{\partial_1, \ldots, \partial_m\}$, is a commutative associative ring with unit 1 and commuting differentiations $\partial_i : R \to R$ such that

$$\partial_i(a+b) = \partial_i(a) + \partial_i(b), \quad \partial_i(ab) = \partial_i(a)b + a\partial_i(b)$$

for all $a, b \in R$. If $\mathbf{k}$ is a field and a Δ-ring then $\mathbf{k}$ is called a Δ-field. We restrict ourselves to the case of

$$\operatorname{char} \mathbf{k} = 0.$$

If $\Delta = \{\partial\}$ then a Δ-field is called a ∂-field. For example, $\mathbb{Q}$ is a ∂-field with a unique possible differentiation (which is the zero one). The field $\mathbb{C}(t)$ is also a ∂-field with $\partial(t) = f$, and this f can be any rational function in $\mathbb{C}(t)$. For simplicity, we will mostly discuss the case of $\Delta = \{\partial\}$ in this paper and come back to the general case of m commuting differentiations in Section 7. Let

$$\Theta = \left\{\partial^i \mid i \in \mathbb{Z}_{\geqslant 0}\right\}.$$

Since ∂ acts on a ∂-ring R, there is a natural action of Θ on R.

A non-commutative ring $R[\partial]$ of linear differential operators is generated as a left R-module by the monoid Θ. A typical element of $R[\partial]$ is a polynomial

$$D = \sum_{i=1}^{n} a_i\partial^i,\ a_i \in R.$$

The right R-module structure follows from the formula

$$\partial \cdot a = a \cdot \partial + \partial(a)$$

for all $a \in R$. We denote the set of operators in $R[\partial]$ of order less than or equal to p by $R[\partial]_{\leqslant p}$.

Let R be a ∂-ring. If B is an R-algebra, then B is a ∂-R-algebra if the action of ∂ on B extends the action of ∂ on R. If R_1 and R_2 are ∂-rings then a ring homomorphism $\varphi : R_1 \to R_2$ is called a ∂-homomorphism if it commutes with ∂, that is,

$$\varphi \circ \partial = \partial \circ \varphi.$$

We denote these homomorphisms simply by $\mathrm{Hom}(R_1, R_2)$. If A_1 and A_2 are ∂-$\mathbf{k}$-algebras then a ∂-$\mathbf{k}$-homomorhism simply means a $\mathbf{k}[\partial]$-homomorphism. Let $Y = \{y_1, \ldots, y_n\}$ be a set of variables. We differentiate them:

$$\Theta Y := \left\{ \partial^i y_j \,\middle|\, i \in \mathbb{Z}_{\geqslant 0},\ 1 \leqslant j \leqslant n \right\}.$$

The ring of differential polynomials $R\{Y\}$ in differential indeterminates Y over a ∂-ring R is the ring of commutative polynomials $R[\Theta Y]$ in infinitely many algebraically independent variables ΘY with the differentiation ∂, which naturally extends ∂-action on R as follows:

$$\partial\left(\partial^i y_j\right) := \partial^{i+1} y_j$$

for all $i \in \mathbb{Z}_{\geqslant 0}$ and $1 \leqslant j \leqslant n$. A ∂-$\mathbf{k}$-algebra A is called finitely ∂-generated over $\mathbf{k}$ if there exists a finite subset $X = \{x_1, \ldots, x_n\} \subset A$ such that A is a $\mathbf{k}$-algebra generated by ΘX.

An ideal I in a ∂-ring R is called differential if it is stable under the action of ∂, that is,

$$\partial(a) \in I$$

for all $a \in I$. If $F \subset R$ then $[F]$ denotes the differential ideal generated by F. If a differential ideal is radical, it is called radical differential ideal. The radical differential ideal generated by F is denoted by $\{F\}$. If a differential ideal is prime, it is called a prime differential ideal.

2.2. **Linear differential algebraic groups.** We shall recall some definitions and results from differential algebra (see for more detailed information [1, 6]) leading up to the "classical definition" of a linear differential algebraic group and its representative functions. Later in the paper we will give a Hopf-theoretic treatment and provide an equivalent definition in terms of representable functors.

Let $\mathbf{k} \subset \mathcal{U}$ be a semi-universal differential field over $\mathbf{k}$, that is, a differential field such that if K is a differential field extension of $\mathbf{k}$, finitely generated in the differential sense, then there exists a $\mathbf{k}$-isomorphism of K into $\mathcal{U}$. We will assume that all differential fields we consider are subfields of $\mathcal{U}$ (the Hopf-theoretic treatment will *not* use semi-universal differential extensions).

Definition 1. For a differential field extension $K \supset \mathbf{k}$ a *Kolchin closed* subset $W(K)$ of K^n over $\mathbf{k}$ is the set of common zeroes of a system of differential algebraic equations with coefficients in $\mathbf{k}$, that is, for $f_1, \ldots, f_k \in \mathbf{k}\{Y\}$ we define

$$W(K) = \{a \in K^n \mid f_1(a) = \ldots = f_k(a) = 0\}.$$

There is a bijective correspondence between Kolchin closed subsets W of $\mathcal{U}^n$ defined over $\mathbf{k}$ and radical differential ideals $\mathbf{I}(W) \subset \mathbf{k}\{y_1, \ldots, y_n\}$ generated by the differential polynomials $f_1, \ldots, f_k$ that define W. In fact, the ∂-ring $\mathbf{k}\{Y\}$ is Ritt-Noetherian, meaning that every radical differential ideal is the radical of a finitely generated differential ideal, by the Ritt-Raudenbush basis theorem. Given a Kolchin closed subset W of $\mathcal{U}^n$ defined over $\mathbf{k}$ we let the *coordinate ring* $\mathbf{k}\{W\}$ be:

$$\mathbf{k}\{W\} = k\{y_1, \ldots, y_n\}\big/\,\mathbf{I}(W).$$

A *differential polynomial* map $\varphi : W_1 \to W_2$ between Kolchin closed subsets of $\mathcal{U}^n$, defined over $\mathbf{k}$, is given in coordinates by differential polynomials from $\mathbf{k}\{y_1, \ldots, y_n\}$. To give $\varphi : W_1 \to W_2$ is equivalent to defining $\varphi^* : \mathbf{k}\{W_2\} \to \mathbf{k}\{W_1\}$.

Definition 2. [1, Chapter II, Section 1, page 905] A *linear differential algebraic group* (or linear ∂-$\mathbf{k}$-group) is a Kolchin closed subgroup G of $\mathbf{GL}_n(\mathcal{U})$, that is, an intersection of a Kolchin closed subset of $\mathcal{U}^{n^2}$ with $\mathbf{GL}_n(\mathcal{U})$, which is closed under the group operations.

Note that we identify $\mathbf{GL}_n(\mathcal{U})$ with a Zarisky closed subset of $\mathcal{U}^{n^2+1}$ given by

$$\big\{(A, a) \;\big|\; (\det(A)) \cdot a - 1 = 0\big\}.$$

If X is an invertible $n \times n$ matrix, we can identify it with the pair $(X, 1/\det(X))$. Hence, we may represent the coordinate ring of $\mathbf{GL}_n(\mathcal{U})$ as

$$\mathbf{k}\{X, 1/\det(X)\}.$$

Denote $\mathbf{GL}_1$ simply by $\mathbf{G_m}$. Its coordinate ring is $\mathbf{k}\{y, 1/y\}$, where y is a differential indeterminate.

Definition 3. [2] A differential polynomial group homomorphism $\rho : G \to \mathbf{GL}(V)$ is called a *differential representation* of a linear differential algebraic group G, where V is a finite dimensional vector space over $\mathbf{k}$. Such space is called a *differential G-module.*

A Hopf algebra A is a commutative associative algebra together with comultiplication $\Delta : A \to A \otimes A$, coinverse $S : A \to A$, and counit $\varepsilon : A \to \mathbf{k}$ satisfying certain axioms [14, 2.1.2]. In [15, page 23] representations of an algebraic group G are viewed as comodules over the Hopf algebra A of regular functions on G. In Section 3 we will develop a similar technique to look at differential representations as *differential comodules* (Definition 7) over *differential* Hopf algebras (Definition 5).

Following [15, page 5] one interprets algebraic groups as representable functors from the category of $\mathbf{k}$-algebras to groups, that is, there must be a $\mathbf{k}$-algebra A such that for any $\mathbf{k}$-algebra B the B-points of G, denoted by $G(B)$, are just $\operatorname{Hom}(A, B)$. Then [15, Theorem, page 6] says that natural maps from one group G to another one G', viewed as functors, correspond to the algebra homomorphisms $A' \to A$. We will develop this in differential setting in Section 3.2 not assuming that all ∂-fields we consider are subfields of $\mathcal{U}$.

2.3. **Representative functions.** Let $W \subset \mathcal{U}^n$ be a Kolchin closed subset defined over $\mathbf{k}$. We define $\mathbf{k}\langle W\rangle$ to be the complete ring of quotients of $\mathbf{k}\{W\}$, that is $\mathbf{k}\{W\}$ is localized with respect to the set of nonzero divisors. We note that $\mathbf{k}\langle W\rangle$ is clearly a ∂-ring.

A linear differential algebraic group $G(\mathcal{U})$ acts on the rational functions $\mathcal{U}\langle G\rangle = \mathcal{U} \otimes \mathbf{k}\langle G\rangle$ by right translations (see [1, page 901], [2, page 227]). We will define this independently of $\mathcal{U}$ later on in formula (3) of Section 4.1. According to [2, page 227] the functions whose orbit generates a finite dimensional vector space are called *representative functions.* Note that the representative functions form a ∂-$\mathbf{k}$-algebra, which we denote by $R(G)$. By ([2, page 230, Theorem]), $R(G) = \mathcal{U}\{G\}$.

We show how these functions (and, hence, the algebra) can be connected with finite dimensional differential representations of G. Let V be a vector space over $\mathbf{k}$ of dimension n and let $\rho : G \to \mathbf{GL}(V)$ be a representation of G. The image group H has coordinate ring $\mathbf{k}\{H\}$. The ∂-$\mathbf{k}$-algebra $\mathbf{k}\{H\}$ is a quotient of $\mathbf{k}\{\mathbf{GL}_n\} = \mathbf{k}\{X, 1/\det(X)\}$. Let Z be the image of X in $\mathbf{k}\{H\}$ with respect to this canonical homormorphism. Then, ρ induces a ∂-$\mathbf{k}$-homomorphism $\rho^* : \mathbf{k}\{H\} \to \mathbf{k}\{G\}$, mapping the entries of Z onto elements φ_{ij} called *coordinate functions* of ρ.

Proposition 1. *Representative functions are the same as the coordinate functions of finite dimensional representations of G.*

Proof. Let $\rho : G \to \mathbf{GL}(V)$. Let also φ_{ij} be a coordinate function of $\mathbf{GL}(V)$. We have

$$G \xrightarrow{\ \rho\ } \mathbf{GL}(V) \xrightarrow{\ \varphi_{ij}\ } \mathcal{U}$$

According to [2, Corollary 1, page 231] we have $\varphi_{ij} \circ \rho \in \mathcal{U}\{G\}$. By [2, Theorem, page 230] we have $R(G) = \mathcal{U}\{G\}$. We show that any $f \in R(G)$ is of the form $\varphi_{ij} \circ \rho_f$ for a finite dimensional representation ρ_f of G.

Take any $f \in R(G)$ and consider its G-orbit $Gf =: V$, which is finite dimensional, under the action $\rho : G \to \mathbf{GL}(\mathcal{U}\{G\})$ with

$$\rho(g)(f)(x) = f(xg).$$

Let V be spanned by $\{f = f_1, \ldots, f_n\}$ over $\mathcal{U}$. So,

$$\rho(g)(f_1) = \sum_{i=1}^{n} c_i(g) f_i.$$

Evaluating the last equality at the point $e \in G$ for all $g \in G$ we get

$$f(g) = (\rho(g)(f))(e) = \sum_{i=1}^{n} c_i(g) \cdot f_i(e).$$

It remains to change the basis which correspond to the conjugation of the representation matrices. Assume that $f_1(e) \neq 0$. A conjugation matrix is

$$C = \begin{pmatrix} f_1(e) & f_2(e) & f_3(e) & \ldots & f_n(e) \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

If $B(g)$ is a representation matrix then it goes to $CB(g)C^{-1}$ under such a change of coordinates. Thus, we have obtained the function $f(g)$ as a coordinate function of some finite dimensional differential representation of the linear differential algebraic group G. □

In the following we will eliminate semi-universal ∂-extensions of $\mathbf{k}$ by using a functorial approach.

3. The category $\mathcal{V}$ and representations

We start with introducing a *differentiation functor* on finite dimensional vector spaces and investigate its essential properties.

