Global Identifiability of Differential Models

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Abstract

Many real-world processes and phenomena are modeled using systems of ordinary differential equations with parameters. Given such a system, we say that a parameter is globally identifiable if it can be uniquely recovered from input and output data. The main contribution of this paper is to provide theory, an algorithm, and software for deciding global identifiability. First, we rigorously derive an algebraic criterion for global identifiability (this is an analytic property), which yields a deterministic algorithm. Second, we improve the efficiency by randomizing the algorithm while guaranteeing the probability of correctness. With our new algorithm, we can tackle problems that could not be tackled before. A software based on the algorithm (called SIAN) is available at https://github.com/pogudingleb/SIAN. © 2000 Wiley Periodicals, Inc.

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1 Introduction

Many real-world processes and phenomena are modeled using systems of parametric ordinary differential equations (ODEs). There are multiple challenges in designing such a model. In this paper, we address one of them: structural global identifiability. One might be interested in knowing the values of some select parameters due to their importance. Usually, one tries to determine (uniquely identify) them by collecting input and output data. However, due to the structure of the model, it can be impossible to determine the parameters from input and output data. When this happens, the parameters are said to be "not globally identifiable". In the process of model design, it is crucial to know whether the parameters of interest in a potential model are globally identifiable, which is the topic of this paper.

Checking global identifiability is challenging. Thus, a weaker notion, called "local identifiability", has been introduced, which means that a parameter can be identified up to finitely many options. There has been remarkable progress in checking local identifiability, resulting in efficient algorithms, including rigorously justified probabilistic algorithms (see [17, 46, 9, 22] and the references given there).

If a parameter is not locally identifiable, then it is not globally identifiable. If it is locally identifiable, then it still remains to check whether it is globally identifiable. Thus, it is desirable and remains a *challenge* to have an algorithm for checking whether a locally identifiable parameter is also globally identifiable.

Due to the importance of the challenge, there have been intensive effort on it. Roughly stated, there have been three approaches.

- Taylor series approaches [37] with termination bounds in several particular cases [50, 51, 30]. Such bounds lead to algorithms in these particular cases, and these algorithms can be practical if applied to systems of small sizes.
- Generating series approaches based on composing the vector fields associated to the model equations as well as a recursive approach based on integrals [53]. See [54] for a comparison of the Taylor and generating series approaches as well as recent version of software GenSSI based on the generating series approach [4, 8, 26]. Having correct termination criteria for algorithms based on this approach has been an open problem (see also Example 2.16).
- Differential algebra-based approaches can be divided in two groups. One group is to treat the parameters as functions with zero derivatives and use differential elimination (see, for example, [28]). This approach can be practical if applied to systems of small sizes.

  The other group is to treat the parameters as elements of the field of coefficients and produce so-called input-output equations. This approach is followed in COMBOS and DAISY [7, 41, 43, 32, 33] under an additional assumption on "solvability" (see [40, Remark 3] and Example 2.14). Having an efficient algorithm that could verify, for a system, whether this assumption holds is an open problem. One of the software packages based on this approach, DAISY, uses randomization to increase efficiency. This can become a randomized algorithm once a probability analysis is carried out.

Summarizing, there has been significant progress (both in theory and algorithms) toward the challenge of checking global identifiability.

The main contribution of this paper is to make further progress by providing a complete and rigorous theory (Theorems 3.16,4.2,5.5) and a symbolic-numeric-randomized algorithm (Algorithm 1) that is general, reliable, and can tackle problems that could not be tackled before using the existing algorithms and software. The algorithm is "complete" in the sense that, for any input that satisfies our syntactically-checkable requirements, it halts with an answer that is correct with any user-chosen probability of correctness (the standard correctness notion in the theory of randomized algorithms). The algorithm is the first one for global identifiability with such guarantees.

We briefly sketch our approach, which could be viewed as a combination of the differential algebra and Taylor series approaches with a correct termination criterion. Informally, identifiability problem can be formulated as a question about fibers of the map that sends the parameter values and the initial conditions of the system of ODEs to the output data, which are functions of the corresponding solution. This observation is formalized using differential algebra in Proposition 3.4. One way to analyze this map is to reduce it to a map between finite-dimensional spaces. We do this by replacing the output functions by truncations of their Taylor series. Theorem 3.16 provides a criterion to find the order of truncation that contains enough information for the identifiability checking. This criterion can be applied efficiently using rank computation due to Proposition 3.20. After that, the identifiability question is reduced to the question about the generic fiber of a map between finite-dimensional varieties.

To significantly increase the efficiency at this step, instead of considering the generic fiber, we consider a fiber over a randomly chosen point. We estimate the probability of correctness of such an algorithm in Theorem 4.2. We do it by first carefully analyzing the set of special points of this map and then applying the Demillo-Lipton-Schwartz-Zippel lemma (we believe that our analysis could be extended to improve other software for global identifiability such as DAISY so that the output of DAISY would be correct with user-specified probability p).

After considering the fiber over a random point, the problem turns into checking the consistency of a system of polynomial equations and inequations ($\neq$). This can be performed using symbolic, symbolic-numeric, or numeric methods. It turns out that, in practice, Gröbner bases computations are efficient enough for moderate-size problems that we encountered and significantly outperformed (unexpectedly!) numerical algebraic geometry software such as Bertini [6] in several examples that we considered (see Method (iv) in Remark 5.3). Thus, in our implementation, we used Gröbner bases. Any new method for checking the consistency of a system of polynomial equations and inequations can be potentially used to make our algorithm even more efficient.

We have implemented the resulting algorithm into a software called SIAN [19], which stands for "Structural Identifiability ANalyser". The software is available at `https://github.com/pogudingleb/SIAN`. The article [19] focuses on the functionality of the software. The present paper focuses on a rigorous theoretical and algorithmic foundation underlying the software.

The paper is structured as follows. In Section 2, we state several notational conventions, give a precise statement of the global identifiability problem and illustrate it by several examples. In Section 3, we give an algebraic criterion for global identifiability. In Section 4, we give a probabilistic criterion for global identifiability. In Section 5, we give an algorithm based on the criteria developed in the previous two sections. In Section 6, we discuss the performance of our algorithm using challenging examples taken from the literature and discuss how other existing software packages perform at those examples.

2 Identifiability Problem

In this section, we give a precise statement of the global identifiability problem of algebraic differential models. For this, we introduce several notions.

2.1 Conventions

Throughout the paper, we will use the following notational conventions.

(a) We will use plain face for scalar variables and bold face for vector variables. We will decorate variables (for instance by ˆ or ˜) to indicate particular values of the variables. For example:

	scalar	vector
variable name	θ	$\boldsymbol{\theta}$
variable value	$\hat{\theta}$	$\hat{\boldsymbol{\theta}}$

(b) For two sets A and B, the notation $A \subset B$ will be used to denote that A is a subset of B (not necessarily a proper subset). The notation $A \subsetneq B$ will be used to denote that $A \subset B$ & $A \neq B$.

(c) For the convenience of the reader, some of the notation is supplied with hyperlinks, highlighting the first occurrence and subsequent uses in statements of results and in Definitions, Examples, etc.

2.2 Definition of Identifiability

We will start with stating what type of differential models we will be working with.

Definition 2.1 (Algebraic Differential Model)**.** An algebraic differential model is a system

$$\Sigma := \begin{cases} \boldsymbol{x}' &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}, u), \\ \boldsymbol{y} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\mu}, u), \\ \boldsymbol{x}(0) &= \boldsymbol{x}^*, \end{cases} \tag{2.1}$$

where $\boldsymbol{f} = (f_1, \ldots, f_n)$ and $\boldsymbol{g} = (g_1, \ldots, g_m)$ with $f_i = f_i(\boldsymbol{x}, \boldsymbol{\mu}, u)$'s and $g_j = g_j(\boldsymbol{x}, \boldsymbol{\mu}, u)$'s being rational functions over the field of complex numbers $\mathbb{C}$.

The components of the n-vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ are the state variables; the derivative of $\boldsymbol{x}$ with respect to time is denoted by $\boldsymbol{x}'$. The scalar function u is the input variable. The components of the m-vector $\boldsymbol{y} = (y_1, \ldots, y_m)$ are the output variables. The components of the λ-vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_\lambda)$ are the system parameters. Finally, the components of the n-vector $\boldsymbol{x}^* = (x_1^*, \ldots, x_n^*)$ are the initial states.

Remark 2.2. We assume that there is only one input function u for a simpler presentation. However, all of our results and proofs can be generalized straightforwardly to the case of several input functions. If the input u is fixed (known) or u just does not appear, then u can be simply omitted when our algorithms/theoretical results are used.

Remark 2.3. In our approach, the initial conditions in (2.1) form a part of the parameters, as in [14, 53, 49, 50, 28, 54, 2, 30, 7, 9, 8, 32, 52], among other works. Other approaches to introducing (2.1) include considering inequalities, specified initial conditions, etc., see e.g. [14, 2, 42, 40], among others.

Notation 2.4. In stating the notion of identifiability precisely, the following notions and notation are useful.

(a) We will study the identifiability of both $\boldsymbol{\mu}$ and $\boldsymbol{x}^*$. Hence we collectively denote them by

$$\boldsymbol{\theta} := (\boldsymbol{\mu}, \boldsymbol{x}^*) = (\theta_1, \ldots, \theta_s),$$

where $s = \lambda + n$.

(b) Let $\mathbb{C}^\infty(0)$ denote the set of all functions that are complex analytic in some neighborhood of $t = 0$.

(c) We will need to avoid division by zero while evaluating $\boldsymbol{f}$ and $\boldsymbol{g}$. For this, let Ω denote the set of all $(\hat{\boldsymbol{\theta}}, \hat{u}) \in \mathbb{C}^s \times \mathbb{C}^\infty(0)$ such that none of the denominators of $\boldsymbol{f}$ and $\boldsymbol{g}$ in Σ vanishes at $t = 0$ after the substitution of $(\hat{\boldsymbol{\theta}}, \hat{u})$ into $(\boldsymbol{\theta}, u)$.

(d) For $(\hat{\boldsymbol{\theta}}, \hat{u}) \in \Omega$, let $X(\hat{\boldsymbol{\theta}}, \hat{u}), Y(\hat{\boldsymbol{\theta}}, \hat{u})$ denote the unique solution of the instance $\Sigma(\hat{\boldsymbol{\theta}}, \hat{u})$ with entries in $\mathbb{C}^\infty(0)$, which are functions of t (see [18, Theorem 2.2.2]).

(e) We will need to capture precisely the informal notions such as "almost always" and "generically". We will do so by using the notion of Zariski open subset. Define
   - A subset $\Theta \subset \mathbb{C}^s$ is called *Zariski open* if there is a non-zero polynomial $P \in \mathbb{C}[\boldsymbol{\theta}]$ such that
   $$\Theta = \{\hat{\boldsymbol{\theta}} \in \mathbb{C}^s \mid P(\hat{\boldsymbol{\theta}}) \neq 0\}.$$
   - A subset $U \subset \mathbb{C}^\infty(0)$ is called *Zariski open* if there exist $h \in \mathbb{Z}_{\geqslant 0}$ and a non-zero polynomial $P(u_0, u_1, \ldots, u_h) \in \mathbb{C}[u_0, \ldots, u_h]$ such that
   $$U = \left\{\hat{u} \in \mathbb{C}^\infty(0) \mid P\left(\hat{u}, \hat{u}^{(1)}, \ldots, \hat{u}^{(h)}\right)|_{t=0} \neq 0\right\}.$$
   - Let $\tau(\mathbb{C}^s)$ denote the set of all Zariski open non-empty subsets of $\mathbb{C}^s$.
   - Let $\tau(\mathbb{C}^\infty(0))$ denote the set of all Zariski open non-empty subsets of $\mathbb{C}^\infty(0)$.

   Note that the complement to every nonempty Zariski open set is of Lebesque measure zero [36, p. 83]. Thus, if some property is true on a nonempty Zariski open subset, it is true "almost always" in the sense of measure theory.

(f) For a parameter $\theta \in \boldsymbol{\theta}$ and a subset $S \subset \mathbb{C}^s$, let $\operatorname{proj}_\theta S$ denote the projection of S onto the θ-coordinate, that is,
$$\operatorname{proj}_\theta S := \left\{\tilde{\theta}_1 \,\middle|\, \exists\, \tilde{\theta}_2, \ldots, \tilde{\theta}_s \text{ such that } (\tilde{\theta}_1, \ldots, \tilde{\theta}_s) \in S\right\},$$
where we assumed that $\theta = \theta_1$ for notational simplicity.

Now we are ready to give a precise definition of identifiability.

Definition 2.5 (Identifiability)**.** Let Σ be an algebraic differential model. A parameter $\theta \in \boldsymbol{\theta}$ is *globally (resp., locally) identifiable* if
$$\exists\, \Theta \in \tau(\mathbb{C}^s)\ \ \exists\, U \in \tau(\mathbb{C}^\infty(0))\ \ \forall\, (\hat{\boldsymbol{\theta}}, \hat{u}) \in \Omega \cap (\Theta \times U)$$
$$\text{the size of } S_\theta\left(\hat{\boldsymbol{\theta}}, \hat{u}\right) \text{ is one (resp., finite),}$$
where
$$S_\theta\left(\hat{\boldsymbol{\theta}}, \hat{u}\right) := \operatorname{proj}_\theta \left\{\tilde{\boldsymbol{\theta}} \,\middle|\, (\tilde{\boldsymbol{\theta}}, \hat{u}) \in \Omega \text{ and } Y\left(\hat{\boldsymbol{\theta}}, \hat{u}\right) = Y\left(\tilde{\boldsymbol{\theta}}, \hat{u}\right)\right\}.$$

Remark 2.6. One may notice that the above definition looks a bit different from (although mathematically equivalent to, as we show in Proposition 3.4) the ones used in the previous literature [14, 24, 53, 55, 30, 40]. Below, we provide more details and explain the differences and our motivation:

(a) "$\theta \in \boldsymbol{\theta}$": Some authors [49, 50, 28, 54, 30, 7, 34] defined the identifiability of all the parameters. We simplify the presentation by defining the identifiability of a single parameter. Of course, we can naturally extend it to a subset of parameters by saying that $\boldsymbol{\theta}^\# \subset \boldsymbol{\theta}$ is globally (resp., locally) identifiable if every parameter in $\boldsymbol{\theta}^\#$ is globally (resp., locally) identifiable. See also Example 2.13.

(b) "$(\hat{\boldsymbol{\theta}}, \hat{u})$, $(\tilde{\boldsymbol{\theta}}, \hat{u}) \in \Omega$": In the previous literature, the values of $\boldsymbol{\theta}, u$ are usually assumed to be chosen so that the denominators of $\boldsymbol{f}$ and $\boldsymbol{g}$ are non-zero. We make the assumption precise by using the "$\in \Omega$".

(c) "$\exists\, \Theta \in \tau(\mathbb{C}^s)\ \ \exists\, U \in \tau(\mathbb{C}^\infty(0))$": Most of the papers about identifiability we have seen use the notion "almost all/generic" parameter values or inputs while defining identifiability. In Examples 2.10 and 2.11, we remind the reader why it is desirable to allow such a notion. We make the notion precise by using the phrase "$\exists\, \Theta \in \tau(\mathbb{C}^s)\ \ \exists\, U \in \tau(\mathbb{C}^\infty(0))$", that is, to restrict the parameter values and inputs to Zariski open sets.

(d) "$\forall\, \hat{u}$": In most of the papers about identifiability we have seen (for example, [14, 53, 49, 50, 30, 7, 55]), the quantification "$\forall\, \hat{u}$" is put inside the definition of the set S or its analogue (see Proposition 3.4(b) for a precise formulation). We decided to put "$\forall\, \hat{u}$" outside, due to the following reasons:
  (i) The two resulting definitions of identifiability, though looking different, turn out to be equivalent (see Proposition 3.4).
  (ii) The new definition putting "$\forall\, \hat{u}$" outside could be more appealing to the model users because of the following reasons:
    - The definition putting "$\forall\, \hat{u}$" inside makes the model users reason that more than one (in the worst case, infinitely many) inputs might be needed in order to identify a parameter.
    - The definition using "$\forall\, \hat{u}$" outside makes the model users reason that only one generically chosen input will be sufficient to identify a parameter.

Summarizing, Definition 2.5 is mathematically equivalent to the previous definition in the literature, is precisely stated, and could be more appealing to the model users.

2.3 Problem statement and illustrating examples

Now we are ready to state the problem of identifiability.

Problem 2.7 (Global Identifiability)**.**

IN: Σ : an algebraic differential model given by rational functions $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}, u)$, and $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\mu}, u)$

$\boldsymbol{\theta}^\ell$: a subset of $\boldsymbol{\theta} = \boldsymbol{\mu} \cup \boldsymbol{x}^*$ such that every parameter in $\boldsymbol{\theta}^\ell$ is locally identifiable

OUT: $\boldsymbol{\theta}^g$: the set of all globally identifiable parameters in $\boldsymbol{\theta}^\ell$

Remark 2.8. Note that we require every parameter in $\boldsymbol{\theta}^\ell$ to be locally identifiable. The requirement is not essential for the theory developed in this paper. It is only for the sake of simple presentation of the theory. Furthermore, the resulting algorithm is not practically restrictive because there are already very efficient algorithms for assessing local identifiability (see, for example, [46, 22, 1, 38]).

In the following, we will illustrate Problem 2.7 on several simple examples.

Example 2.9. Consider the system

$$\text{In:} \quad \Sigma = \begin{cases} x_1' = \theta_1, \\ y_1 = x_1, \\ x_1(0) = \theta_2 \end{cases}$$
$$\boldsymbol{\theta}^\ell = \{\theta_1, \theta_2\}$$
$$\text{Out:} \quad \boldsymbol{\theta}^g = \{\theta_1, \theta_2\}$$

Reason: We will explicitly show that θ_1 is globally identifiable, thus also showing that it is locally identifiable (a proof that θ_2 is globally identifiable can be done mutatis mutandi). For this, first note that $Y(\boldsymbol{\theta}) = \theta_1 t + \theta_2$ and choose $\Theta = \mathbb{C}^2$. For all $(\hat\theta_1, \hat\theta_2) \in \Theta$, we have

$$\begin{aligned} S_{\theta_1}(\hat\theta_1, \hat\theta_2) &= \left\{\tilde\theta_1 \in \mathbb{C} \,\middle|\, \exists\, \tilde\theta_2 \in \mathbb{C} \text{ such that } \hat\theta_1 t + \hat\theta_2 = \tilde\theta_1 t + \tilde\theta_2\right\} \\ &= \left\{\tilde\theta_1 \in \mathbb{C} \,\middle|\, \exists\, \tilde\theta_2 \in \mathbb{C} \text{ such that } \hat\theta_1 = \tilde\theta_1, \hat\theta_2 = \tilde\theta_2\right\} = \left\{\hat\theta_1\right\}. \end{aligned}$$

Therefore, $|S_{\theta_1}(\hat\theta_1, \hat\theta_2)| = 1$, and thus θ_1 is globally identifiable.