3.1. Definition.

Definition 4. The category $\mathcal{V}$ over a ∂-field $\mathbf{k}$ is the category of finite dimensional vector spaces over $\mathbf{k}$:

(1) objects are finite dimensional $\mathbf{k}$-vector spaces,
(2) morphisms are $\mathbf{k}$-linear maps;

with tensor product $\otimes$, direct sum $\oplus$, dual $*$, and additional operations:

$$\partial^p : V \mapsto V^{(p)} := \mathbf{k}[\partial]_{\leqslant p} \otimes V,$$

which we call differentiation (or prolongation) functors. If $\varphi \in \mathrm{Hom}(V, W)$ then we define

$$\partial^p(\varphi) : V^{(p)} \to W^{(p)},\ \varphi(\partial^q \otimes v) = \partial^q \otimes \varphi(v),\ 0 \leqslant q \leqslant p.$$

Remark 1. Note that $\mathbf{k}[\partial]$ is a non-commutative $\mathbf{k}$-algebra, so in this tensor product we think of $\mathbf{k}[\partial]$ as a right $\mathbf{k}$-space and V as a left $\mathbf{k}$-space.

Remark 2. If $\{v_1, \ldots, v_n\}$ is a basis of V then $\{v_1, \ldots, v_n, \ldots, \partial^p \otimes v_1, \ldots, \partial^p \otimes v_n\}$ is a basis of $V^{(p)}$.

We denote $\partial \otimes v$ simply by ∂v.

3.2. **Linear differential algebraic groups.** In the following Section 3.3 we will introduce another definition of a differential representation of a linear differential algebraic group (see Definition 7). For this purpose we will view linear differential algebraic groups as *representable functors* from the category of ∂-$\mathbf{k}$-algebras to groups (see (2)). So, throughout this section we are defining a linear differential algebraic group not as a subgroup of $\mathbf{GL}_n(\mathcal{U})$ but functorially.

We take a Hopf-theoretic approach to the study of linear differential algebraic groups as in [2]. Let A be a (finitely generated) ∂-$\mathbf{k}$-algebra. Following [2, page 226] one defines the set

$$G(\mathcal{U}) = \mathrm{Hom}(A, \mathcal{U}),$$

where $\mathcal{U}$ is the semi-universal differential field as before, to get $G(\mathcal{U})$ back from A as a Kolchin closed subset of $\mathcal{U}^n$. Assume that A is supplied with the following operations:

- differential algebra homomorphism $m : A \otimes A \to A$ is the multiplication map on A,
- differential algebra homomorphism $\Delta : A \to A \otimes A$, which is a comultiplication,
- differential algebra homomorphism $\varepsilon : A \to k$, which is a counit,
- differential algebra homomorphism $S : A \to A$, which is a coinverse.

We also assume that these maps satisfy commutative diagrams (see [2, page 225]):

$$\begin{array}{ccccccccc} A & \xrightarrow{\Delta} & A \otimes A & A & \xrightarrow{\Delta} & A \otimes A & A & \xrightarrow{\Delta} & A \otimes A \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \Delta} & {\scriptstyle \mathrm{id}_A}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \varepsilon} & {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle m \circ (S \otimes \mathrm{id}_A)} \\ A \otimes A & \xrightarrow{\Delta \otimes \mathrm{id}_A} & A \otimes A \otimes A & A & \xrightarrow{\sim} & A \otimes k & k & \hookrightarrow & A \end{array} \tag{1}$$

Definition 5. Such a commutative associative ∂-$\mathbf{k}$-algebra A with the unity and operations m, Δ, S, and ε satisfying axioms (1) is called a *differential Hopf algebra* (or *Hopf ∂-$\mathbf{k}$-algebra*).

Remark 3. Introduced in this way $G(\mathcal{U})$ with operations corresponding to Δ, S, and ε is not only a Kolchin-closed subset of some $\mathcal{U}^n$ but a group at the same time. Indeed, the group multiplication is defined in the usual way:

$$(\varphi_1 \cdot \varphi_2)(a) := (\varphi_1 \otimes \varphi_2)(\Delta(a))$$

for all $\varphi_1, \varphi_2 \in \operatorname{Hom}(A,\mathcal{U}) = G(\mathcal{U})$ and $a \in A$; the group inverse is given by a similar formula. Moreover, all these operations are continuous in the Kolchin topology (see [1] for proofs). Finally, since A is ∂-finitely generated, the differential algebraic group $G(\mathcal{U})$ is linear [1, Proposition 12, page 914].

Recall that a linear ∂-$\mathbf{k}$-group G is defined by a system of differential polynomial equations $F = 0$ with coefficients in $\mathbf{k}$ or by the radical differential ideal I of $\mathbf{k}\{y_1, \ldots, y_n\}$ generated by F (see [1, page 895]). If we represent $0 \to I \to \mathbf{k}\{y_1, \ldots, y_n\} \to A \to 0$ then we can consider the group in a ∂-$\mathbf{k}$-algebra B, that is,

$$G(B) = \operatorname{Hom}_{\mathbf{k}[\partial]}(A, B) =: \operatorname{Hom}(A, B).$$

Note that here B is not necessarily a subring of $\mathcal{U}$. For convenience we gather all such $G(B)$ and form the following *representable* functor

$$G : \{\partial\text{-}\mathbf{k}\text{-algebras}\} \to \{\text{Groups}\}, \quad B \mapsto \operatorname{Hom}_{\mathbf{k}[\partial]}(A, B), \tag{2}$$

which we call the *affine differential algebraic group* defined by A.

Among all differential algebraic groups we distinguish the differential general linear group. Consider a finite dimensional vector space V over $\mathbf{k}$ of dimension m. We define $\mathbf{GL}(V)$ by the functor

$$\mathbf{GL}(V) : B \mapsto \operatorname{Hom}_{\mathbf{k}[\partial]}(\mathbf{k}\{X_{11}, \ldots, X_{mm}, 1/\det\}, B).$$

One may consider $\mathbf{GL}(V)(B)$ as the set of $m \times m$ invertible matrices with coefficients in the ∂-$\mathbf{k}$-algebra B.

Example 1. Recall, that the coordinate ring of $\mathbf{G_m}$ is $\mathbf{k}\{y, 1/y\}$. Its ∂-$\mathbf{k}$-Hopf algebra structure is given by

$$\Delta(y) = y \otimes y,$$
$$S(y) = 1/y.$$

These maps are ∂-homomorphisms. Therefore,

$$\Delta(\partial y) = \partial(\Delta(y)) = \partial y \otimes y + y \otimes \partial y,$$
$$S(\partial y) = \partial(S(y)) = \partial(1/y) = -\partial y/y^2$$

By differentiating these expressions further one gets the action of Δ and S on higher derivatives of y.

Example 2. The differential additive group $\mathbf{G_a}$ is represented by the ∂-$\mathbf{k}$-Hopf algebra $\mathbf{k}\{y\}$ with

$$\Delta(\partial^p y) = \partial^p y \otimes 1 + 1 \otimes \partial^p y,$$
$$S(\partial^p y) = (-1)^{p+1} y$$

for all $p \in \mathbb{Z}_{\geqslant 0}$.

The linear differential algebraic groups we defined earlier correspond to subgroups of $\mathbf{GL}(V)$. Unlike the situation for algebraic groups, there are affine differential algebraic groups that are not isomorphic to linear differential algebraic groups (see [1, page 911]).

Morphisms of differential algebraic groups then in our sense are *morphisms of their representable functors.* We need the following result (a corollary of Yoneda's Lemma) from the theory of categories to see that this corresponds to the morphisms of algebras defining the groups.

Lemma 1. [11, Corollary 2, page 44],[5, 30.7, Corollary, page 224] *Let $\mathcal{C}$ be a category such that $\operatorname{Hom}(A, B)$ is a set for all objects A, B of $\mathcal{C}$. Let E and F be functors from the category $\mathcal{C}$ to the category of sets represented by some objects A and B, that is, $E = \operatorname{Hom}(A, \_)$ and $F = \operatorname{Hom}(B, \_)$. Then morphisms of functors E and F correspond to homomorphisms of B and A.*

It remains to note that the category of ∂-$\mathbf{k}$-algebras satisfies the assumptions of Lemma 1.

3.3. **Representations.** We will take a careful look at differential representations of a linear differential algebraic group G. These are differential algebraic group homomorphisms

$$\Phi : G \to \mathbf{GL}(V) =: \operatorname{Aut}(V)$$

for some finite dimensional $\mathbf{k}$-vector space V.

Remark 4. Here, G and $\mathbf{GL}(V)$ are considered as functors whose points vary with ∂-$\mathbf{k}$-algebras as it was explained in Section 3.2. As mentioned earlier, G is determined by its $\mathcal{U}$-points and we can *identify* the two concepts: $G(\mathcal{U})$ and G. If $\mathbf{k}$ is a differentially closed field then $G(\mathbf{k})$ determines G and one does not need to look at $G(\mathcal{U})$.

By Lemma 1 the morphism Φ corresponds to the homomorphism of the ∂-$\mathbf{k}$-algebras. In [2] a representation of the group G defined over $\mathbf{k}$ is a rational differential algebraic group homomorphism

$$G(\mathcal{U}) \to \mathbf{GL}(V)(\mathcal{U}).$$

But such a morphism is a differential polynomial map by [2, Corollary 1, page 231] and so corresponds to a homomorphism of the associated ∂-$\mathbf{k}$-algebras. Hence, we can freely use our language of functors.

Definition 6. For a linear differential algebraic group G an object $V \in \mathcal{O}\mathrm{b}(\mathcal{V})$ together with a natural map of group functors $r_V : G \to \operatorname{Aut}(V)$ which is a group homomorphism is called a *differential G-module.* The map r_V is called a ∂-$\mathbf{k}$-representation of G in V.

3.4. **Differential comodules.** We are going to restate this in the language of comodules which we introduce now. For this we define a differential analogue of an algebraic comodule. Let A be a differential Hopf algebra.

Definition 7. A finite dimensional vector space V over $\mathbf{k}$ is called an *A-differential comodule* if there is a given $\mathbf{k}$-linear morphism

$$\rho : V \to V \otimes A$$

satisfying the axioms:

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V\otimes A \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle \mathrm{id}_V\otimes\Delta} \\ V\otimes A & \xrightarrow{\rho\otimes\mathrm{id}_A} & V\otimes A\otimes A \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\rho} & V\otimes A \\ \downarrow{\scriptstyle\mathrm{id}_V} & & \downarrow{\scriptstyle \mathrm{id}_V\otimes\varepsilon} \\ V & \xrightarrow{\sim} & V\otimes \mathbf{k} \end{array}$$

together with the prolongation of ρ on $V^{(i)}$ commuting with ∂.

The definition and correctness of the prolongation are given in Theorem 1 and Lemma 2, which follows the theorem. We will show that A-differential comodules are in one-to-one correspondence with differential G-modules, where G is the functor represented by A.

3.5. **Equivalent definitions of differential representations.**

Theorem 1. *Let A be a ∂-$\mathbf{k}$-Hopf algebra and G be the linear ∂-$\mathbf{k}$-group (∂-$\mathbf{k}$-group functor) represented by A. Let V be an object in $\mathcal{V}$. Then, there is a bijective correspondence between the set of ∂-representations from G into $\mathbf{GL}(V)$ and the set ∂-A-comodule structures*

$$\rho : V \to V\otimes A$$

on V. If $\{v_1,\ldots,v_n\}$ is a basis of V then in coordinates we have

$$\rho(v_j)=\sum_{i=1}^{n} v_i\otimes a_{ij},\quad \Delta(a_{ij})=\sum_{r=1}^{n} a_{ir}\otimes a_{rj}.$$

Moreover,

$$\rho(\partial^p v_j)=\sum_{i=1}^{n}\sum_{q=0}^{p}\binom{p}{q}\partial^q v_i\otimes(\partial^{p-q}a_{ij})$$

gives a prolongation of ρ on $V^{(p)}$ for all $p\in\mathbb{Z}_{\geqslant 1}$.

Proof. The representation Φ is a morphism of group functors $G\to \mathrm{Aut}(V)$. According to [15, Theorem, Section 3.2] such a representation Φ of the affine group scheme G corresponds to the $\mathbf{k}$-linear map ρ with the following commuting diagrams:

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V\otimes A \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle \mathrm{id}_V\otimes\Delta} \\ V\otimes A & \xrightarrow{\rho\otimes\mathrm{id}_A} & V\otimes A\otimes A \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\rho} & V\otimes A \\ \downarrow{\scriptstyle\mathrm{id}_V} & & \downarrow{\scriptstyle \mathrm{id}_V\otimes\varepsilon} \\ V & \xrightarrow{\sim} & V\otimes \mathbf{k} \end{array}$$

More precisely, the map ρ comes from the restriction of the A-linear map

$$\Phi(\mathrm{id}_A): V\otimes A\to V\otimes A$$

to $V\otimes\mathbf{k}\cong V$. By [15, Corollary, Section 3.2] we have

$$\rho(v_j)=\sum_{i=1}^{n} v_i\otimes a_{ij},\quad \Delta(a_{ij})=\sum_{r=1}^{n} a_{ir}\otimes a_{rj}.$$

Let us demonstrate the last differential identity. We have

$$\Phi(\mathrm{id}_A)\in \mathrm{Hom}_A(V\otimes A, V\otimes A)$$

and $\Phi(\mathrm{id}_A)$ can be extended to a map $V^{(p)} \otimes A \to V^{(p)} \otimes A$ commuting with the ∂-structure. The first step gives us the following:

$$\begin{array}{ccc}
v_j \otimes 1 & \xrightarrow{\partial_{V\otimes A}} & \partial v_j \otimes 1 \\
\Big\downarrow {\scriptstyle \Phi(\mathrm{id}_A)=\rho} & & \Big\downarrow {\scriptstyle \Phi(\mathrm{id}_A)=\rho} \\
\sum\limits_{i=1}^{n} v_i \otimes a_{ij} & \xrightarrow{\partial_{V\otimes A}} & \sum\limits_{i=1}^{n}((\partial v_i)\otimes a_{ij} + v_i \otimes \partial a_{ij})
\end{array}$$

The formula for higher order derivatives can be obtained by induction. This, indeed, makes $V^{(p)}$ an A-comodule (see Lemma 2).