In Examples 2.10 and 2.11, we will see the importance of genericity (choosing open subsets) that appears in Definition 2.5.

Example 2.10. Consider the system

$$\text{In:} \quad \Sigma = \begin{cases} x_1' = \theta_1 x_1, \\ y_1 = x_1, \\ x_1(0) = \theta_2 \end{cases}$$
$$\boldsymbol{\theta}^\ell = \{\theta_1, \theta_2\}$$
$$\text{Out:} \quad \boldsymbol{\theta}^g = \{\theta_1, \theta_2\}$$

Reason: We will explicitly show that θ_1 is globally identifiable, thus also showing that it is locally identifiable (a proof that θ_2 is globally identifiable can be done mutatis mutandi). For this, first note that $Y(\boldsymbol{\theta}) = \theta_2 e^{\theta_1 t}$ and choose $\Theta = \left\{(z_1, z_2) \in \mathbb{C}^2 \,\middle|\, z_2 \neq 0\right\}$. For all $(\hat\theta_1, \hat\theta_2) \in \Theta$, we have

$$\begin{aligned} S_{\theta_1}(\hat\theta_1, \hat\theta_2) &= \left\{\tilde\theta_1 \in \mathbb{C} \,\middle|\, \exists\, \tilde\theta_2 \in \mathbb{C} \text{ such that } \hat\theta_2 e^{\hat\theta_1 t} = \tilde\theta_2 e^{\tilde\theta_1 t}\right\} \\ &= \left\{\tilde\theta_1 \in \mathbb{C} \,\middle|\, \exists\, \tilde\theta_2 \in \mathbb{C} \text{ such that } \hat\theta_1 = \tilde\theta_1, \hat\theta_2 = \tilde\theta_2\right\} = \left\{\hat\theta_1\right\}. \end{aligned}$$

Therefore, $|S_{\theta_1}(\hat\theta_1, \hat\theta_2)| = 1$, and so θ_1 is globally identifiable according to Definition 2.5.

However, note that choosing $\Theta = \mathbb{C}^2$ would result in requiring us to consider $S_{\theta_1}(\hat\theta_1, 0) = \mathbb{C}$, which is infinite. Therefore, if the genericity (restriction to open sets) for $\hat{\boldsymbol{\theta}}$ in Definition 2.5 were not used, then this system would not be globally/locally identifiable.

Example 2.11. Consider the system

$$
\begin{aligned}
\text{In:}\qquad & \Sigma = \begin{cases} x_1' = \theta_1 u, \\ y_1 = x_1, \\ x_1(0) = \theta_2 \end{cases} \\
& \boldsymbol{\theta}^\ell = \{\theta_1\} \\
\text{Out:}\qquad & \boldsymbol{\theta}^g = \{\theta_1\}
\end{aligned}
$$

Reason: Note that $Y(\boldsymbol{\theta}, u)' = \theta_1 u$ and choose $\Theta = \mathbb{C}^2$ and $U = \{\hat{u} \in \mathbb{C}^\infty(0) \mid \hat{u}(0) \neq 0\}$. For all $(\hat{\theta}_1, \hat{\theta}_2) \in \Theta$ and $\hat{u} \in U$, we have

$$
\begin{aligned}
S_{\theta_1}(\hat{\theta}_1, \hat{\theta}_2, \hat{u}) &= \left\{\tilde{\theta}_1 \in \mathbb{C} \,\middle|\, \exists\, \tilde{\theta}_2 \in \mathbb{C} \text{ such that } Y(\hat{\boldsymbol{\theta}}, \hat{u}) = Y(\tilde{\boldsymbol{\theta}}, \hat{u})\right\} \\
&\subset \left\{\tilde{\theta}_1 \in \mathbb{C} \,\middle|\, \exists\, \tilde{\theta}_2 \in \mathbb{C} \text{ such that } Y(\hat{\boldsymbol{\theta}}, \hat{u})' = Y(\tilde{\boldsymbol{\theta}}, \hat{u})'\right\} \\
&\subset \left\{\tilde{\theta}_1 \in \mathbb{C} \,\middle|\, \hat{\theta}_1 \hat{u}(0) = \tilde{\theta}_1 \hat{u}(0)\right\} = \left\{\hat{\theta}_1\right\}.
\end{aligned}
$$

Therefore, $|S_{\theta_1}(\hat{\theta}_1, \hat{\theta}_2, \hat{u})| = 1$, and so θ_1 is globally identifiable according to Definition 2.5.

However, note that choosing $U = \mathbb{C}^\infty(0)$ would result in requiring us to consider $S_{\theta_1}(\hat{\boldsymbol{\theta}}, 0) = \mathbb{C}$, which is infinite. Therefore, if the genericity (restriction to open sets) for $\hat{u}$ in Definition 2.5 were not used, then this system would not be globally/locally identifiable.

Example 2.12. Consider the system

$$
\begin{aligned}
\text{In:}\qquad & \Sigma = \begin{cases} x_1' = \theta_1^2, \\ y_1 = x_1, \\ x_1(0) = \theta_2 \end{cases} \\
& \boldsymbol{\theta}^\ell = \{\theta_1, \theta_2\} \\
\text{Out:}\qquad & \boldsymbol{\theta}^g = \{\theta_2\}
\end{aligned}
$$

Reason: To see that θ_1 is locally identifiable, note that $Y(\boldsymbol{\theta}) = \theta_1^2 t + \theta_2$ and choose $\Theta = \mathbb{C}^2$. For all $(\hat{\theta}_1, \hat{\theta}_2) \in \Theta$,

$$
\begin{aligned}
& S_{\theta_1}(\hat{\theta}_1, \hat{\theta}_2) \\
={}& \left\{\tilde{\theta}_1 \in \mathbb{C} \,\middle|\, \exists\, \tilde{\theta}_2 \in \mathbb{C} \text{ such that } \hat{\theta}_1^2 t + \hat{\theta}_2 = \tilde{\theta}_1^2 t + \tilde{\theta}_2\right\} \\
={}& \left\{\tilde{\theta}_1 \in \mathbb{C} \,\middle|\, \exists\, \tilde{\theta}_2 \in \mathbb{C} \text{ such that } \hat{\theta}_1^2 = \tilde{\theta}_1^2, \hat{\theta}_2 = \tilde{\theta}_2\right\} \\
={}& \left\{\hat{\theta}_1, -\hat{\theta}_1\right\}.
\end{aligned}
$$

Therefore, $|S_{\theta_1}(\hat{\theta}_1, \hat{\theta}_2)| \leqslant 2$. Similarly, we conclude that $|S_{\theta_2}(\hat{\theta}_1, \hat{\theta}_2)| = 1$, thus showing that $\boldsymbol{\theta}^\ell = \{\theta_1, \theta_2\}$ and $\theta_2 \in \boldsymbol{\theta}^g$.

To see that $\theta_1 \notin \boldsymbol{\theta}^g$, let $\Theta \subset \mathbb{C}^2$ be non-empty open such that, for all $(\hat{\theta}_1, \hat{\theta}_2) \in \Theta$, $|S_{\theta_1}(\hat{\theta}_1, \hat{\theta}_2)| = 1$. Since, for all $(\hat{\theta}_1, \hat{\theta}_2) \in \mathbb{C}^2$, $\hat{\theta}_1 \neq 0$ implies $|S_{\hat{\theta}_1}(\hat{\theta}_1, \hat{\theta}_2)| = 2$,

$$
\Theta \subset \left\{(\hat{\theta}_1, \hat{\theta}_2) \in \mathbb{C}^2 \mid \hat{\theta}_1 = 0\right\},
$$

so Θ cannot be a non-empty open set.

In the above elementary examples (see also Examples 2.14 and 2.16), we were able to solve the ODEs explicitly and decide the identifiability directly. We now give an example with a system to which explicit solutions are not known to us but we are still able to check identifiability using the algorithm from this paper. The example also illustrates that our approach allows working with *proper subset* of the parameters.

Example 2.13. Consider the system (a predator-prey model)

$$\text{In:}\qquad \Sigma = \begin{cases} x_1' = \theta_1 x_1 - \theta_2 x_1 x_2, \\ x_2' = -\theta_3 x_2 + \theta_4 x_1 x_2, \\ y_1 = x_1, \\ x_1(0) = \theta_5, \\ x_2(0) = \theta_6. \end{cases}$$

$$\text{Out:}\qquad \begin{aligned} \boldsymbol{\theta}^{\ell} &= \{\theta_1, \theta_3, \theta_4, \theta_5\} \\ \boldsymbol{\theta}^{g} &= \{\theta_1, \theta_3, \theta_4, \theta_5\} \end{aligned}$$

Reason: This has been determined by Algorithm 1 (see Section 5). Additionally, a calculation based on differential elimination terminates with representations for $\theta_1, \theta_3, \theta_4$ that are consequences of Σ of the form

$$\theta_i = \frac{P_i(y, \ldots, y^{(5)})}{Q_i(y, \ldots, y^{(5)})}, \quad \text{for } i = 1, 3, 4.$$

Note that, if the values of θ_1 and θ_3 are known, we can also write in a simpler form:

$$\theta_4 = \frac{\theta_1 \theta_3 y^2 - \theta_3 y y' - y'' y + y'^2}{\theta_1 y^3 - y'^2}.$$

One can also show directly by definition that neither θ_2 nor θ_6 is locally identifiable. This result tells us that, if we can only observe the prey population (x_1), then it is impossible to identify the rate of decrease (θ_2) of the prey population due to the prey-predator interactions ($x_1 x_2$). But we can identify the rate of increase (θ_4) of the predator population.

2.4 Subtleties with other approaches

In this section, we will consider a few subtleties in applying DAISY and GenSSI, two existing software packages to test for global identifiability (for more details on them, see the beginning of Section 6).

Example 2.14. Consider the system

$$\text{In:}\qquad \Sigma = \begin{cases} x_1' = 0, \\ y_1 = x_1, \\ y_2 = \theta_1 x_1 + \theta_1^2, \\ x_1(0) = \theta_2. \end{cases}$$

$$\boldsymbol{\theta}^{\ell} = \{\theta_1, \theta_2\}$$

OUT: $\boldsymbol{\theta}^g = \{\theta_2\}$

REASON: To see that θ_1 is locally identifiable, note that

$$Y(\boldsymbol{\theta}) = \begin{pmatrix} \theta_2 \\ \theta_1\theta_2 + \theta_1^2 \end{pmatrix}$$

and $\Omega = \mathbb{C}^2$ and choose $\Theta = \mathbb{C}^2$. For all $(\hat\theta_1, \hat\theta_2) \in \Theta$, we have

$$\begin{aligned} S_{\theta_1}(\hat\theta_1, \hat\theta_2) &= \left\{\tilde\theta_1 \in \mathbb{C} \,\middle|\, \exists\, \tilde\theta_2 \in \mathbb{C} \text{ s.t. } \tilde\theta_2 = \hat\theta_2, \tilde\theta_1\tilde\theta_2 + \tilde\theta_1^2 = \hat\theta_1\hat\theta_2 + \hat\theta_1^2\right\} \\ &= \left\{\tilde\theta_1 \in \mathbb{C} \,\middle|\, \tilde\theta_1\hat\theta_2 + \tilde\theta_1^2 = \hat\theta_1\hat\theta_2 + \hat\theta_1^2\right\} = \left\{\hat\theta_1, -\hat\theta_1 - \hat\theta_2\right\}, \end{aligned} \tag{2.2}$$

and so $|S_{\theta_1}(\hat\theta_1, \hat\theta_2)| \leqslant 2$. Similarly, we conclude that $|S_{\theta_2}(\hat\theta_1, \hat\theta_2)| = 1$, thus showing that $\boldsymbol{\theta}^\ell = \{\theta_1, \theta_2\}$ and $\theta_2 \in \boldsymbol{\theta}^g$.

To see that θ_1 is not globally identifiable, let $\Theta \subset \mathbb{C}^2$ be non-empty open such that, for all $(\hat\theta_1, \hat\theta_2) \in \Theta$, $|S_{\theta_1}(\hat\theta_1, \hat\theta_2)| = 1$. Formula (2.2) implies that, for all $(\hat\theta_1, \hat\theta_2) \in \mathbb{C}^2$,

$$|S_{\theta_1}(\hat\theta_1, \hat\theta_2)| = 2 \iff \hat\theta_1 \neq -\hat\theta_1 - \hat\theta_2.$$

Therefore,

$$\Theta \subset \left\{(\hat\theta_1, \hat\theta_2) \in \mathbb{C}^2 \mid \hat\theta_1 = -\hat\theta_1 - \hat\theta_2\right\},$$

so Θ cannot be non-empty open.

However, the software tool DAISY (see [7, 43, 41]), which is based on theory that uses input-output equations, returned that θ_1 is *globally identifiable*. A possible explanation for this is as follows. The equation

$$y_2 - \theta_1 y_1 - \theta_1^2 = 0, \tag{2.3}$$

which is a consequence of Σ, is an input-output equation (see [40, formula (11)]) in this case. The approach based on input-output equations assumes that the coefficients of input-output equations as polynomials with respect to the y's and u's and their derivatives are globally identifiable (such an additional condition is called "solvability" in [40, Remark 3]). In this example, however, this condition does not hold. Indeed, if the coefficients of (2.3), 1, $-\theta_1$, and $-\theta_1^2$, were globally identifiable, θ_1 would be identifiable. However, we showed above that the values of y_1 and y_2 are not sufficient for a unique recovery of the value of θ_1 in the generic case.

Remark 2.15. One can modify Example 2.14 so that so that the solutions of the state variables are non-constant but with the same conclusion about "solvability" and the output of DAISY, e.g., by considering

$$\Sigma = \begin{cases} x_1' = 1, \\ x_2' = -1, \\ y_1 = x_1 + x_2, \\ y_2 = \theta_1(x_1 + x_2) + \theta_1^2, \\ x_1(0) = \theta_2, \\ x_2(0) = \theta_3. \end{cases}$$

The same argument as in Example 2.14 shows that θ_1 is locally but not globally identifiable, while DAISY would output that θ_1 is globally identifiable.

Example 2.16. Consider the system

IN: $\Sigma = \begin{cases} x_1' = x_1, \\ y_1 = x_1, \\ y_2 = \theta_1 + \theta_1^2 x_1, \\ x_1(0) = \theta_2 \end{cases}$

$\boldsymbol{\theta}^{\ell} = \{\theta_1, \theta_2\}$

OUT: $\boldsymbol{\theta}^{g} = \{\theta_1, \theta_2\}$

REASON: First note that

$$Y(\boldsymbol{\theta}) = \begin{pmatrix} \theta_2 \cdot e^t \\ \theta_1 + \theta_1^2\theta_2 \cdot e^t \end{pmatrix}.$$

To show that θ_2 is *globally identifiable*, choose $\Theta = \mathbb{C}^2$. For all $(\hat{\theta}_1, \hat{\theta}_2) \in \Theta$, we have

$$S_{\theta_2}(\hat{\theta}_1, \hat{\theta}_2) \subset \left\{\tilde{\theta}_2 \in \mathbb{C} \mid \exists\, \tilde{\theta}_1 \in \mathbb{C} \text{ such that } \hat{\theta}_2 \cdot e^t = \tilde{\theta}_2 \cdot e^t\right\} = \left\{\hat{\theta}_2\right\}.$$

Therefore, $|S_{\theta_2}(\hat{\theta}_1, \hat{\theta}_2)| \leqslant 1$. To show that θ_1 is *globally identifiable*, choose $\Theta = \mathbb{C}^2$. For all $(\hat{\theta}_1, \hat{\theta}_2) \in \Theta$, we have

$$\begin{aligned} & S_{\theta_1}(\hat{\theta}_1, \hat{\theta}_2) \\ = & \left\{\tilde{\theta}_1 \in \mathbb{C} \;\middle|\; \exists\, \tilde{\theta}_2 \in \mathbb{C} \text{ such that } \begin{array}{l} \hat{\theta}_2 \cdot e^t = \tilde{\theta}_2 \cdot e^t, \\ \hat{\theta}_1 + \hat{\theta}_1^2 \cdot \hat{\theta}_2 \cdot e^t = \tilde{\theta}_1 + \tilde{\theta}_1^2 \cdot \tilde{\theta}_2 \cdot e^t \end{array}\right\} \\ = & \left\{\tilde{\theta}_1 \in \mathbb{C} \mid \hat{\theta}_1 + \hat{\theta}_1^2 \cdot \hat{\theta}_2 \cdot e^t = \tilde{\theta}_1 + \tilde{\theta}_1^2 \cdot \hat{\theta}_2 \cdot e^t\right\} \\ = & \left\{\tilde{\theta}_1 \in \mathbb{C} \mid \hat{\theta}_1 = \tilde{\theta}_1 \;\&\; \hat{\theta}_1^2 \cdot \hat{\theta}_2 = \tilde{\theta}_1^2 \cdot \hat{\theta}_2\right\} \\ = & \left\{\hat{\theta}_1\right\}. \end{aligned}$$

Therefore, $|S_{\theta_2}(\hat{\theta}_1, \hat{\theta}_2)| = 1$.

However, the software GenSSI 2.0 (see [4, 8, 26]) has returned that θ_1 is *locally identifiable* and is not able to conclude the global identifiability because of limitations of the method that is used in the software. More precisely, GenSSI 2.0

checks that the Jacobian of

$$\begin{cases} y_1(0) = \theta_2, \\ y_2(0) = \theta_1 + \theta_1^2\theta_2 \end{cases} \tag{2.4}$$

w.r.t. θ_1 and θ_2, which is

$$\begin{pmatrix} 0 & 1 \\ 1 + 2\theta_1\theta_2 & \theta_1^2 \end{pmatrix},$$

has rank generically equal to the number of parameters. Having positively checked this equality, following the steps outlined in "Electronic supplementary material" of [4] as well as by running the code, GenSSI 2.0 does not differentiate $y_1 = x_1$, $y_2 = \theta_1 + \theta_1^2 x_1$ and checks whether θ_1 and θ_2 are uniquely determined by $y_1(0)$ and $y_2(0)$ in (2.4). GenSSI 2.0 observes that θ_1 is not uniquely determined by $y_1(0)$ and $y_2(0)$ from (2.4) and thus outputs that θ_1 is only locally identifiable, while θ_2 is uniquely determined by $y_1(0)$ and so is globally identifiable.