On the other hand, having such a $\rho : V \to V \otimes A$ one extends it by the A-linearity to $\rho_A : V \otimes A \to V \otimes A$ and then to $V^{(p)} \otimes A \to V^{(p)} \otimes A$ commuting with the ∂-structure (see Lemma 2 for correctness). Consider a ∂-$\mathbf{k}$-algebra B. For any $g \in \mathrm{Hom}_{\mathbf{k}[\partial]}(A, B)$ in $G(B)$ we have the following commutative diagram:

$$\begin{array}{ccc}
V \otimes A & \xrightarrow{\Phi(\mathrm{id}_A)} & V \otimes A \\
\Big\downarrow {\scriptstyle \mathrm{id}_V \otimes g} & & \Big\downarrow {\scriptstyle \mathrm{id}_V \otimes g} \\
V \otimes B & \xrightarrow{\Phi(g)} & V \otimes B
\end{array}$$

meaning that $\Phi(g)$ is determined by $(\mathrm{id}_V \otimes g) \circ \rho$ using B-linearity of $\Phi(g)$, where ρ is the restriction of $\rho_A := \Phi(\mathrm{id}_A)$ to V. Since $\Phi(\mathrm{id}_A)$ is constructed so as to preserve the ∂-structure, the map $\Phi(g)$ does the same thing, since g is a ∂-$\mathbf{k}$-algebra homomorphism $A \to B$. Indeed, take $v = \sum c_i \partial^i v_i$. Then

$$\begin{aligned}
\Phi(g)(v) &= (\mathrm{id}_V \otimes g) \circ \Phi(\mathrm{id}_A)\left(\sum c_i \partial^i v_i\right) = \sum c_i \cdot (\mathrm{id}_V \otimes g)\partial^i(\Phi(\mathrm{id}_A)(v_i)) = \\
&= \sum c_i \cdot (\mathrm{id}_V \otimes g)\partial^i\left(\sum_{j=1}^{n} v_j \otimes b_{ji}\right) = \\
&= \sum c_i \cdot (\mathrm{id}_V \otimes g)\left(\sum_{j=1}^{n}\sum_{r=0}^{i}\binom{i}{r}\partial^r v_j \otimes \partial^{i-r} b_{ji}\right) = \\
&= \sum c_i \left(\sum_{j=1}^{n}\sum_{r=0}^{i}\binom{i}{r}\partial^r v_j \otimes \partial^{i-r} g(b_{ji})\right) = \\
&= \sum c_i \partial^i \left(\sum_{j=1}^{n} v_j \otimes g(b_{ji})\right) = \sum c_i \partial^i (\Phi(g)(v_i)).
\end{aligned}$$

Here we denoted $\Phi(\mathrm{id}_A)(v_i) = \sum_j v_j \otimes b_{ji}$ for some elements $b_{ij} \in A$. From this it follows that

$$\Phi(g) \in \mathrm{Hom}_{\mathbf{k}[\partial]}\left(\mathbf{k}\left\{X_{11}, \ldots, X_{nn}, 1/\det\right\}, B\right).$$

This finally establishes a bijection between differential representations and differential comodules. $\square$

3.6. **Prolongation of representations.** Let G be a linear differential algebraic group represented by A and $\rho : G \to \mathbf{GL}(V)$ be its differential representation.

Lemma 2. *The action of G on each $V^{(i)}$ is algebraic, that is, $V^{(i)}$ is an A-comodule for all $i \geqslant 0$.*

Proof. We have a differential algebraic action on V. Let $\{v_1, \ldots, e_v\}$ be a $\mathbf{k}$-basis of V. For $\rho : V \to V \otimes A$ from Theorem 1 we have

$$\begin{aligned}
\rho(v_j) &= \sum_{i=1}^{n} v_i \otimes a_{i,j},\\
\Delta(a_{i,j}) &= \sum_{r=1}^{n} a_{i,r} \otimes a_{r,j},\\
\Delta(\partial^p a_{i,j}) = \partial^p(\Delta(a_{i,j})) &= \sum_{r=1}^{n}\sum_{q=0}^{p}\binom{p}{q}(\partial^q a_{i,r}) \otimes (\partial^{p-q} a_{r,j}),\\
\rho(\partial^p v_j) = \partial^p \rho(v_j) &= \sum_{i=1}^{n}\sum_{q=0}^{p}\binom{p}{q}(\partial^q v_i) \otimes (\partial^{p-q} a_{i,j}).
\end{aligned}$$

One can show by induction that if $B = (a_{i,j})_{i,j=1}^{n}$ is the "representation" matrix for $G \to \mathbf{GL}(V)$ then for a fixed number $i \geqslant 0$ we have

$$B_i := \begin{pmatrix} B & 0 & 0 & \ldots & 0\\ \binom{i}{1}\cdot B_t & B & 0 & \ldots & 0\\ \binom{i}{2}\cdot B_{tt} & \binom{i-1}{1}\cdot B_t & B & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ B_{t^i} & \ldots & \ldots & \ldots & B \end{pmatrix} = (c_{r,s})_{r,s=1}^{i\cdot n}$$

is the one for $G \to \mathbf{GL}\left(V^{(i)}\right)$, where B_{t^k} means the matrix $\left(\partial^k a_{i,j}\right)_{i,j=1}^{n}$.

It remains to show we do have a group action. Let us do this. The ordered basis of $V^{(i)}$ is

$$\{\partial^i v_1, \ldots, \partial^i v_n, \partial^{i-1} v_1, \ldots, \partial^{i-1} v_n, \ldots, v_1, \ldots, v_n\}.$$

By induction, using the facts that B_i has:

(1) B on the main diagonal,
(2) 0 above the main diagonal,
(3) derivatives of B below the diagonal,

we conclude that the only part of matrix we need to take care of is the set of first n columns. Let $1 \leqslant q \leqslant n$ and $n \cdot m + 1 \leqslant p \leqslant n \cdot (m+1)$. We have:

$$\begin{aligned}
\Delta(c_{p,q}) &= \Delta\left(\binom{i}{m}\partial^m b_{p-m\cdot n,q}\right) = \binom{i}{m}(\partial^m \Delta(b_{p-m\cdot n,q})) =\\
&= \binom{i}{m}\partial^m\left(\sum_{l=1}^{n} b_{p-m\cdot n,l} \otimes b_{l,q}\right) =\\
&= \sum_{l=1}^{n}\sum_{r=0}^{m}\binom{i}{m}\binom{m}{r}(\partial^r b_{p-m\cdot n,l}) \otimes (\partial^{m-r} b_{l,q}) =\\
&= \sum_{l=1}^{n}\sum_{r=0}^{m}\binom{i-r}{m-r}\binom{i}{r}(\partial^r b_{p-m\cdot n,l}) \otimes (\partial^{m-r} b_{l,q}),
\end{aligned}$$

because
$$\binom{i}{m}\binom{m}{r} = \frac{i!m!}{m!(i-m)!r!(m-r)!} = \frac{(i-r)!i!}{(m-r)!(i-m)!r!(i-r)!}$$
and this is exactly what we needed. $\square$

4. Essential properties of differential representations

In the following we will use different equivalent definitions of differential representations of a linear differential algebraic group. Summarizing the previous sections, we see that a differential representation $V \in \mathcal{O}\mathrm{b}(\mathcal{V})$ of a linear differential algebraic group G with Hopf algebra A can be defined in the following ways:

- by a differential morphism $G(\mathcal{U}) \to \mathbf{GL}(V)(\mathcal{U})$ (Section 3.3);
- by a natural map of group functors $G \to \mathrm{Aut}(V)$ (Definition 6);
- by a differential A-comodule structure on V (Definition 7, Theorem 1).

Differential representations of G form a category which we denote by $\mathbf{Rep}_G$. The objects $\mathcal{O}\mathrm{b}(\mathbf{Rep}_G)$ are the underlying vector spaces. If $V, W \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)$ and $r_V : G \to \mathbf{GL}(V)$ and $r_W : G \to \mathbf{GL}(W)$ are the corresponding representations then $\mathrm{Hom}(V, W)$ consists of those $\mathbf{k}$-linear maps between V and W that commute with the action of G.

4.1. Recovering representations.

Let G be a linear differential algebraic group with $A := \mathbf{k}\{G\}$ and $G \to \mathrm{Aut}(V)$ be its faithful representation. The ∂-coalgebra structure on A makes A a ∂-A comodule:

$$\rho_A := \Delta : A \to A \otimes A \tag{3}$$

called the *regular representation* of G.

Lemma 3. *Every finite dimensional differential representation $r_U : G \to \mathbf{GL}(U)$ embeds in a finite sum of copies of the regular representation of G.*

Proof. Denote $M = U \otimes A$. Then M is a differential comodule with
$$\mathrm{id}_U \otimes \Delta : M \to M \otimes A.$$
Since $(\mathrm{id} \otimes \Delta) \circ \rho = (\rho \otimes \mathrm{id}) \circ \rho$, the map $\rho : U \to M$ is a map of A-comodules. It is injective, because $v = (\mathrm{id} \otimes \varepsilon) \circ \rho(v)$. Finally, $M \cong A^{\dim(U)}$. $\square$

Proposition 2. *Every differential representation U of G is a subquotient of several copies of a G-module*
$$V^{(i_1)} \otimes \ldots \otimes V^{(i_k)} \otimes V^* \otimes \ldots \otimes V^*.$$

Proof. Fix a basis $\{u_1, \ldots, u_m\}$ of U. By Lemma 3 the representation U is an $A := \mathbf{k}\{G\}$-subcomodule of
$$U \otimes A = (u_1 \otimes A) \oplus \ldots \oplus (u_m \otimes A) \cong A^m.$$
Consider the canonical projections $\pi_i : A^m \to A$, which are G-equivariant maps with respect to the comultiplication $\Delta : A \to A \otimes A$. Since $U \subset A^m$, we have
$$U \subset \bigoplus_{i=1}^{m} \pi_i(U)$$
and each $\pi_i(U)$ is a G-module, because π_i is G-equivariant.

Consider the following surjection

$$\pi : B := \mathbf{k}\{X_{11}, \ldots, X_{nn}, 1/\det\} \to A \to 0.$$

Since $\pi_i(U)$ is a finite dimensional G-subspace of A, there exist numbers $r, s, p \in \mathbb{Z}_{\geqslant 0}$ such that $\pi_i(U)$ is contained in $\pi(L_{r,s,p})$, where

$$L_{r,s,p} := (1/\det)^r \{f(X_{ij}) \mid \deg(f) \leqslant s, \operatorname{ord}(f) \leqslant p\}.$$

There is a B-comodule structure on B given by

$$\Delta(X_{ij}) = \sum_{l=1}^{n} X_{il} \otimes X_{lj},$$
$$\Delta(\partial X_{ij}) = \sum_{l=1}^{n} ((\partial X_{il}) \otimes X_{lj} + X_{il} \otimes (\partial X_{lj}))$$

and $L_{r,s,p}$ is a B-subcomodule of B, because of

$$\Delta(X_{ij}X_{pq}) = \sum_{l,r=1}^{n} X_{il}X_{pr} \otimes X_{lj}X_{rq}$$

and Lemma 2. We then have that $L_{r,s,p}$ is also an A-subcomodule of B. Hence, each $\pi_i(U)$ is a subquotient of some $L_{r,s,p}$. Thus, we only need to show how to construct these $L_{r,s,p}$.

Fix a basis $\{v_1, \ldots, v_n\}$ of V. We have a B-comodule V with respect to

$$\rho(v_j) = \sum_{i=1}^{n} v_i \otimes X_{ij}.$$

For each i, $1 \leqslant i \leqslant n$, the map $\varphi_i : v_j \mapsto X_{ij}$ is $\mathbf{GL}_n$ (hence, G)-equivariant, because

$$\varphi_i(\rho_V(v_j)) = \varphi_i\left(\sum_{l=1}^{n} v_l \otimes X_{lj}\right) = \sum_{l=1}^{n} X_{il} \otimes X_{lj} = \Delta(X_{ij}) = \rho_B(\varphi_i(v_j))$$

and both ρ and Δ preserve the product rule with respect to ∂.

Consider the space of linear polynomials $L_{0,1,p}$ in the variables $\{X_{ij}\}$ and their derivatives of order up to p. An element f of such a space is of the form

$$f = \sum_{i,j=1}^{n} \sum_{q=0}^{p} c_{ij} X_{ij}^{(q)},$$

where $c_{ij} \in \mathbf{k}$. As it has been noticed above this space is an A-subcomodule of B. The map $(\varphi_1, \ldots, \varphi_n)$ gives an A-comodule isomorphisms between the nth power $\left(V^{(p)}\right)^n$ of the pth derivative of the original representation of G and $L_{0,1,p}$. Hence, one can construct $L_{0,1,p}$.