We will show how to remedy this limitation in checking global identifiability for this example based on the above particular stopping criterion used in GenSSI. It turns out that one additional derivative $y_2' = \theta_1^2 x_1' = \theta_1^2 x_1$ of $y_2 = \theta_1 + \theta_1^2 x_1$ is sufficient to conclude the global identifiability of θ_1 in addition to using (2.4):

$$\theta_1 = y_2(0) - \theta_1^2 \cdot y_1(0) = y_2(0) - \theta_1^2 \cdot x_1(0) = y_2(0) - y_2'(0).$$

3 Algebraic Criteria

In this section, the analytic definition of global identifiability from the previous section will be characterized algebraically. First, we provide an equivalence in terms of field extensions in Proposition 3.4 of Section 3.3. This section culminates in Theorem 3.16 of Section 3.4, which gives a constructive algebraic criterion for global identifiability.

3.1 Basic Terminology

A derivation δ on a commutative ring R is a map $\delta : R \longrightarrow R$ such that, for all $a, b \in R$,

$$\delta(a + b) = \delta(a) + \delta(b), \qquad \delta(ab) = \delta(a)b + a\delta(b). \tag{3.1}$$

Call (R, δ) a differential ring. For a domain R, $\mathrm{Quot}(R)$ denotes the field of fractions of R. The derivation δ can be extended uniquely to a derivation on $\mathrm{Quot}(R)$ using the quotient rule. An ideal $I \subset R$ is said to be a differential ideal if $a' \in I$ for every $a \in I$.

The ring of differential polynomials in $z_1, \ldots, z_s$ with coefficients in $\mathbb{C}$ is denoted by $\mathbb{C}\{z_1, \ldots, z_s\}$. As a ring, it is the ring of polynomials in the algebraic indeterminates

$$z_1, \ldots, z_s, z_1', \ldots, z_s', z_1'', \ldots, z_s'', \ldots, z_1^{(q)}, \ldots, z_s^{(q)}, \ldots .$$

A differential ring structure is defined by, for all i, $1 \leqslant i \leqslant s$ and $q \geqslant 0$,

$$\big(z_i^{(q)}\big)' := z_i^{(q+1)}, \quad z_i^{(0)} := z_i$$

and extended to $\mathbb{C}\{z_1, \ldots, z_s\}$ by the Leibniz rule, additivity (see (3.1)) and $c' = 0$ for all $c \in \mathbb{C}$. For example,

$$\left(2z_1'^2 z_3 + 3z_5''\right)' = 4z_1' z_1'' z_3 + 2z_1'^2 z_3' + 3z_5'''.$$

For all i, $1 \leqslant i \leqslant s$, and $P \in \mathbb{C}\{z_1, \ldots, z_s\}$, we define $\operatorname{ord}_{z_i} P$ to be the largest integer q such that $z_i^{(q)}$ appears in P if such a q exists and -1 if such a q does not exist. For non-empty $S, T \subset R$, we define

$$T : S^\infty = \{r \in R \mid \text{there exist } s \in S \text{ and } n \in \mathbb{Z}_{\geqslant 0} \text{ such that } s^n r \in T\}.$$

If T is an ideal of R, then $T : S^\infty$ is an ideal of R. For subsets $X \subset \mathbb{C}^n$ and $J \subset \mathbb{C}[x_1, \ldots, x_n]$, we denote

$$\begin{aligned} I(X) &:= \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f(\boldsymbol{p}) = 0 \text{ for all } \boldsymbol{p} \in X\}, \\ Z(J) &:= \{\boldsymbol{p} \in \mathbb{C}^n \mid f(\boldsymbol{p}) = 0 \text{ for all } f \in J\}. \end{aligned}$$

3.2 Algebraic Preparation

We will start with introducing technical notation and proving several auxiliary statements in this subsection.

Fix Σ from Definition 2.1 (see Notation 2.4 as well). Let Q be the lcm of all denominators in $\boldsymbol{f}$ and $\boldsymbol{g}$. Then all these rational functions can be written as

$$f_i = \frac{F_i}{Q} \text{ for } 1 \leqslant i \leqslant n, \text{ and } g_j = \frac{G_j}{Q} \text{ for } 1 \leqslant j \leqslant m.$$

Denote $\boldsymbol{F} := (F_1, \ldots, F_n)$ and $\boldsymbol{G} := (G_1, \ldots, G_m)$. Let

$$\mathcal{R} := \mathbb{C}[\boldsymbol{\mu}]\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}\} \tag{3.2}$$

considered as a differential ring with $\boldsymbol{\mu}' = 0$ (which is set up this way due to the time-independence of the parameters). For all i, $1 \leqslant i \leqslant n$ and j, $1 \leqslant j \leqslant m$, we consider F_i, G_j, and Q as elements of $\mathbb{C}[\boldsymbol{\mu}, \boldsymbol{x}, \boldsymbol{u}] \subset \mathcal{R}$. Let

$$\mathcal{J} = \left\{r \in \mathcal{R} \mid \forall((\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*), \hat{u}) \in \Omega \;\; r\big(\hat{\boldsymbol{\mu}}, X\big((\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*), \hat{u}\big), Y\big((\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*), \hat{u}\big), \hat{u}\big) = 0\right\}, \tag{3.3}$$

which is a (differential) ideal in $\mathcal{R}$. For all $((\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*), \hat{u}) \in \Omega$, we have

$$Q(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*, \hat{u}) \neq 0.$$

Therefore, for every $P \in \mathcal{R}$,

$$Q \cdot P \in \mathcal{J} \implies P \in \mathcal{J}. \tag{3.4}$$

Lemma 3.1. $\mathcal{J} \cap \mathbb{C}[\boldsymbol{\mu}, \boldsymbol{x}]\{\boldsymbol{u}\} = \{0\}$.

Proof. Assume that there is nonzero $P\big(\boldsymbol{\mu}, \boldsymbol{x}, u, u', \ldots, u^{(h)}\big) \in \mathcal{J} \cap \mathbb{C}[\boldsymbol{\mu}, \boldsymbol{x}]\{u\}$. Since P and Q are nonzero polynomials, there exist $\hat{\boldsymbol{\theta}} := (\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*) \in \mathbb{C}^s$ and $\hat{u} \in \mathbb{C}[t] \subset \mathbb{C}^\infty(0)$ such that

$$Q\big(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*, \hat{u}(0)\big) \neq 0 \quad \text{and} \quad P\big(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*, \hat{u}(0), \hat{u}'(0), \ldots, \hat{u}^{(h)}(0)\big) \neq 0,$$

which contradicts (3.3). □

Lemma 3.2. *We have*

$$\mathcal{J} = \Big(\big(Qx_i' - F_i\big)^{(j)}, (Qy_k - G_k)^{(j)} \;\Big|\; 1 \leqslant i \leqslant n, 1 \leqslant k \leqslant m, j \geqslant 0\Big) : Q^\infty \tag{3.5}$$

and $\mathcal{J}$ is a prime ideal.

Proof. Consider any ordering $>$ of the variables in $\mathcal{R}$ such that

(a) $y_i^{(k+1)} > y_j^{(k)}$ for every $1 \leqslant i, j \leqslant m$ and $k \geqslant 0$;
(b) $y_i^{(k)}$ is larger that any variable in $\mathbb{C}[\boldsymbol{\mu}]\{\boldsymbol{x}, u\}$ for every $1 \leqslant i \leqslant m$ and $k \geqslant 0$;
(c) $x_i^{(k+1)} > x_j^{(k)}$ for every $1 \leqslant i, j \leqslant n$ and $k \geqslant 0$;
(d) $x_i^{(k)}$ is larger that any variable in $\mathbb{C}[\boldsymbol{\mu}]\{u\}$ for every $1 \leqslant i \leqslant n$ and $k \geqslant 0$;

Then the set of polynomials

$$\mathcal{S} := \Big\{\big(Qx_i' - F_i\big)^{(j)}, (Qy_k - G_k)^{(j)} \mid 1 \leqslant i \leqslant n, 1 \leqslant k \leqslant m, j \geqslant 0\Big\} \tag{3.6}$$

is a triangular set (see [20, Definition 4.1 and page 10]).

In order to prove (3.5), we consider $\widetilde{\mathcal{J}} := (\mathcal{S}) \colon Q^\infty$. For all $((\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*), \hat{u}) \in \Omega$, it follows from the definition of $X((\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*), \hat{u}), Y((\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*), \hat{u})$ that

$$Q(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*, \hat{u}) \cdot X((\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*), \hat{u})' - \boldsymbol{F}\big(\hat{\boldsymbol{\mu}}, X((\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*), \hat{u}), \hat{u}\big) = 0,$$
$$Q(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*, \hat{u}) \cdot Y((\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*), \hat{u}) - \boldsymbol{G}\big(\hat{\boldsymbol{\mu}}, X((\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*), \hat{u}), \hat{u}\big) = 0.$$

Since $\mathcal{J}$ is a differential ideal, we therefore have $\mathcal{S} \subset \mathcal{J}$. By (3.4), we moreover obtain $\widetilde{\mathcal{J}} \subset \mathcal{J}$. For the reverse containment, consider $P \in \mathcal{J}$. Since $\mathcal{S}$ is a triangular set, let N be a positive integer and $P_0 \in \mathbb{C}[\boldsymbol{\mu}, \boldsymbol{x}]\{u\}$ (see [20, Section 4.2]) such that

$$Q^N P - P_0 \in (\mathcal{S}) \subset \mathcal{J}.$$

Hence, $P_0 \in \mathcal{J}$, so $P_0 = 0$ by Lemma 3.1. Then $Q^N P \in \widetilde{\mathcal{J}}$, so $P \in \widetilde{\mathcal{J}}$.