Consider any $s \in \mathbb{Z}_{\geqslant 2}$. The G-space $L_{0,s,p}$ is the quotient of $(L_{0,1,p})^{\otimes s}$ by the symmetric relations. So, we have all $L_{0,s,p}$. Let now $s = n = \dim_{\mathbf{k}} V$. Then the one-dimensional representation $\det : G \to \mathbf{k}$ with $g \mapsto \det(g)$ is in $L_{0,n,p}$. For $f \in \mathbf{k}^*$ we have

$$\det(g)(f)(x) = f(x/\det(g)) = \frac{1}{\det(g)} f(x).$$

Thus,

$$L_{r,s,p} = (\det{}^*)^{\otimes r} \otimes L_{0,s,p},$$

which is what we wanted to construct. □

4.2. **Example.** We will show how Proposition 2 works step by step.

Example 3. Consider the differential representation $\rho : \mathbf{G_m} \to \mathbf{G_a}$ by

$$\mathbf{G_m} \ni y \mapsto \begin{pmatrix} 1 & \frac{\partial y}{y} \\ 0 & 1 \end{pmatrix} \in \mathbf{G_a}\,.$$

So, the underlying vector space U is $\mathbf{k}^2$. The representation ρ corresponds to the map of ∂-$\mathbf{k}$-algebras

$$\rho^* : B := \mathbf{k}\left\{X_{11}, X_{12}, X_{21}, X_{22}, 1/\det\right\} \to A := \mathbf{k}\left\{y, 1/y\right\} \to 0$$

with

$$\begin{aligned} &X_{11} \mapsto 1, X_{12} \mapsto y'/y, \\ &X_{21} \mapsto 0, X_{22} \mapsto 1. \end{aligned}$$

Take the standard basis $\{u_1, u_2\}$ of $\mathbf{k}^2$. We then have $\rho : U \to U \otimes A$ given by

$$\begin{aligned} \rho(u_1) &= u_1 \otimes 1, \\ \rho(u_2) &= u_1 \otimes (y'/y) + u_2 \otimes 1. \end{aligned}$$

So, as an A-comodule

$$\mathbf{k}^2 \subset \operatorname{span}_{\mathbf{k}}\left\{u_1 \otimes 1, u_1 \otimes (y'/y)\right\} \oplus \operatorname{span}_{\mathbf{k}}\left\{\{u_2 \otimes 1\right\}.$$

Hence, it is sufficient to construct $\operatorname{span}_{\mathbf{k}}\left\{u_1 \otimes 1, u_1 \otimes (y'/y)\right\}$ or, equivalently,

$$W := \operatorname{span}_{\mathbf{k}}\left\{1, \frac{y'}{y}\right\}.$$

Consider the B-subcomodule $L_{0,1,0}$ of linear polynomials in $X_{11}, X_{12}, X_{21}, X_{22}$ with coefficients in $\mathbf{k}$ which is also an A-subcomodule of B. The A-comodule W is contained in the image of $L_{0,1,1}$ with respect to ρ^*. Hence, W is a subquotient of $L_{0,1,1}$. The A-comodule $L_{0,1,1}$ is constructed as follows. It is enough to get $L_{0,1,0}$, as $L_{0,1,1} = (L_{0,1,0})^{(1)}$.

The group $\mathbf{G_m}$ has a representation on $V = \operatorname{span}_{\mathbf{k}}\{v_1, v_2\}$ as

$$\begin{aligned} v_1 &\mapsto v_1 \otimes y, \\ v_2 &\mapsto v_2 \otimes y. \end{aligned}$$

Consider the two A-comodule maps

$$\begin{aligned} &\varphi_1 : v_1 \mapsto X_{11},\ v_2 \mapsto X_{12}, \\ &\varphi_2 : v_1 \mapsto X_{21},\ v_2 \mapsto X_{22}. \end{aligned}$$

The map (φ_1, φ_2) provides an A-comodule isomorphism between V^2 and $L_{0,1,0}$. Summarizing, we need to take the 4th power of the original faithful representation of $\mathbf{G_m}$ on $\mathbf{k}$, compute several subquotients, and then sum up ($\oplus$) the result, take a subrepresentation, and differentiate to obtain the representation ρ.

5. Linear groups of constant elements

Let $\mathbf{k}$ be a differential field of characteristic zero with field of constants C. We say that H is a *group of constant matrices* if it is a subgroup of some $\mathbf{GL}_n(C)$.

Proposition 3. *A linear differential algebraic group $G \subset \mathbf{GL}(V)$ is conjugate to a group $H \subset \mathbf{GL}(V)(C)$ of constant matrices iff*

$$\bar{\mathbf{k}} \otimes V^{(p)} = \bar{\mathbf{k}} \otimes \left(V \oplus \bigoplus_{i=1}^{p} V_i \right)$$

for all $p \geqslant 1$, where V is a faithful representation of G, $V_i \cong V$, and $\bar{\mathbf{k}}$ is the differential closure of $\mathbf{k}$ (see, for instance, [3, page 120]*).*

Proof. Let $\{v_1, \ldots, v_n\}$ be a $\mathbf{k}$-basis of V. Assume that there exists a matrix $D \in \mathbf{GL}_n\left(\bar{\mathbf{k}}\right)$ such that

$$D^{-1} G D = H.$$

For $g \in G$ let A_g be the corresponding matrix with respect to the basis $\{v_1, \ldots, v_n\}$. The matrix of g with respect to the basis

$$(w_1, \ldots, w_n) = (v_1, \ldots, v_n) \cdot D$$

is given by

$$B_g = D^{-1} A_g D.$$

Hence, $B_g \in \mathbf{GL}_n(C)$. We have

$$g \cdot (\partial^p \otimes w_i) = \partial^p \otimes (g \cdot w_i) = \partial \otimes (B_g w_i) = B_g(\partial \otimes w_i).$$

Thus,

$$\begin{aligned} \bar{\mathbf{k}} \otimes V^{(p)} &= \operatorname{span}_{\bar{\mathbf{k}}}\{w_1, \ldots, w_n\} \oplus \ldots \oplus \operatorname{span}_{\bar{\mathbf{k}}}\{\partial^p \otimes w_1, \ldots, \partial^p \otimes w_n\} = \\ &= \bar{\mathbf{k}} \otimes \left(V \oplus \bigoplus_{i=1}^{p} V_i \right), \end{aligned}$$

where $V_i = \operatorname{span}_{\mathbf{k}}\{\partial^i \otimes w_1, \ldots, \partial^i \otimes w_n\}$.

Conversely, let

$$\bar{\mathbf{k}} \otimes V^{(1)} = (\bar{\mathbf{k}} \otimes V) \oplus V_1,$$

where $V_1 \cong \bar{\mathbf{k}} \otimes V$. Choose bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ in V and V_1, respectively. There exists a matrix $A \in \mathbf{M}_{(2\cdot n)\times n}(\mathbf{k})$ such that

$$(\partial \otimes v_1, \ldots, \partial \otimes v_n) = (v_1, \ldots, v_n, w_1, \ldots, w_n) \cdot A.$$

For a matrix $B \in \mathbf{GL}_n\left(\bar{\mathbf{k}}\right)$ we have

$$\begin{aligned} \partial \otimes ((v_1, \ldots, v_n) \cdot B) &= (\partial \otimes (v_1, \ldots, v_n)) \cdot B + (v_1, \ldots, v_n) \cdot \partial(B) = \\ &= (v_1, \ldots, v_n, w_1, \ldots, w_n) \cdot AB + (v_1, \ldots, v_n) \cdot \partial(B). \end{aligned}$$

We will first show that there are $n \times n$ invertible matrices B and D such that

$$\partial \otimes ((v_1, \ldots, v_n) \cdot B) = (w_1, \ldots, w_n) \cdot D. \tag{4}$$

It follows then that

$$AB + \begin{pmatrix} \partial B \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ D \end{pmatrix}.$$

Let $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$. We then obtain the following system:

$$\begin{cases} \partial B & = -A_1 B; \\ A_2 B & = D. \end{cases} \tag{5}$$

Since the field $\bar{\mathbf{k}}$ is differentially closed, there exists a matrix $B \in \mathbf{GL}_n\left(\bar{\mathbf{k}}\right)$ satisfying the first equation of system (5). Now, equation (4) forces D to be invertible as well, as the dimension of the span of $\partial \otimes ((v_1, \ldots, v_n) \cdot B)$ is equal to n.

Let

$$(u_1, \ldots, u_n) = (v_1, \ldots, v_n) \cdot B.$$

Consider $g \in G$ and its matrix $A_g = (a_{ij})$ with respect to the basis $\{u_1, \ldots, u_n\}$. For each j, $1 \leqslant j \leqslant n$, we have:

$$g \cdot \partial \otimes u_j = \partial \otimes \left(\sum_{i=1}^{n} a_{ij} u_i \right) = \sum_{i=1}^{n} \left(\partial(a_{ij}) \otimes u_i + a_{ij} \cdot \partial \otimes u_i \right).$$

By the construction, the space

$$\operatorname{span}_{\mathbf{k}} \{\partial \otimes u_1, \ldots, \partial \otimes u_n\}$$

is G invariant. This implies that $\partial(a_{ij}) = 0$ for all i and j, $1 \leqslant i, j \leqslant n$. Thus, $A_g \in \mathbf{M}_n(C)$. □

Corollary 1. *A linear differential algebraic group $G \subset \mathbf{GL}(V)$ is differentially isomorphic over $\bar{\mathbf{k}}$ to an algebraic subgroup of some $\mathbf{GL}_m(C)$ if and only if* **there exists** *a faithful representation W of G such that*

$$\bar{\mathbf{k}} \otimes W^{(1)} = \bar{\mathbf{k}} \otimes (W \oplus W_1),$$

where $W_1 \cong W$ as differential representations of the group G.

Proof. Let $\bar{\mathbf{k}} \otimes W^{(1)} = \bar{\mathbf{k}} \otimes (W \oplus W_1)$. The representation morphism $r_W : G \to \mathbf{GL}(W)$ is a differential algebraic group homomorphism and is a differential isomorphism between G and the image $r_W(G)$. By Proposition 3 the group $r_W(G)$ is conjugate to a group of matrices with constant coefficients. Composition of this isomorphism with r_W gives the desired differential algebraic group isomorphism.

Let now G be differentially isomorphic to a subgroup of $\mathbf{GL}_m(C)$. This gives a faithful representation W of G. Moreover, since the matrices have constant entries, we have $\bar{\mathbf{k}} \otimes W^{(1)} = \bar{\mathbf{k}} \otimes (W \oplus W)$. □

6. Tannaka's theorem for linear differential algebraic groups

In this section we will show how one can recover a linear differential algebraic group knowing all its representations. For this we first prove Tannaka's theorem (Theorem 2). Then using this fact we reconstruct the differential Hopf algebra of functions on the group in Sections 6.3 and 6.4. Parts of the proof closely follow [14, Section 2.5]. The novelty lies in the fact that we can recover the differential structure on the Hopf algebra.

6.1. **Preliminaries.** Let G be a linear differential algebraic group with the Hopf algebra $A := \mathbf{k}\{G\}$. Note that A is also a locally finite G-module with the action (3) by [2, Theorem, page 230]. Let $\omega : \mathbf{Rep}_G \to \mathcal{V}$ be the forgetful functor.

Definition 8. For a ∂-$\mathbf{k}$-algebra B we define the group $\mathrm{Aut}^{\otimes,\partial}(\omega)(B)$ to be the set of sequences

$$\lambda(B) = (\lambda_X \mid X \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)) \in \mathrm{Aut}^{\otimes,\partial}(\omega)(B)$$

such that λ_X is a B-linear automorphism of $\omega(X)\otimes B$ for each G-space X, $\omega(X) \in \mathcal{O}\mathrm{b}(\mathcal{V})$, that is, $\lambda_X \in \mathrm{Aut}_B(\omega(X)\otimes B)$, such that

- for all X_1, X_2 we have

$$\lambda_{X_1\otimes X_2} = \lambda_{X_1}\otimes\lambda_{X_2}, \tag{6}$$

- $\lambda_{\underline{1}}$ is the identity map on $\underline{1}\otimes B = B$,
- for every $\alpha \in \mathrm{Hom}_G(X,Y)$ we have

$$\lambda_Y\circ(\alpha\otimes\mathrm{id}_B) = (\alpha\otimes\mathrm{id}_B)\circ\lambda_X : X\otimes B \to Y\otimes B, \tag{7}$$

- for every X we have

$$\partial\circ\lambda_X = \lambda_{X^{(1)}}\circ\partial, \tag{8}$$

- the group operation $\lambda_1(B)\cdot\lambda_2(B)$ is defined by composition in each set $\mathrm{Aut}_B(\omega(X)\otimes B)$.