For the primality of $\mathcal{J}$, consider P_1 and P_2 such that $P_1 \cdot P_2 \in \mathcal{J}$. Let N be such that $Q^N P_1$ and $Q^N P_2$ are equivalent to elements $\widetilde{P}_1$ and $\widetilde{P}_2$ of $\mathbb{C}[\boldsymbol{\mu}, \boldsymbol{x}]\{u\}$ modulo $\mathcal{J}$. If $\widetilde{P}_1\widetilde{P}_2 = 0$, then $P_1 \in \mathcal{J}$ or $P_2 \in \mathcal{J}$. If $\widetilde{P}_1\widetilde{P}_2 \neq 0$, then, by Lemma 3.1 and a straightforward argument,

$$Q^N P_1 \cdot Q^N P_2 \notin \mathcal{J},$$

so $P_1P_2 \notin \mathcal{J}$, which is a contradiction. □

3.3 Algebraic Criterion: Non-constructive Version

Notation 3.3. We will use the following notation.

(i) Let $\mathcal{T} := \mathcal{R}/\mathcal{J}$ and $\mathcal{F} := \mathrm{Quot}(\mathcal{T})$ (Recall (3.2) and (3.3)). The latter is well-defined because $\mathcal{J}$ is prime, so $\mathcal{T}$ is a domain.
(ii) Note that $\mathcal{F}$ is generated by the images of $\boldsymbol{\mu}, \boldsymbol{x}, u, u', \ldots$.
(iii) Let $\mathcal{E}$ denote the subfield of $\mathcal{F}$ generated by the image of $\mathbb{C}\{\boldsymbol{y}, u\}$.
(iv) We will denote elements of $\mathcal{R}$ and their images in $\mathcal{T}$ by the same symbols.
(v) $\boldsymbol{\theta}$ will be understood as a tuple $(\boldsymbol{\mu}, \boldsymbol{x}^*)$ if it is considered as the tuple of parameters of Σ and as a tuple of variables $(\boldsymbol{\mu}, x_1, \ldots, x_n)$ if it is considered as a tuple of elements of $\mathcal{R}$ or its subalgebras.
(vi) For every $\hat{\boldsymbol{\theta}} \in \mathbb{C}^s$, let $\Omega_{\hat{\boldsymbol{\theta}}} := \{\hat{u} \in \mathbb{C}^\infty(0) \mid Q(\hat{\boldsymbol{\theta}}, \hat{u}) \neq 0\} \subset \mathbb{C}^\infty(0)$.

Proposition 3.4. *For every parameter θ of system Σ, the following statements are equivalent*

(a) θ is globally (resp., locally) identifiable according to Definition 2.5, that is,

$$\exists\, \Theta \in \tau(\mathbb{C}^s)\ \exists\, U \in \tau(\mathbb{C}^\infty(0))\ \forall\, (\hat{\boldsymbol{\theta}}, \hat{u}) \in \Omega \cap (\Theta \times U)$$
$$\text{the size of } S_\theta\left(\hat{\boldsymbol{\theta}}, \hat{u}\right) \text{ is one (resp., finite),}$$

where

$$S_\theta(\hat{\boldsymbol{\theta}}, \hat{u}) := \mathrm{proj}_\theta \left\{\tilde{\boldsymbol{\theta}} \,\middle|\, (\tilde{\boldsymbol{\theta}}, \hat{u}) \in \Omega \text{ and } Y\left(\hat{\boldsymbol{\theta}}, \hat{u}\right) = Y\left(\tilde{\boldsymbol{\theta}}, \hat{u}\right)\right\}.$$

(b) θ is globally (resp., locally) identifiable according to much of the previous literature [14, 53, 49, 50, 30, 7, 55]*, that is,*

$$\exists\, \Theta \in \tau(\mathbb{C}^s)\ \forall\, \hat{\boldsymbol{\theta}} \in \Theta \text{ the size of } S'_\theta\left(\hat{\boldsymbol{\theta}}\right) \text{ is one (resp., finite),}$$

where

$$S'_\theta(\hat{\boldsymbol{\theta}}) := \mathrm{proj}_\theta \left\{\tilde{\boldsymbol{\theta}} \,\middle|\, \Omega_{\tilde{\boldsymbol{\theta}}} \neq \varnothing \text{ and } \forall\, \hat{u} \in \Omega_{\tilde{\boldsymbol{\theta}}} \cap \Omega_{\hat{\boldsymbol{\theta}}}\ Y(\tilde{\boldsymbol{\theta}}, \hat{u}) = Y(\hat{\boldsymbol{\theta}}, \hat{u})\right\}.$$

Note that the main difference with Definition 2.5 is that "$\forall\, \hat{u}$" has been put inside of S.

(c) the fields $\mathcal{E}$ and $\mathcal{E}(\theta)$ coincide (resp., the extension $\mathcal{E} \subset \mathcal{E}(\theta)$ is algebraic).

Proof. (c) $\implies$ (a). Assume (c). Considering the preimage of a minimal polynomial of θ over $\mathcal{T}$ in $\mathcal{R}$, we obtain a nonzero polynomial (resp., nonzero polynomial linear in θ) in $\mathbb{C}\{\boldsymbol{y}, u\}[\theta] \cap \mathcal{J}$ such that its leading coefficient ℓ does not lie in $\mathcal{J}$. From Lemma 3.2 and the triangular set constructed in its proof, we obtain that there exists M such that $Q^M \ell$ is equal to some $\ell_0 \in \mathbb{C}[\boldsymbol{\theta}]\{u\}$ modulo $\mathcal{J}$. We apply Lemma 3.5 (see next) to the polynomial ℓ and obtain nonempty open $\Theta \subset \mathbb{C}^s$ and $U \subset \mathbb{C}^\infty(0)$. We claim that Definition 2.5 holds for this choice of open sets. Consider any $(\hat{\boldsymbol{\theta}}, \hat{u}) \in (\Theta \times U) \cap \Omega$. The choice of Θ and U implies that ℓ_0 does not vanish at $(\hat{\boldsymbol{\theta}}, \hat{u})$. Since Q does not vanish at $(\hat{\boldsymbol{\theta}}, \hat{u})$, then ℓ does not vanish at this point, so there are only finitely many (resp., only one) possible values for every θ provided that $\hat{\boldsymbol{y}} := Y(\hat{\boldsymbol{\theta}}, \hat{u})$ and $\hat{u}$ are fixed. Thus, we have (a).

(a) $\implies$ (b). Assume (a) and let $\Theta_0 \subset \mathbb{C}^s$ and $U \subset \mathbb{C}^\infty(0)$ be the open subsets from the definition. We claim that (b) holds with

$$\Theta_1 := \Theta_0 \cap \{\hat{\boldsymbol{\theta}} \mid Q(\hat{\boldsymbol{\theta}}, u) \neq 0\}.$$

Assume the contrary. Then there exists $\hat{\boldsymbol{\theta}} \in \Theta_1$ such that $|S'_\theta(\hat{\boldsymbol{\theta}})| > 1$ (resp., $S'_\theta(\hat{\boldsymbol{\theta}})$ is infinite). The inequation $Q(\hat{\boldsymbol{\theta}}, u) \neq 0$ implies $\Omega_{\hat{\boldsymbol{\theta}}} \neq \varnothing$. Now consider local and global identifiability separately

(i) Global identifiability. Let $\tilde{\theta} \in S'_\theta(\hat{\boldsymbol{\theta}})$ such that $\tilde{\theta} \neq \hat{\theta}$. Consider the corresponding $\tilde{\boldsymbol{\theta}}$ from the definition of S'_θ. Since $U, \Omega_{\hat{\boldsymbol{\theta}}}, \Omega_{\tilde{\boldsymbol{\theta}}}$ are nonempty Zariski open sets, there exists $\hat{u}$ in their intersection by Lemma 3.10. Then $\hat{\theta}, \tilde{\theta} \in S_\theta(\hat{\boldsymbol{\theta}}, \hat{u})$, but this contradicts (a).

(ii) Local identifiability. Let $\tilde{\theta}_1, \tilde{\theta}_2, \ldots \in S'_\theta(\hat{\boldsymbol{\theta}})$ be distinct elements. Consider the corresponding $\tilde{\boldsymbol{\theta}}_1, \tilde{\boldsymbol{\theta}}_2, \ldots$ from the definition of S'_θ. Since $U, \Omega_{\tilde{\boldsymbol{\theta}}_1}, \Omega_{\tilde{\boldsymbol{\theta}}_2}, \ldots$ are nonempty Zariski open sets, Lemma 3.10 implies that there exists $\hat{u}$ in their intersection. Then $\tilde{\theta}_1, \tilde{\theta}_2, \ldots \in S_\theta(\hat{\boldsymbol{\theta}}, \hat{u})$, but this contradicts (a).