$\mathrm{Aut}^{\otimes,\partial}(\omega)$ is a functor from the category of ∂-$\mathbf{k}$-algebras to groups:

$$B \mapsto \mathrm{Aut}^{\otimes,\partial}(\omega)(B).$$

Any $g \in G(B)$ determines an element $\lambda_g \in \mathrm{Aut}^{\otimes,\partial}(\omega)(B)$, because if $X \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)$ and $\Phi_B : G(B) \to \mathbf{GL}(\omega(X)\otimes B)$ then $\Phi(g)$ is a B-linear automorphism of $\omega(X)\otimes B$ and the property (7) is satisfied by the definition of G-equivariance of α. So, we have a morphism Φ of functors

$$G \to \mathrm{Aut}^{\otimes,\partial}(\omega), \quad g \in G(B) \mapsto \Phi(g) \in \mathrm{Aut}^{\otimes,\partial}(\omega)(B)$$

for any ∂-$\mathbf{k}$-algebra B as for any $\varphi : B_1 \to B_2$ we have the following commutative diagram:

$$\begin{array}{ccc} G(B_1) & \xrightarrow{\Phi_{B_1}} & \mathrm{Aut}^{\otimes,\partial}(\omega)(B_1) \\ \Big\downarrow{\scriptstyle G(\varphi)} & & \Big\downarrow{\scriptstyle \mathrm{Aut}^{\otimes,\partial}(\omega)(\varphi)} \\ G(B_2) & \xrightarrow{\Phi_{B_2}} & \mathrm{Aut}^{\otimes,\partial}(\omega)(B_2) \end{array}$$

where $G(\varphi)$ and $\mathrm{Aut}^{\otimes,\partial}(\omega)(\varphi)$ denote the morphisms

$$[\psi : A \to B_1] \mapsto [G(\varphi)(\psi) = \varphi\circ\psi : A \to B_2];$$

$$[\lambda(B_1) = (\lambda_X(B_1) : \omega(X)\otimes B_1 \to \omega(X)\otimes B_1)] \mapsto$$

$$\Big[\mathrm{Aut}^{\otimes,\partial}(\omega)(\varphi)(\lambda(B_1)) = \lambda(B_2) =$$

$$= \big(\big(\mathrm{id}_{\omega(X)}\otimes\varphi\big)\circ\lambda_X(B_1) : \omega(X)\otimes B_2 \to \omega(X)\otimes B_2\big)\Big],$$

respectively. The latter means that we take the restriction of $\lambda_X(B_1)$ to $\omega(X)$ and map it to $\omega(X)\otimes B_1$ and then apply $\mathrm{id}_{\omega(X)}\otimes\varphi$ mapping it to $\omega(X)\otimes B_2$. At the end we prolong such a map to $\omega(X)\otimes B_2$ by B_2-linearity.

6.2. **The theorem.**

Theorem 2. *For a linear differential algebraic group G let $\omega : \mathbf{Rep}_G \to \mathcal{V}$ be the forgetful functor. Then*
$$G \cong \mathrm{Aut}^{\otimes,\partial}(\omega)$$
as functors.

Proof. (Following [14, Theorem 2.5.3] with *differential* modification) Since $g \in G$ determines an element of $\mathrm{Aut}^{\otimes,\partial}(\omega)$, we only need to show the converse. Let B be a ∂-$\mathbf{k}$-algebra and $(\lambda_X) \in \mathrm{Aut}^{\otimes,\partial}(\omega)(B)$. Let $V \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)$ not necessarily finite dimensional but locally finite. This means that every vector $v \in V$ is contained in a differential G-module W with $\dim_{\mathbf{k}} W < \infty$.

We will show that for given B, λ, and a locally finite differential G-module V, there exists a B-linear automorphism of $V \otimes B$ (denoted by λ_V) such that the properties (6), (7), and (8) are satisfied and for any $W \subset V$ we have $\lambda_V|_W = \lambda_W$. For $v \in V$ we take $W \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)$ such that $v \in W$ and $\dim_{\mathbf{k}} W < \infty$.

First, define $\lambda_V(v) := \lambda_W(v)$. We need to show the correctness. Let W' be another representation such that $v \in W'$. Consider the G-module $W \cap W' \ni Gv$. From (7) it follows that
$$\lambda_{W'}(v) = \lambda_{W\cap W'}(v) = \lambda_W(v).$$
Since each λ_W is invertible and linear, the map λ_V is also invertible and linear. We are going to show now that (6), (7), and (8) hold for locally finite modules. For locally finite V and V' we choose $W \ni v$ and $W' \ni v'$, objects of $\mathcal{V}$. Then
$$\lambda_{V\otimes V'}(v \otimes v') = \lambda_{W\otimes W'}(v \otimes v') = \lambda_W(v) \otimes \lambda_{W'}(v') = \lambda_V(v) \otimes \lambda_{V'}(v').$$
Hence, the property (6) is satisfied. Also, let $\alpha \in \mathrm{Hom}_G(V, V')$. Consider $v \in V$ and W such that $v \in W$ and $\dim W < \infty$ and of the same kind $W' \supset \alpha(W)$. We obtain that
$$\lambda_{V'} \circ \alpha(v) = \lambda_{W'}\left(\alpha\big|_W(v)\right) = \alpha\big|_W \circ \lambda_W(v) = \alpha \circ \lambda_V(v).$$
We then have (7). Moreover, since $W^{(1)} \subset V^{(1)}$, we have
$$\partial \circ \lambda_V(v) = \partial \circ \lambda_W(v) = \lambda_{W^{(1)}}(\partial v) = \lambda_{V^{(1)}}(\partial v),$$
which implies (8). Thus, we may say $\lambda_V \in \mathrm{Aut}_B(V \otimes B)$.

Recall that the G-module A is locally finite via
$$\rho : A \times G \to A, \quad (\rho(g)f)(x) = f(x \cdot g)$$
by [2, Theorem, page 230], where $x, g \in G(B)$ and $f \in A = \mathbf{k}\{G\}$. The same is true for $A \otimes A$. Consider the multiplication map
$$m : A \otimes A \to A, \quad f \otimes h \mapsto f \cdot h$$
which is G-equivariant, because each $g \in G(B)$ is a (B-linear) algebra automorphism of $A \otimes B$. According to (6) and (7) we have
$$m \circ (\lambda_A \otimes \lambda_A) = m \circ \lambda_{A\otimes A} = \lambda_A \circ m.$$
Moreover, from (8) we conclude that λ_A is a ∂-$\mathbf{k}$-algebra automorphism $A \to A$. We will show that this must correspond to an element in the group. More precisely, there exists a differential algebraic automorphism $\varphi : G(B) \to G(B)$ such that for all $f \in A$ and $g \in G(B)$ we have
$$\lambda_A(f)(g) = f(\varphi(g)).$$

We will show that this morphism φ is right multiplication by an element of $G(B)$. This will show that an element of the group corresponds to λ_A. After that we demonstrate that the algebra A can be replaced by any G-module.

For every $f \in A$ and g, $h \in G(B)$ we have $\Delta(f)(g,h) = f(gh)$. Take any $f \in A$ and $g, x, y \in G(B)$. We have

$$\Delta \circ \rho(g)(f)(x,y) = f(xyg) = (\mathrm{id}_A \otimes \rho(g)) \circ \Delta(f)(x,y).$$

Hence,

$$\Delta \circ \rho(g) = (\mathrm{id}_A \otimes \rho(g)) \circ \Delta. \tag{9}$$

Consider the locally finite G-module $U := A \otimes A$ via $r_U = \mathrm{id}_A \otimes \rho$. Then, by (9) the map Δ is G-equivariant for ρ and r_U. From (7) we have $\Delta \circ \lambda_A = \lambda_U \circ \Delta$. Because of (6) we obtain that

$$\lambda_U = \lambda_{A,\mathrm{id}_A} \otimes \lambda_{A,\rho} = \mathrm{id} \otimes \lambda_{A,\rho},$$

because $\lambda_I = \mathrm{id}$ and λ is $\mathbf{k}$-linear. Thus,

$$\Delta \circ \lambda_A = (\mathrm{id} \otimes \lambda_A) \circ \Delta.$$

For any $f \in A$ and g, $h \in G$ we have $\Delta \circ \lambda_A(f)(g,h) = f(\varphi(gh))$. On the other hand,

$$(\mathrm{id} \otimes \lambda_A) \circ \Delta(f)(g,h) = f(g\varphi(h)).$$

Thus,

$$\varphi(gh) = g\varphi(h).$$

Let

$$x = \varphi(e),$$

which is a *differential* algebra homomorphism $A \to B$. From this we conclude that for any $g \in G(B)$ one has $\varphi(g) = gx$. Hence, $\lambda_A = \rho(x)$. It remains to show that other automorphisms λ_V look the same (completely determined by this element x).

Consider any $V \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)$ with the action r_V. For any $u \in V^*$ there is a G-homomorphism

$$\varphi_u : V \to A, \quad v \mapsto \varphi_u(v), \quad \varphi_u(v)(g) := u(\rho(g) \cdot v),$$

where $v \in V$. By (7) and the above, we have $\rho(x) \circ \varphi_u = \lambda_A \circ \varphi_u = \varphi_u \circ \lambda_V$. Take any $g \in G(B)$ and $v \in V$. We then have

$$\begin{aligned}\rho(x)(u(r_V(g)(v))) &= u(r_V(gx)(v)),\\ \varphi_u \circ \lambda_V(v)(g) &= u(r_V(g) \circ \lambda_V(v)).\end{aligned}$$

Thus,

$$r_V(x) = r_V(ex) = r_V(e) \circ \lambda_V = \lambda_V,$$

because the elements v, g, and u were arbitrary. $\square$

6.3. **Recovering the differential Hopf algebra of G.** Similar to [14, Sections 2.5.4–2.5.8] we can recover the Hopf algebra $A = \mathbf{k}\{G\}$ in the following way. In addition, we show how to obtain the *differential structure* on A (see Lemma 7).

First step. We will construct the map ψ_V from "some representations" of G to the algebra A of regular differential functions on G and study main properties of ψ_V.

Recall that for $V \in \mathcal{V}$ we denote $V^{(i)} = \mathbf{k}[\partial]_{\leqslant i} \otimes V$ (non-commutative tensor product) and sometimes we write $V^{(0)}$ instead of V for convenience. For $V \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)$ and

$$v \in V,\ \ u \in V^*$$

we have the linear map

$$\psi_V :\ V \otimes V^* \to A, \quad \psi_V(v \otimes u)(g) = u(r_V(g)\cdot v). \tag{10}$$

Also we introduce the following map:

$$F : V^* \to \left(V^{(1)}\right)^*, \quad F(u)(v) = u(v),\ F(u)(\partial \otimes v) = \partial(u(v)),\ v \in V.$$

Lemma 4. *We have the following properties:*

(1) *If* $\phi \in \mathrm{Hom}_G(V,W)$ *then*
$$\psi_V \circ (\mathrm{id} \otimes \phi^*) = \psi_W \circ (\phi \otimes \mathrm{id})$$
as maps of $V \otimes W^* \to A$.

(2) *We have*
$$\psi_{V\otimes W} = m \circ (\psi_V \otimes \psi_W) \circ c,$$
where $c : (V \otimes W) \otimes (V \otimes W)^* \cong (V \otimes V^*) \otimes (W \otimes W^*)$.

(3) *Moreover,*
$$\partial(\psi_V(v \otimes u)) = \psi_{V^{(1)}}\left((\partial v) \otimes F(u)\right).$$

Proof. For $v \in V$, $u \in V^*$, and $g \in G$ we have

$$\begin{aligned}\psi_V \circ (\mathrm{id} \otimes \phi^*)(v \otimes u)(g) &= \psi_V(v \otimes \phi^*(u))(g) = \phi^*(u)(r_V(g)\cdot v) = \\ &= u(\phi(r_V(g)\cdot v)) = u(r_W(g)\cdot \phi(v)) = \\ &= \psi_W \circ (\phi \otimes \mathrm{id})(v \otimes u)(g),\end{aligned}$$

Furthermore, consider $w \in W$ and $t \in W^*$. We then also have

$$\begin{aligned}\psi_{V\otimes W}(v \otimes w \otimes u \otimes t)(g) &= (u \otimes t)(r_{V\otimes W}(g)\cdot(v \otimes w)) = \\ &= (u \otimes t)((r_V(g)\cdot v) \otimes (r_W(g)\cdot w)) = \\ &= u(r_V(g)\cdot v)\cdot t(r_W(g)\cdot w) = \\ &= m \circ (\psi_V \otimes \psi_W)((v \otimes u) \otimes (w \otimes t))(g)) = \\ &= m \circ (\psi_V \otimes \psi_W) \circ c(v \otimes w \otimes u \otimes t)(g).\end{aligned}$$

Let $\{e_1,\ldots,e_n\}$ be a basis of V with the dual basis $\{f_1,\ldots,f_n\}$ and take $f \in \mathbf{k}$. We have:

$$\begin{aligned}\partial(\psi_V(f\cdot e_i \otimes f_j))(g) &= \partial(f\cdot f_j(r_V(g)\cdot e_i)) = \partial\left(f\cdot g_{ij}^V\right) = \\ &= \partial(f)\cdot g_{ij}^V + f\cdot\partial\left(g_{ij}^V\right) = \\ &= \partial(f)\cdot f_j(r_V(g)\cdot e_i) + f\cdot\partial(f_j(r_V(g)e_i)) = \\ &= \partial(f)\cdot\psi_{V^{(1)}}(e_i \otimes f_j)(g) + f\cdot\psi_{V^{(1)}}((\partial e_i)\otimes F(f_j))(g) = \\ &= \psi_{V^{(1)}}\left(\partial(f\cdot e_i)\otimes F(f_j)\right)(g).\end{aligned}$$

$\square$

Second step. Here, we will construct a differential algebra $\mathcal{A}$ (that will be our candidate for A) using representations of G as objects of $\mathcal{V}$ together with morphisms between them and not using any other information from G.

Let

$$\mathcal{F} = \bigoplus_{V \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)} V \otimes V^*.$$

So, the canonical injections

$$i_V : V \otimes V^* \to \mathcal{F}$$

are defined. Consider the subspace $\mathcal{R}$ of $\mathcal{F}$ spanned by

$$\left\{ \left(i_V(\mathrm{id} \otimes \phi^*) - i_W(\phi \otimes \mathrm{id})\right)(z) \,\middle|\, V,\ W \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G), \phi \in \mathrm{Hom}\,(V, W)\,, z \in V \otimes W^* \right\}.$$

We now put $\mathcal{A} = \mathcal{F} \,/\, \mathcal{R}$. For $v \in V$ and $u \in V^*$ we denote by

$$a_V\,(v \otimes u)$$

the image in $\mathcal{A}$ of

$$i_V\,(v \otimes u)\,.$$

So, for any $\phi \in \mathrm{Hom}_G(V, W)$ we have

$$a_V(v \otimes \phi^*(u)) = a_W(\phi(v) \otimes u). \tag{11}$$

Lemma 5. *For all $v \in V$ and $u \in V^*$ we have*

$$a_V(v \otimes u) = a_{V^{(1)}}(\partial v \otimes \varphi^*(u)), \tag{12}$$

where the morphism

$$\varphi : V^{(1)} \to V, \quad 1 \otimes v \mapsto 0,\ \partial \otimes v \mapsto v.$$

Proof. Follows from formula (11). □

Take $v \in V$, $w \in W$, $u \in V^*$, $t \in W^*$. Let also $\{v_i\}$ be a basis of V and $\{u_i\}$ be its dual. Introduce the following operations on $\mathcal{A}$:

$$m(a_V(v \otimes u), a_W(w \otimes t)) = a_{V \otimes W}((v \otimes w) \otimes (u \otimes t)), \tag{13}$$

$$\partial(a_V(v \otimes u)) = a_{V^{(1)}}(\partial v \otimes F(u)), \tag{14}$$

$$\tilde{\Delta}(a_V(v \otimes u)) = \sum_j a_V(v_j \otimes u) \otimes a_V(v \otimes u_j), \tag{15}$$

$$\tilde{S}(a_V(v \otimes u)) = a_{V^*}(u \otimes v). \tag{16}$$

Proposition 4. *The $\mathbf{k}$-vector space $\mathcal{A}$ contains a non-zero vector.*

Proof. Let $\mathcal{C}$ be the category of differential representations of the trivial differential group $G_e = \{e\}$. Note that the algebra of G_e is just the differential field $\mathbf{k}$. Let V, $W \in \mathcal{O}\mathrm{b}(\mathcal{C})$. Choose ordered bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ of V and W, respectively. Take $a_V(v_1 \otimes v_2^*)$ and $a_W(0 \otimes w_2^*)$. Consider the linear map:

$$\phi : V \to W, \quad v_2 \mapsto w_2,\ v_i \mapsto 0,\ i \neq 2.$$

We have $\phi^*(w_2^*) = v_2^*$. Then,

$$a_V(v_1 \otimes v_2^*) = a_V(v_1 \otimes \phi^*(w_2^*)) = a_W(0 \otimes w_2^*) = 0.$$

Let now $0 \neq v \in V$ and $0 \neq w \in W$. Without a loss of generality we may assume that $v = v_1$ and $w = w_1$. Consider the linear map

$$\phi : V \to W, \quad v_1 \mapsto w_1,\ v_i \mapsto 0,\ i \neq 1.$$

We have $\phi^*(w^*) = v^*$. Then,

$$a_V(v\otimes v^*) = a_V\left(v\otimes \phi^*(w^*)\right) = a_W(w\otimes w^*).$$

Thus, if for the trivial representation $\underline{1} = \operatorname{span}_{\mathbf{k}}\{e\}$ then

$$\mathcal{A}_{G_e} = \operatorname{span}_{\mathbf{k}}\left\{a_{\underline{1}}(e\otimes e^*)\right\}.$$

But $a_{\underline{1}}(e\otimes e^*) \neq 0$ in $\mathcal{A}_{G_e}$. The vector space $\mathcal{A}_{G_e}$ is just the quotient of $\mathcal{F}$ by the subspace generated by the relations coming from all possible morphisms $\phi : V \to W$. And the latter subspace is not the whole $\mathcal{F}$ as it has been shown above. Since there is a surjective linear map $\mathcal{A} \to \mathcal{A}_{G_e}$, this implies that $\mathcal{A}$ contains a non-zero vector. □

Lemma 6. *The definition of m is correct and provides a structure of a commutative associative algebra on the $\mathbf{k}$-vector space $\mathcal{A}$ and the unit 1 is given by the trivial representation $\underline{1}$.*

Proof. Consider morphisms $\phi_1 : V \to X$ and $\phi_2 : W \to Y$ as G-vector spaces and vectors

$$v\in V,\ u\in V^*,\ w\in W,\ t\in W^*,\ x\in X^*,\ y\in Y^*$$

such that $\phi_1^*(x) = u$ and $\phi_2^*(y) = t$. We then have:

$$\begin{aligned} a_V(v\otimes u)\cdot a_W(w\otimes t) =& a_V(v\otimes \phi_1^*(x))\cdot a_W(w\otimes \phi_2^*(y)) =\\ &= a_X(\phi_1(v)\otimes x)\cdot a_Y(\phi_2(w)\otimes y) =\\ &= a_{X\otimes Y}((\phi_1(v)\otimes \phi_2(w))\otimes (x\otimes y)) =\\ &= a_{X\otimes Y}((\phi_1\otimes\phi_2)(v\otimes w)\otimes (x\otimes y)) =\\ &= a_{V\otimes W}((v\otimes w)\otimes (u\otimes t)). \end{aligned}$$

We now prove that the multiplication is associative and commutative. Consider the morphism

$$\phi \in \operatorname{Hom}(V\otimes W, W\otimes V),\quad v\otimes w\mapsto w\otimes v.$$

We then have

$$\begin{aligned} a_V(v\otimes u)\cdot a_W(w\otimes t) = a_{V\otimes W}((v\otimes w)\otimes(u\otimes t)) = a_{W\otimes V}((w\otimes v)\otimes(t\otimes u)) =\\ = a_W(w\otimes t)\cdot a_V(v\otimes u). \end{aligned}$$

So, the multiplication is commutative.

For $X \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)$ and $x\in X$, $y\in X^*$ we also have

$$\begin{aligned} &(a_V(v\otimes u)\cdot a_W(w\otimes t))\cdot a_X(x\otimes y) =\\ &= a_{V\otimes W}((v\otimes w)\otimes(u\otimes t))\cdot a_X(x\otimes y) =\\ &= a_{(V\otimes W)\otimes X}((v\otimes w)\otimes x)\otimes((t\otimes u)\otimes y)) =\\ &= a_{V\otimes(W\otimes X)}((v\otimes(w\otimes x))\otimes(t\otimes(u\otimes y))) =\\ &= a_V(v\otimes t)\cdot a_{W\otimes X}((w\otimes x)\otimes(u\otimes y)) =\\ &= a_V(v\otimes u)\cdot(a_W(w\otimes t)\cdot a_X(x\otimes y)). \end{aligned}$$

We have shown that the multiplication is associative.

Let $\underline{1}$ be the trivial representation of the group G and $0\neq e\in\underline{1}$, $f\in\underline{1}^*$, $f(e)=1$. We have the morphism

$$\varphi\in\operatorname{Hom}(V, V\otimes\underline{1}),\quad v\mapsto v\otimes e.$$

Then

$$
\begin{aligned}
a_V(v\otimes u)\cdot a_{\underline{1}}(e\otimes f) = a_{V\otimes\underline{1}}((v\otimes e)\otimes(u\otimes f)) = a_{V\otimes\underline{1}}(\varphi(v)\otimes(u\otimes f)) =\\
= a_V(v\otimes\varphi^*(u\otimes f)) = a_V(v\otimes u).
\end{aligned}
$$

Thus, $\mathcal{A}$ is a commutative associative algebra with unity. □

Our main goal is to recover the *differential* Hopf algebra A. We give a differential structure on $\mathcal{A}$ and then show that this structure corresponds to the one of A. Let us describe the *intuition* behind the construction we are going to present.

We are recovering a subgroup of $\mathbf{GL}_n$. Let us denote the corresponding matrix coordinate functions by y_{ij}. We must be able to:

(1) differentiate these functions y_{ij} obtaining $y'_{ij},\ldots,y^{(p)}_{ij},\ldots$;
(2) multiply the results of this differentiation and stay in the algebra.

For bases $v_1,\ldots,v_n$ and $u_1,\ldots,u_n$ of V and V^*, respectively, the coordinate functions mapping $G\to\mathbf{k}$ are given by

$$
y_{ij}(g) = \psi_V(v_i\otimes u_j)(g) = u_j(r_V(g)\cdot v_i),
$$

where $g\in G$. The candidates for these functions in $\mathcal{A}$ are, certainly, $a_V(v_i\otimes u_j)$. Our correspondence between $\mathcal{A}$ and A must preserve differentiation. So, y'_{ij} corresponds to $\partial(a_V(v_i\otimes u_j))$ which we still need to define.

Moreover, such a definition must leave us in the same category and satisfy the product rule for differentiation. We also notice that since $y'_{ij},\ldots,y^{(p)}_{ij},\ldots$ and $y^{(q)}_{ij}\cdot y^{(r)}_{kl}$ are monomials, we should preserve this property for $\partial^q\left(a_V(v_i\otimes u_j)\right)\cdot\partial^r\left(a_V(v_k\otimes u_l)\right)$. These ideas are implemented in Lemma 7 and Theorem 3.

Lemma 7. *The natural differential structure on $\mathcal{A}$ introduced in formula* (14) *makes it a ∂-$\mathbf{k}$-algebra.*

Proof. First of all, recall that

$$
\partial\left(a_V(v\otimes u)\right) = a_{V^{(1)}}((\partial v)\otimes F(u)).
$$

Recall also that we let $F(u)(\partial v)=\partial(u(v))$ for any $v\in V$ and $u\in V^*$. We need to show its correctness with respect to the morphisms. Let $t\in W^*$. We have:

$$
\begin{aligned}
\partial(a_W(\phi(v)\otimes t)) = a_{W^{(1)}}(\partial(\phi(v))\otimes F(t)) = a_{W^{(1)}}(\phi(\partial v)\otimes F(t)) =\\
= a_{V^{(1)}}(\partial v\otimes\phi^*(F(t))) = \partial(a_V(v\otimes\phi^*(t)))
\end{aligned}
$$

for a morphism $\phi: V\to W$ that we naturally prolong to a morphism $\phi: V^{(1)}\to W^{(1)}$ mapping $\partial v\mapsto\partial(\phi(v))$. Hence, the differentiation is correct.

We need to show the product rule. We have:

$$
\begin{aligned}
&\partial\left(a_V(v\otimes u)\cdot a_W(w\otimes t)\right) = \partial\left(a_{V\otimes W}((v\otimes w)\otimes(u\otimes t))\right) =\\
&= a_{(V\otimes W)^{(1)}}(\partial(v\otimes w)\otimes F(u\otimes t)) =\\
&= a_{V^{(1)}\otimes W^{(1)}}((\partial v\otimes w)\otimes(F(u)\otimes F(t)) + (v\otimes\partial w)\otimes(F(u)\otimes F(t))) =\\
&= a_{V^{(1)}}(\partial v\otimes F(u))\cdot a_W(w\otimes t) + a_V(v\otimes u)\cdot a_{W^{(1)}}(\partial w\otimes F(t)) =\\
&= (\partial a_V(v\otimes u))\cdot a_W(w\otimes t) + a_V(v\otimes u)\cdot(\partial a_W(w\otimes t)),
\end{aligned}
$$

where we have used the morphism:

$$\begin{aligned}
&\psi : (V\otimes W)^{(1)} \to V^{(1)}\otimes W^{(1)},\\
&\psi : 1\otimes x\otimes z \mapsto (1\otimes x)\otimes(1\otimes z),\\
&\psi : \partial\otimes x\otimes z \mapsto (\partial x)\otimes z + x\otimes(\partial z).
\end{aligned}$$

Its dual

$$\psi^* : \left(V^{(1)}\otimes W^{(1)}\right)^* \to \left((V\otimes W)^{(1)}\right)^*$$

maps

$$F(u)\otimes F(t)\mapsto F(u\otimes t)$$

Indeed,

$$\begin{aligned}
&\psi^*(F(u)\otimes F(t))(1\otimes x\otimes z) = (F(u)\otimes F(t))((1\otimes x)\otimes(1\otimes z)) =\\
&= F(u)(1\otimes x)\cdot F(t)(1\otimes z) = u(x)\cdot t(z) = F(u\otimes t)(1\otimes x\otimes z)
\end{aligned}$$

and

$$\begin{aligned}
&\psi^*(F(u)\otimes F(t))(\partial\otimes x\otimes z) = (F(u)\otimes F(t))(\partial x\otimes z + x\otimes\partial z)) =\\
&= F(u)(\partial x)\cdot F(t)(1\otimes z) + F(u)(1\otimes x)\cdot F(t)(\partial z) =\\
&= \partial(u(x))\cdot t(z) + u(x)\cdot\partial(t(z)) = \partial(u(x)\cdot t(z)) = F(u\otimes t)(\partial\otimes x\otimes z).
\end{aligned}$$

□

Third step. Now, we can show that the differential algebra $\mathcal{A}$ we have constructed is what we were looking for.