(b) $\implies$ (c) for global identifiability. Assume (b) but the containment $\mathcal{E} \subset \mathcal{E}(\theta)$ is proper. Let $Q_1 \in \mathbb{C}[\boldsymbol{\theta}]$ be a polynomial defining the complement to Θ from (b) and $Q_2 \in \mathbb{C}[\boldsymbol{\theta}]$ be any non-zero coefficient of Q viewed as a polynomial in u. Due to [21, Theorem 2.6] applied with K being $\mathcal{E}$, L being $\mathcal{F}$, and s being θ, there is a differential field $\widehat{\mathcal{F}} \supset \mathcal{F}$ and a differential automorphism $\alpha \colon \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}$ over $\mathcal{E}$ such that $\alpha(\theta) \neq \theta$. For a finitely generated subalgebra

$$A := \mathbb{C}[1/Q_1(\boldsymbol{\theta}), \boldsymbol{\theta}, \alpha(\boldsymbol{\theta}), 1/(\theta - \alpha(\theta)), 1/Q_2(\boldsymbol{\theta}), 1/Q_2(\alpha(\boldsymbol{\theta}))]$$

of $\widehat{\mathcal{F}}$, there exists a $\mathbb{C}$-algebra homomorphism $\varepsilon \colon A \to \mathbb{C}$. Let $\hat{\boldsymbol{\theta}} := \varepsilon(\boldsymbol{\theta})$ and $\tilde{\boldsymbol{\theta}} := \varepsilon(\alpha(\boldsymbol{\theta}))$. Then $\hat{\boldsymbol{\theta}} \in \Theta$, $\hat{\theta} \neq \tilde{\theta}$, and $\Omega_{\hat{\boldsymbol{\theta}}}$ and $\Omega_{\tilde{\boldsymbol{\theta}}}$ are nonempty. Applying Lemma 3.9 to the natural embedding $\mathcal{T} \to \widehat{\mathcal{F}}$ and the restriction of α to $\mathcal{T} \to \widehat{\mathcal{F}}$ as β_2 and β_1 and to the appropriate restriction of ε as γ, we show that $\tilde{\theta} \in S'_\theta(\hat{\boldsymbol{\theta}})$. Thus, $|S'_\theta(\hat{\boldsymbol{\theta}})| \geqslant 2$, and this contradicts (b).

(b) $\implies$ (c) for local identifiability. Assume (b) but θ is transcendental over $\mathcal{E}$. Let $Q_1 \in \mathbb{C}[\boldsymbol{\theta}]$ be a polynomial defining the complement to Θ from (b) and $Q_2 \in \mathbb{C}[\boldsymbol{\theta}]$ be any non-zero coefficient of Q viewed as a polynomial in u. Lemma 3.11 implies that there exists a differential field $\widetilde{\mathcal{F}} \supset \mathcal{F}$ and automorphisms $\alpha_1 = \mathrm{id}, \alpha_2, \alpha_3, \ldots$ of $\widetilde{\mathcal{F}}$ over its differential subfield $\mathcal{E}$ such that $\alpha_1(\theta), \alpha_2(\theta), \ldots$ are all distinct. Since the subalgebra $\mathcal{A} := \mathbb{C}[S_1, S_2]$, where

$$S_1 := \{1/Q_1(\boldsymbol{\theta}), \alpha_1(\boldsymbol{\theta}), \alpha_2(\boldsymbol{\theta}), \ldots, 1/Q_2(\alpha_1(\boldsymbol{\theta})), 1/Q_2(\alpha_2(\boldsymbol{\theta})), \ldots\},$$
$$S_2 := \{1/(\alpha_i(\theta) - \alpha_j(\theta)) \mid 1 \leqslant i < j\},$$

of $\widetilde{\mathcal{F}}$ is countably generated, there exists a $\mathbb{C}$-algebra homomorphism $\varepsilon \colon \mathcal{A} \to \mathbb{C}$ [48, Theorem 10.34.11]. Let $\hat{\boldsymbol{\theta}} := \varepsilon(\boldsymbol{\theta})$, then $\hat{\boldsymbol{\theta}} \in \Theta$ and all $\varepsilon(\alpha_i(\theta))$ are distinct. Moreover, for every $i \geqslant 0$, $\Omega_{\varepsilon(\alpha_i(\theta))} \neq \varnothing$. Consider $i \geqslant 1$ and apply Lemma 3.9

to the restrictions of α_1 and α_i to $\mathcal{T} \to \widetilde{\mathcal{F}}$ as β_2 and β_1 and to the appropriate restriction of ε as γ. We obtain that, for every $i \geqslant 1$, $\varepsilon(\alpha_i(\theta)) \in S'_\theta(\hat{\boldsymbol{\theta}})$, which is a contradiction. ◻

Lemma 3.5. *Let $P(\boldsymbol{\theta}, u, \ldots, u^{(N)}) \in \mathbb{C}[\boldsymbol{\theta}]\{u\}$ be nonzero. Then there exist nonempty Zariski open subsets $\Theta \in \mathbb{C}^s$ and $U \subset \mathbb{C}^\infty(0)$ such that, for every $\hat{\boldsymbol{\theta}} \in \Theta$ and $\hat{u} \in U$, the function $P(\hat{\boldsymbol{\theta}}, \hat{u}, \ldots, (\hat{u})^{(N)})$ is a nonzero element of $\mathbb{C}^\infty(0)$.*

Proof. We write $P(\boldsymbol{\theta}, u, \ldots, u^{(N)}) = \sum_{i=1}^{\ell} c_i(\boldsymbol{\theta}) m_i(u)$, where $m_1, \ldots, m_\ell$ are distinct monomials from $\mathbb{C}\{u\}$ and $c_1(\boldsymbol{\theta}), \ldots, c_\ell(\boldsymbol{\theta}) \in \mathbb{C}[\boldsymbol{\theta}]$. Let $W(u) \in \mathbb{C}\{u\}$ be the determinant of the Wronskian matrix of $m_1, \ldots, m_\ell$. We set

$$\Theta := \{\boldsymbol{\theta} \in \mathbb{C}^s \mid c_1(\boldsymbol{\theta}) \neq 0\} \quad \text{and} \quad U = \{u \in \mathbb{C}^\infty(0) \mid W(u)|_{t=0} \neq 0\}.$$

Since $c_1(\boldsymbol{\theta})$ is a nonzero polynomial, $\Theta \neq \varnothing$. The differential polynomial W can be considered as an algebraic polynomial $W\big(u, u', \ldots, u^{(M)}\big) \in \mathbb{C}\big[u, u', \ldots, u^{(M)}\big]$ for some M. Let $(a_0, \ldots, a_M) \in \mathbb{C}^{M+1}$ be such that $W(a_0, \ldots, a_M) \neq 0$. We set $\hat{u}(t) := \sum_{i=0}^{M} a_i \frac{t^i}{i!}$. A direct computation shows that

$$W\big(\hat{u}, \ldots, \hat{u}^{(M)}\big)|_{t=0} = W(a_0, \ldots, a_M) \neq 0.$$

Thus, $\hat{u} \in U$, so $U \neq \varnothing$. Let $\hat{\boldsymbol{\theta}} \in \Theta$ and $\hat{u} \in U$. If the function $P\big(\hat{\boldsymbol{\theta}}, \hat{u}, \ldots, \hat{u}^{(N)}\big)$ is zero, it provides a nontrivial (because of $c_1(\hat{\boldsymbol{\theta}}) \neq 0$) linear dependence of $m_1(\hat{u}), \ldots, m_\ell(\hat{u})$. Since $W(\hat{u}) \neq 0$, such a dependence does not exist due to [29, Proposition 2.8]. ◻

Lemma 3.6. *Let $\varphi\colon \mathcal{R} \to \mathbb{C}$ be a $\mathbb{C}$-algebra homomorphism such that $\varphi(Q) \neq 0$ and $\mathcal{J} \subset \operatorname{Ker} \varphi$. We define power series*

$$u_\varphi(t) = \sum_{i=0}^{\infty} \frac{\varphi\big(u^{(i)}\big)}{i!} t^i, \quad \boldsymbol{y}_\varphi = \sum_{i=0}^{\infty} \frac{\varphi\big(\boldsymbol{y}^{(i)}\big)}{i!} t^i. \tag{3.7}$$

If u_φ converges in some neighborhood of 0, then $\boldsymbol{y}_\varphi$ defines a function in $\mathbb{C}^\infty(0)$ and

$$\boldsymbol{y}_\varphi = Y\big(\varphi(\boldsymbol{\theta}), u_\varphi\big).$$

Proof. A direct computation shows that, for every $k \geqslant 0$, $u_\varphi^{(k)}(0) = \varphi(u^{(k)})$. We also define

$$\boldsymbol{x}_\varphi := \sum_{i=0}^{\infty} \frac{\varphi\left(\boldsymbol{x}^{(i)}\right)}{i!} t^i.$$

By the theorem of existence and uniqueness of solutions for differential equations [18, Theorem 2.2.2], there exist unique

$$X\big((\varphi(\boldsymbol{\mu}), \varphi(\boldsymbol{x})), u_\varphi\big), Y\big((\varphi(\boldsymbol{\mu}), \varphi(\boldsymbol{x})), u_\varphi\big) \in \mathbb{C}^\infty(0)$$

satisfying the instance $\Sigma((\varphi(\boldsymbol{\mu}, \varphi(\boldsymbol{x})), u_\varphi)$, as in Notation 2.4. We denote these functions by $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$, respectively. We prove that

$$(\hat{x}_i)^{(j)}(0) = \varphi\big(x_i^{(j)}\big)$$

for every $1 \leqslant i \leqslant n$ and $j \geqslant 0$ by induction on j. The base case is $j = 0$. Then $(\hat{x}_i)(0) = \varphi(x_i)$, because $\varphi(x_i)$ is the initial condition. Assume that, for all i and k, $1 \leqslant i \leqslant n$ and $0 \leqslant k \leqslant j$,

$$(\hat{x}_i)^{(k)}(0) = \varphi\big(x_i^{(k)}\big).$$

We write the differential polynomial $\big(Qx_i' - F_i\big)^{(j)}$ in the form $Qx_i^{(j+1)} + P$, where P only involves derivatives of $\boldsymbol{x}$ of order at most j. Since this differential polynomial belongs to $\mathcal{J}$,

$$\varphi\left(x_i^{(j+1)}\right) = -\frac{\varphi(P)}{\varphi(Q)}.$$

The inductive hypothesis implies that the right-hand side is equal to

$$-\frac{P(\hat{\boldsymbol{x}}, \boldsymbol{\mu}, u_\varphi)}{Q(\hat{\boldsymbol{x}}, \boldsymbol{\mu}, u_\varphi)}\bigg|_{t=0} = (\hat{x}_i)^{(j+1)}(0).$$

Hence, $\hat{\boldsymbol{x}}$ and $\boldsymbol{x}_\varphi$ have the same Taylor expansion, so coincide. Using this, one can analogously prove that $\hat{\boldsymbol{y}}$ and $\boldsymbol{y}_\varphi$ coincide. ◻

Notation 3.7. We denote $\mathcal{F}_{\boldsymbol{\theta}} := \mathbb{C}(\boldsymbol{\theta})$ and $\mathcal{F}_u := \mathbb{C}(u, u', \ldots)$, which are subfields of $\mathcal{F}$.