Lemma 8. *Let $\mathcal{C}$ be a rigid abelian tensor category with a tensor $\mathbf{k}$-linear functor $\omega : \mathcal{C}\to\mathcal{V}$ then*

$$\mathrm{End}^{\otimes}(\omega) = \mathrm{Aut}^{\otimes}(\omega).$$

Proof. We expand the proof that appears in [4, Proposition 1.13]. Let $\lambda : \omega\to\omega$. For each $X\in\mathcal{C}$ there exists a morphism $t_X : \omega(X)^*\to\omega(X)^*$ such that the following diagram is commutative:

$$\begin{tikzcd}
\omega(X^*) \arrow[r, "\lambda_{X^*}"] \arrow[d, "\varphi"', "\cong"] & \omega(X^*) \arrow[d, "\varphi"', "\cong"] \\
\omega(X)^* \arrow[r, "t_X"] & \omega(X)^*
\end{tikzcd}$$

The category $\mathcal{V}$ is rigid. So, for all $U,V\in\mathcal{O}\mathrm{b}(\mathcal{V})$ we have $\underline{\mathrm{Hom}}(U,V)\cong\underline{\mathrm{Hom}}(V^*,U^*)$. We then let $\mu_X := (t_X)^* : \omega(X)\to\omega(X)$. For any $f : X\to Y$ the following diagram commutes:

$$\begin{tikzcd}
Y \arrow[r] & Y^* \arrow[r, "\omega"] \arrow[d, "f^*"] & \omega(Y^*) \arrow[r, "\lambda_{Y^*}"] \arrow[d, "\omega(f^*)"] & \omega(Y^*) \arrow[d, "\omega(f^*)"] \\
X \arrow[r] \arrow[u, "f"] & X^* \arrow[r, "\omega"] & \omega(X^*) \arrow[r, "\lambda_{X^*}"] & \omega(X^*)
\end{tikzcd}$$

Gathering all commutative diagrams together we obtain that $\mu = (\mu_X) \in \operatorname{End}^{\otimes}(\omega)$. We now show that $\mu = \lambda^{-1}$. We have

$$\begin{array}{ccccc} X^*\otimes X & \xrightarrow{\ \omega\ } & \omega(X^*\otimes X) & \xrightarrow{\lambda_{X^*\otimes X}} & \omega(X^*\otimes X) \\ \downarrow{\scriptstyle \mathrm{ev}_X} & & \downarrow{\scriptstyle \omega(\mathrm{ev}_X)} & & \downarrow{\scriptstyle \omega(\mathrm{ev}_X)} \\ \underline{1} & \xrightarrow{\ \omega\ } & \omega(\underline{1}) & \xrightarrow{\ \mathrm{id}\ } & \omega(\underline{1}) \end{array}$$

The evaluation morphism takes $f \otimes x \in \omega(X)^* \otimes \omega(X)$ and evaluates providing $f(x) \in \omega(\underline{1})$ as its output. Take any $y \in \omega(X)^*$ and $x \in \omega(X)$. Then

$$\begin{aligned} (y, \mu_X \circ \lambda_X(x)) &= (t_X(y), \lambda_X(x)) = (\varphi \circ \lambda_{X^*} \circ \varphi^{-1}(y), \lambda_X(x)) = \\ &= \mathrm{ev}_{\omega(X)} \circ (\varphi \otimes \mathrm{id}) \circ (\lambda_{X^*} \otimes \lambda_X)(\varphi^{-1}(y), x) = \\ &= \mathrm{ev}_{\omega(X)} \circ (\varphi \otimes \mathrm{id}) \circ \lambda_{X^*\otimes X}(\varphi^{-1}(y), x) = (y, x) \end{aligned}$$

and as a result λ_X is injective. Thus, λ_X is also surjective and μ_X is its inverse. We can show this differently:

$$(y, \lambda_X \circ \mu_X(x)) = \mathrm{ev}_{\omega(X)} \circ (\varphi \otimes \mathrm{id}) \circ \mu_{X^*\otimes X}(\varphi^{-1}(y), x) = (y, x)$$

as $\mu \in \operatorname{End}^{\otimes}(\omega)(R)$. □

The proof of the following result that we give differs from the similar one in [14]. Our goal was to provide a correct differential structure. Also, if one follows our proof in a non-differential case one finds that it does *not* depend on $\operatorname{char} \mathbf{k}$. For this commutative case the change (in comparison to [14]) that we make is at the end of the proof where we take the "generic point".

Theorem 3. *We have*

(1) *the algebra $\mathcal{A}$ is a finitely generated ∂-$\mathbf{k}$-algebra;*
(2) *there is a surjective ∂-$\mathbf{k}$-algebra homomorphism $\Phi : \mathcal{A} \to A$ such that*

$$\Phi \circ a_V = \psi_V$$

for all differential G-modules V, where ψ_V is defined in formula (10)*;*
(3) *the map Φ is a ∂-$\mathbf{k}$-algebra isomorphism $\mathcal{A} \to A$.*

Proof. Let $V \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)$. Fix a basis $\{u_1, \ldots, u_n\}$ of V^*. Since A is locally finite and

$$\phi_{u_i} : v \mapsto u_i(r_V(\cdot) \cdot v)$$

is a G module morphism $V \to A$, there is $W \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)$, $W \subset A$, and $\dim W < \infty$ containing the images of ϕ_{u_i} for all i, $1 \leqslant i \leqslant n$. According to the proof of Lemma 3 the induced G-morphism $\phi : V \to W^n$ is injective. Hence, the map ϕ^* is surjective and for $u \in V^*$ there exists $t = (t_1, \ldots, t_n) \in W^n$ such that $u = \phi^*(t)$. We then have

$$a_V(v \otimes u) = a_V(v \otimes \phi^*((t_1, \ldots, t_n))) = a_W(\phi(v) \otimes (t_1, \ldots, t_n)).$$

Thus, the differential algebra $\mathcal{A}$ is generated by the images of the a_V for $A \supset V \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)$ and $\dim V < \infty$. Let V be such a G-submodule of A which also contains 1 and a finite set of generators of A as a ∂-$\mathbf{k}$-algebra. The multiplication on the algebra A defines for any $l \in \mathbb{Z}_{\geqslant 1}$ a surjective G-morphism ϕ_l from $V^{\otimes l}$ onto some $V(l) \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)$ with $\dim V(l) < \infty$. We then have $V(l) \subset V(l+1)$, because $1 \in V$, and $A = \bigcup_{l \geqslant 1} V(l)$.

Consider any $\mathcal{O}\mathrm{b}(\mathbf{Rep}_G) \ni W \subset A$ with $\dim W < \infty$. There exists $l \in \mathbb{Z}_{\geqslant 1}$ such that $W \subset V(l)$. Since ϕ_l is surjective, we have

$$\operatorname{Im} a_W \subset \operatorname{Im} a_{V(l)} \subset \operatorname{Im} a_{V^{\otimes l}}.$$

Because of the multiplication structure of $\mathcal{A}$ the set $\operatorname{Im} a_{V^{\otimes l}}$ lies in the ∂-$\mathbf{k}$-subalgebra generated by $\operatorname{Im} a_V$. Hence, this subalgebra is the whole $\mathcal{A}$.

The homomorphism Φ of the second statement is constructed as follows. We take an element $a_V(v \otimes u)$ and map it to $\psi_V(v, u)$ for all $V \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)$. Since

$$\begin{aligned}
\Phi\left(a_V(v \otimes u) \cdot a_W(w \otimes t)\right) &= \Phi\left(a_{V\otimes W}((v \otimes w) \otimes (u \otimes t))\right) = \\
&= \psi_{V\otimes W}(v \otimes w, u \otimes t) = \\
&= m \circ (\psi_V \otimes \psi_W) \circ c(v \otimes w, u \otimes t) = \\
&= m \circ (\psi_V \otimes \psi_W)(v \otimes u, w \otimes t) = \\
&= m\left(\Phi(a_V(v \otimes u)), \Phi(a_W(w \otimes t))\right),
\end{aligned}$$

the map Φ is a $\mathbf{k}$-algebra homomorphism. Let us show that it is differential. From Lemma 4 we have:

$$\begin{aligned}
\Phi\left(\partial\left(a_V(v \otimes u)\right)\right) = \Phi\left(a_{V^{(1)}}(\partial v \otimes F(u)\right) = \psi_{V^{(1)}}(\partial v \otimes F(u)) = \\
= \partial\left(\psi_V(v \otimes u)\right) = \partial\left(\Phi\left(a_V(v \otimes u)\right)\right).
\end{aligned}$$

We now show the last statement. Let B be a ∂-$\mathbf{k}$-algebra. Consider a point $\xi \in \operatorname{Hom}_{\mathbf{k}[\partial]}(\mathcal{A}, B)$ and $V \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)$. Fix bases $\{v_i\}$ and $\{u_j\}$ of V and V^*, respectively. There is an endomorphism λ_V of $V \otimes B$ such that

$$\langle \lambda_V(v_i), u_j \rangle = u_j(\lambda_V(v_i)) = \xi(a_V(v_i \otimes u_j)).$$

We show now that $(\lambda_V \mid V \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G))$ satisfies the conditions of Theorem 2.

Let $V, W \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)$. Then we have

$$\begin{aligned}
\langle \lambda_{V\otimes W}(v_i \otimes w_j), u_r \otimes t_l \rangle &= \xi(a_{V\otimes W}((v_i \otimes w_j) \otimes (u_r \otimes t_l))) = \\
&= \xi(a_V(v_i \otimes u_r) \cdot a_W(w_j \otimes t_l)) = \\
&= \xi(a_V(v_i \otimes u_r)) \cdot \xi(a_W(w_j \otimes t_l)) = \\
&= \langle \lambda_V(v_i), u_r \rangle \cdot \langle \lambda_W(w_j), t_l \rangle = \\
&= \langle (\lambda_V \otimes \lambda_W)(v_i \otimes w_j), u_r \otimes t_l \rangle.
\end{aligned}$$

Hence, $\lambda_{V\otimes W} = \lambda_V \otimes \lambda_W$. Since $a_{\underline{1}}$ is the identity in $\mathcal{A}$ and $\xi(1_{\mathcal{A}}) = 1$, we have $\lambda_{\underline{1}}$ is the identity. Let us show the functoriality of (λ_V). For a G-equivariant map $\phi : V \to W$ we have

$$\begin{aligned}
\langle (\lambda_W \circ \phi)(v_i), t_j \rangle = \xi(a_W(\phi(v_i) \otimes t_j)) = \xi(a_V(v_i \otimes \phi^*(t_j))) = \\
= \langle \lambda_V(v_i), \phi^*(t_j) \rangle = \langle (\phi \circ \lambda_V)(v_i), t_j \rangle.
\end{aligned}$$

Hence, $\lambda_W \circ \phi = \phi \circ \lambda_V$. Finally, since $\{v_i\}$ and $\{u_j\}$ are dual to each other, we have:

$$\begin{aligned}
\langle \partial \circ \lambda_V(v_i), u_j \rangle = \partial(u_j(\lambda_V(v_i))) = \partial \circ \xi(a_V(v_i \otimes u_j)) = \\
= \xi(\partial a_V(v_i \otimes u_j)) = \xi(a_{V^{(1)}}((\partial v_i) \otimes F(u_j))) = \langle \lambda_{V^{(1)}}(\partial v_i), F(u_j) \rangle.
\end{aligned}$$

Moreover, let $\lambda_V(v_i) = \sum c_{ik} v_k$. Due to (12) we have:

$$\begin{aligned}\langle \partial \circ \lambda_V(v_i), (\partial v_j)^* \rangle = (\partial v_j)^*(\partial(c_{ik}v_k)) &= (\partial v_j)^*(\partial(c_{ik})v_k + c_{ik}\partial v_k) = \\ &= c_{ik} = \xi(a_V(v_i \otimes v_j^*)) = \xi\left(a_{V^{(1)}}\left(\partial v_i \otimes \varphi^*\left(v_j^*\right)\right)\right) = \\ &= \xi(a_{V^{(1)}}(\partial v_i \otimes (\partial v_j)^*)) = \langle \lambda_{V^{(1)}}(\partial v_i), (\partial v_j)^* \rangle.\end{aligned}$$

We then conclude that $\lambda \in \mathrm{End}^{\otimes,\partial}(\omega)$. Since the category $\mathbf{Rep}_G$ is rigid, $\lambda \in \mathrm{Aut}^{\otimes,\partial}(\omega)$ by Lemma 8.

Take $B = \mathcal{A}$ and the generic point $\xi = \mathrm{id}_{\mathcal{A}}$. By Theorem 2 there exists $x \in \mathrm{Hom}_{\mathbf{k}[\partial]}(A, \mathcal{A}) = G(\mathcal{A})$ such that $\lambda_V = r_V(x)$ for all $V \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)$. We have

$$\begin{aligned}x \circ \Phi \circ a_V(v_i \otimes u_j)) = x \circ \psi_V(v_i \otimes u_j) &= \langle r_V(x)v_i, u_j \rangle = \\ &= \langle \lambda_V(v_i), u_j \rangle = \xi(a_V(v_i \otimes u_j)).\end{aligned}$$

Hence, $x \circ \Phi = \xi = \mathrm{id}_{\mathcal{A}}$. This implies that Φ is injective. Since Φ is also surjective, we obtain that $\Phi : \mathcal{A} \to A$ is a ∂-$\mathbf{k}$-algebra isomorphism. $\square$

6.4. **Recovering Δ and S.** We provide a differential Hopf algebra structure to $\mathcal{A}$. Let $V \in \mathcal{O}\mathrm{b}(\mathbf{Rep}_G)$ and $\{v_i\}$ be its basis with the dual basis $\{u_j\}$ of V^*. Recall the $\mathbf{k}$-linear map

$$\tilde{\Delta} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \quad a_V(v \otimes u) \mapsto \sum a_V(v_i \otimes u) \otimes a_V(v \otimes u_i).$$

Lemma 9. *The map $\tilde{\Delta}$ is a ∂-$\mathbf{k}$-algebra homomorphism and is a comultiplication.*

Proof. We first check the basis independence. Let $\{e_1, \ldots, e_n\}$ be another basis for V and $\{f_1, \ldots, f_n\}$ be its dual. Hence, there exits a matrix $C = (c_{ij}) \in \mathbf{GL}_n(\mathbf{k})$ such that $v_i = \sum e_j c_{ji}$. We then have:

$$\begin{aligned}\sum_{i=1}^n a_V(v_i \otimes u) \otimes a_V(v \otimes u_i) &= \sum_{i=1}^n a_V\left(\sum_{j=1}^n e_j c_{ji} \otimes u\right) \otimes a_V(v \otimes u_i) = \\ &= \sum_{j=1}^n a_V\left(e_j \otimes u\right) \otimes a_V\left(v \otimes \sum_{i=1}^n c_{ji} u_i\right) = \\ &= \sum_{j=1}^n a_V\left(e_j \otimes u\right) \otimes a_V(v \otimes f_j).\end{aligned}$$

We check that it is an algebra homomorphism. We have:

$$\begin{aligned}&\tilde{\Delta}(a_V(v \otimes u) \cdot a_W(w \otimes t)) = \tilde{\Delta}(a_{V \otimes W}((v \otimes w) \otimes (u \otimes t))) = \\ &= \sum_{i,j} a_{V \otimes W}((v_i \otimes w_j) \otimes (u \otimes t)) \otimes a_{V \otimes W}((v \otimes w) \otimes (u_i \otimes t_j)) = \\ &= \sum_{i,j} (a_V(v_i \otimes u) \otimes a_V(v \otimes u_i)) \cdot (a_W(w_j \otimes t) \otimes a_W(w \otimes t_j)) = \\ &= \left(\sum_i a_V(v_i \otimes u) \otimes a_V(v \otimes u_i)\right) \cdot \left(\sum_j a_W(w_j \otimes t) \otimes a_W(w \otimes t_j)\right) = \\ &= \tilde{\Delta}(a_V(v \otimes u)) \cdot \tilde{\Delta}(a_W(w \otimes t)).\end{aligned}$$

Moreover, due to identity (12) and the imbedding

$$V \mapsto V^{(1)}, \quad v \mapsto 1 \otimes v,\ v \in V,$$

we have

$$\begin{aligned}\partial \circ \tilde{\Delta}(a_V(v \otimes u)) =& \sum \left(a_{V^{(1)}}(\partial v_i \otimes F(u)) \otimes a_V(v \otimes u_i) + \right.\\ &\left. + a_V(v_i \otimes u) \otimes a_{V^{(1)}}(\partial v \otimes F(u_i))\right) = \\ =& \sum \left(a_{V^{(1)}}(\partial v_i \otimes F(u)) \otimes a_{V^{(1)}}(\partial v \otimes (\partial v_i)^*) + \right.\\ &\left. + a_{V^{(1)}}(v_i \otimes u) \otimes a_{V^{(1)}}(\partial v \otimes F(u_i))\right) = \\ =& \tilde{\Delta}(\partial(a_V(v \otimes u))).\end{aligned}$$

We finally show the coassociativity:

$$\begin{aligned}\left(\tilde{\Delta} \otimes \mathrm{id}\right) \circ \tilde{\Delta}(a_V(v \otimes u)) &= \sum_i \tilde{\Delta}(a_V(v_i \otimes u)) \otimes a_V(v \otimes u_i) = \\ &= \sum_i \left(\sum_j a_V(v_j \otimes u) \otimes a_V(v_i \otimes u_j)\right) \otimes a_V(v \otimes u_i) = \\ &= \sum_i a_V(v_i \otimes u) \otimes \left(\sum_j a_V(v_j \otimes u_i) \otimes a_V(v \otimes u_j)\right) = \\ &= \sum_i a_V(v_i \otimes u) \otimes \tilde{\Delta}(a_V(v \otimes u_i)) = \\ &= \left(\mathrm{id} \otimes \tilde{\Delta}\right) \circ \tilde{\Delta}(a_V(v \otimes u)).\end{aligned}$$

□

Lemma 10. *The map $\tilde{\varepsilon} : a_V(v \otimes u) \mapsto u(v)$ is a counit for $\mathcal{A}$ corresponding to the counit of A.*

Proof. We show that $\tilde{\varepsilon}$ and $\tilde{\Delta}$ satisfy $m \circ (\mathrm{id}_{\mathcal{A}} \otimes \tilde{\varepsilon}) \circ \tilde{\Delta} = \mathrm{id}_{\mathcal{A}}$. We have:

$$\begin{aligned}m \circ (\mathrm{id}_{\mathcal{A}} \otimes \tilde{\varepsilon}) \circ \tilde{\Delta}(a_V(v \otimes u)) &= m \circ (\mathrm{id}_{\mathcal{A}} \otimes \tilde{\varepsilon}) \left(\sum a_V(v_i \otimes u) \otimes a_V(v \otimes u_i)\right) = \\ &= \sum (a_V(v_i \otimes u) \cdot u_i(v)) = \\ &= a_V\left(\left(\sum u_i(v) \cdot v_i\right) \otimes u\right) = a_V(v \otimes u).\end{aligned}$$

In addition, $\tilde{\varepsilon}$ is a differential homomorphism. Indeed,

$$\tilde{\varepsilon}\,(\partial a_V(v \otimes u)) = \tilde{\varepsilon}\,(a_{V^{(1)}}(\partial v \otimes F(u))) = F(u)(\partial v) = \partial(u(v)) = \partial(\tilde{\varepsilon}(a_V(v \otimes u))).$$

Finally, we show that Φ maps $\tilde{\varepsilon}$ to ε.

$$\varepsilon \circ \Phi(a_V(v \otimes u)) = \Phi(a_V(v \otimes u))(e) = u(r_V(e) \cdot v) = u(v) = \Phi(\tilde{\varepsilon}(a_V(v \otimes u)).$$

□

Proposition 5. *The map $\Phi : \mathcal{A} \to A$ is a differential Hopf algebra homomorphism.*

Proof. We show that $\tilde{\Delta}$ is mapped to Δ. We have:

$$
\begin{aligned}
&(\Phi\otimes\Phi)\circ\tilde{\Delta}(a_V(v\otimes u))(g_1,g_2)=\\
&=(\Phi\otimes\Phi)\left(\sum_i(a_V(v_i\otimes u)\otimes a_V(v\otimes u_i)\right)(g_1,g_2)=\\
&=\sum_i u(r_V(g_1)\cdot v_i)\cdot u_i(r_V(g_2)\cdot v);\\
&\Delta\circ\Phi(a_V(v\otimes u))(g_1,g_2)=u(r_V(g_1\cdot g_2)\cdot v)=\\
&=u(r_V(g_1)\cdot(r_V(g_2)\cdot v)).
\end{aligned}
$$

Let $r_V(g_2)\cdot v=\sum c_jv_j$. Then

$$
\begin{aligned}
\sum_i u(r_V(g_1)\cdot v_i)\cdot u_i(r_V(g_2)\cdot v) &= \sum_i u(r_V(g_1)\cdot v_i)\cdot u_i\left(\sum_j c_jv_j\right)=\\
&=\sum_i u(r_V(g_1)\cdot v_i)\cdot c_i=\\
&=\sum_i u(r_V(g_1)\cdot c_iv_i)=\\
&=u(r_V(g_1)\cdot(r_V(g_2)\cdot v)).
\end{aligned}
$$

□

Recall the $\mathbf{k}$-linear map

$$\tilde{S}:\mathcal{A}\to\mathcal{A},\quad a_V(v\otimes u)\mapsto a_{V^*}(u\otimes v).$$

Lemma 11. *The map $\tilde{S}$ is a ∂-$\mathbf{k}$-algebra homomorphism and together with $\tilde{\Delta}$ gives a differential Hopf algebra structure on $\mathcal{A}$.*

Proof. Let $\{v_1,\ldots,v_n\}$ be a basis of V and $\{v_1^*,\ldots,v_n^*\}$ be its dual. We show that $\tilde{\mathrm{S}}$ commutes with ∂ :

$$
\begin{aligned}
\partial\left(\tilde{S}(a_V(v_i\otimes v_j^*))\right) &= \partial(a_{V^*}(v_j^*\otimes v_i))=a_{(V^*)^{(1)}}\left(\partial(v_j^*)\otimes F(v_i)\right)=\\
&=a_{\left(V^{(1)}\right)^*}(F(v_j^*)\otimes\partial v_i)=\tilde{S}\left(a_{V^{(1)}}\left(\partial v_i\otimes F(v_j^*)\right)\right)=\\
&=\tilde{S}(\partial a_V(v_i\otimes v_j^*)),
\end{aligned}
$$

where we use the morphism

$$\phi:(V^*)^{(1)}\to\left(V^{(1)}\right)^*,\quad \partial v_j^*\mapsto F(v_j^*),\ v_j^*\mapsto(\partial v_j)^*$$

commuting with the G-action. Moreover, $\tilde{\mathrm{S}}$ is an algebra homomorphism:

$$
\begin{aligned}
\tilde{S}(a_V(v\otimes u)\cdot a_W(w\otimes t)) &= \tilde{S}(a_{V\otimes W}((v\otimes w)\otimes(u\otimes t)))=\\
&=a_{V^*\otimes W^*}((u\otimes t)\otimes(v\otimes w))=\\
&=a_{V^*}(u\otimes v)\cdot a_{W^*}(t\otimes w)=\\
&=\tilde{S}(a_V(v\otimes u))\cdot\tilde{S}(a_W(w\otimes t)).
\end{aligned}
$$

We show that it respects comultiplication:

$$\begin{aligned} m\circ(\tilde S\otimes \mathrm{id})\circ\tilde\Delta(a_V(v\otimes u)) &= m\circ(\tilde S\otimes \mathrm{id})\left(\sum_i a_V(v_i\otimes u)\otimes a_V(v\otimes u_i)\right) = \\ &= m\left(\sum_i a_{V^*}(u\otimes v_i)\otimes a_V(v\otimes u_i)\right) = \\ &= \sum_i a_{V^*\otimes V}((u\otimes v)\otimes(v_i\otimes u_i)) = \\ &= u(v)\cdot a_I(e\otimes f) = \\ &= \tilde\varepsilon(a_V(v\otimes u)). \end{aligned}$$

We have denoted a basis of the trivial representation $\underline{1}$ by $\{e\}$ and its dual by $\{f\}$. We have also used the morphism $V^*\otimes V\to\underline{1}$ mapping $u\otimes v$ to $u(v)$. □

7. Partial differential case

We have only used elementary ring theoretic properties of $\mathbf{k}[\partial]$ and none of its special properties as a left and right Euclidean domain. In particular, all statements concerning recovering the differential Hopf algebra from representations hold true in the partial case. We just restate Definition 4 for the case of several commuting differentiations.

Definition 9. The category $\mathcal{V}_{\mathbf{k}}(\partial_1,\ldots,\partial_m)$ over a $\{\partial_1,\ldots,\partial_m\}$-field $\mathbf{k}$ is the category of finite dimensional vector spaces together with the usual operations $\otimes$, $\oplus$, *, and additional differentiation functors

$$\partial_1^{p_1}\cdot\ldots\cdot\partial_m^{p_m} : V\mapsto \mathbf{k}[\partial_1,\ldots,\partial_m]_{\leqslant(p_1,\ldots,p_m)}\otimes V$$

for all m-tuples $(p_1,\ldots,p_m)\in(\mathbb{Z}_{\geqslant 0})^m$.

8. Conclusions

The results of the previous section allow us to recover a differential algebraic group from the category of its finite dimensional differential representations. From Proposition 2 this category can be generated by one faithful representation of the group applying certain operations of linear algebra and the prolongation functor we introduced in this paper.

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