Lemma 3.8. *For all*

- *differential fields K containing $\mathcal{F}_u$ and*
- *$\psi_i \colon \mathcal{F} \to K$, $i = 1, 2$, homomorphisms of differential fields over $\mathcal{F}_u$,*

$u, u', \ldots$ are algebraically independent over K_0, where

$$K_0 := \mathbb{C}(\psi_1(\mathcal{F}_{\boldsymbol{\theta}}) \cup \psi_2(\mathcal{F}_{\boldsymbol{\theta}})) \subset K.$$

Proof. Consider the following set of indices

$$H := \big\{h \in \mathbb{Z}_{\geqslant 0} \mid u^{(h)} \text{ is algebraic over } \mathbb{C}\big(K_0, u, \ldots, u^{(h-1)}\big)\big\} \subset \mathbb{Z}_{\geqslant 0}.$$

Claim: *The set H is finite.* Assume the contrary and let $H = \{h_1, h_2, \ldots\}$, where $h_1 < h_2 < \ldots$. Consider $M := 2s + 1$, where $s = |\boldsymbol{\theta}|$. Then the definition of H implies that

$$\operatorname{trdeg}_{\mathbb{C}} \mathbb{C}\big(K_0, u, \ldots, u^{(h_M)}\big) \leqslant \operatorname{trdeg}_{\mathbb{C}} K_0 + h_M + 1 - M.$$

Since $\operatorname{trdeg}_{\mathbb{C}} K_0 \leqslant 2s$, the latter expression does not exceed h_M. On the other hand, since $\mathbb{C}\big(K_0, u, \ldots, u^{(h_M)}\big)$ contains $\mathbb{C}\big(u, \ldots, u^{(h_M)}\big)$, its transcendence degree over $\mathbb{C}$ is at least $h_M + 1$. The obtained contradiction proves the claim.

Claim: *The set H is empty.* Assume the contrary. Since we already know that H is finite, we can consider the maximal element in H, say h. Let $P(z_0, \ldots, z_h) \in$

$K_0[z_0, \ldots, z_h]$ be a polynomial of the smallest degree such that $P(u, u', \ldots, u^{(h)}) = 0$ and P depends on z_h. Then $\frac{\partial P}{\partial z_h}(u, u', \ldots, u^{(h)}) \neq 0$. Using $K_0' \subset K_0(u)$, we obtain

$$u^{(h+1)} \cdot \frac{\partial P}{\partial z_h}(u, u', \ldots, u^{(h)}) \in K_0(u, u', \ldots, u^{(h)}).$$

Hence, $h + 1 \in H$, so the contradiction with the maximality of h proves the claim.

The latter claim proves the lemma. □

Lemma 3.9. *Let $Q_2 \in \mathbb{C}[\boldsymbol{\theta}]$ be any non-zero coefficient of Q viewed as a polynomial in $\boldsymbol{u}$. For every*

- *differential field K over $\mathcal{F}_{\boldsymbol{u}}$,*
- *injective $\mathbb{C}\{\boldsymbol{u}\}$-algebra homomorphisms $\beta_1, \beta_2 \colon \mathcal{T} \to K$ such that $\beta_1|_{\mathbb{C}\{\boldsymbol{y},\boldsymbol{u}\}} = \beta_2|_{\mathbb{C}\{\boldsymbol{y},\boldsymbol{u}\}}$,*
- *$\mathbb{C}$-algebra homomorphism $\gamma \colon B \to \mathbb{C}$, where*

$$B := \mathbb{C}[\beta_1(\boldsymbol{\theta}), \beta_2(\boldsymbol{\theta}), 1/Q_2(\beta_1(\boldsymbol{\theta})), 1/Q_2(\beta_2(\boldsymbol{\theta}))] \subset K,$$

- *parameter $\theta \in \boldsymbol{\theta}$,*

$$\gamma \circ \beta_1(\theta) \in S_\theta'(\gamma \circ \beta_2(\boldsymbol{\theta})).$$

Proof. We set

$$\tilde{\boldsymbol{\theta}} := \gamma \circ \beta_1(\boldsymbol{\theta}) \quad \text{and} \quad \hat{\boldsymbol{\theta}} := \gamma \circ \beta_2(\boldsymbol{\theta}).$$

Since $\gamma(Q_2(\beta_1(\boldsymbol{\theta}))) \neq 0$ and $\gamma(Q_2(\beta_2(\boldsymbol{\theta}))) \neq 0$, the sets $\Omega_{\hat{\boldsymbol{\theta}}}$ and $\Omega_{\tilde{\boldsymbol{\theta}}}$ are nonempty. Consider $\hat{u} \in \Omega_{\hat{\boldsymbol{\theta}}} \cap \Omega_{\tilde{\boldsymbol{\theta}}}$. Lemma 3.8 applied to the extensions of β_1 and β_2 to $\mathcal{F}$, implies that $u, u', \ldots$ are algebraically independent over B. Hence, the homomorphism γ can be extended to $\gamma \colon B\{u\} \to \mathbb{C}$ in such a way that $\gamma(u^{(h)}) = \hat{u}^{(h)}(0)$ for every $h \geqslant 0$. Hence, since $\hat{u} \in \Omega_{\hat{\boldsymbol{\theta}}} \cap \Omega_{\tilde{\boldsymbol{\theta}}}$,

$$\gamma(Q(\beta_1(\boldsymbol{\theta}), u)) \neq 0 \quad \text{and} \quad \gamma(Q(\beta_2(\boldsymbol{\theta}), u)) \neq 0.$$

So, γ can be extended to

$$\gamma \colon B\{u\}\,[1/\beta_1(Q), 1/\beta_2(Q)] \to \mathbb{C},$$

where the domain contains both $\beta_1(\mathcal{T})$ and $\beta_2(\mathcal{T})$ because $\mathcal{T} \subset \mathbb{C}\{u\}[\boldsymbol{\theta}, 1/Q]$. Let $\pi \colon \mathcal{R} \to \mathcal{T}$ be the natural surjection. We apply Lemma 3.6 to $\gamma \circ \beta_1 \circ \pi$ and $\gamma \circ \beta_2 \circ \pi$ and obtain

$$Y(\tilde{\boldsymbol{\theta}}, \hat{u}) = \sum_{j=0} \frac{\gamma \circ \beta_1 \circ \pi(\boldsymbol{y}^{(j)})}{j!} t^j \quad \text{and} \quad Y(\hat{\boldsymbol{\theta}}, \hat{u}) = \sum_{j=0} \frac{\gamma \circ \beta_2 \circ \pi(\boldsymbol{y}^{(j)})}{j!} t^j.$$

Since $\beta_1(\pi(\boldsymbol{y}^{(j)})) = \beta_2(\pi(\boldsymbol{y}^{(j)}))$, we have $Y(\hat{\boldsymbol{\theta}}, \hat{u}) = Y(\tilde{\boldsymbol{\theta}}, \hat{u})$. Hence, $\tilde{\theta} \in S_\theta'(\hat{\boldsymbol{\theta}})$. □

Lemma 3.10. *Let $A_1, A_2, \ldots$ be nonempty Zariski open subsets of $\mathbb{C}^\infty(0)$. Then $\bigcap\limits_{i=1}^{\infty} A_i$ is nonempty.*

Proof. For every $i \geqslant 0$, let $P_i(u) \in \mathbb{C}\{u\}$ denote a differential polynomial defining the complement to A_i. Let $h_i := \operatorname{ord}_u P_i$. We will inductively construct an infinite sequence of complex numbers $u_0, u_1, u_2, \ldots$ such that

- $|u_i| < 1$ for every $i \geqslant 0$;
- for every $i \geqslant 0$ and $j \geqslant 1$, P_j does not vanish after substituting u_k instead of $u^{(k)}$ for every $k \leqslant i$.

Assume that we have already constructed first $i \geqslant 0$ elements $u_0, \ldots, u_{i-1}$ of the sequence. For every $j \geqslant 1$, there are only finitely many complex numbers z such that P_j vanishes after substituting u_k instead of $u^{(k)}$ for every $k < i$ and z instead of $u^{(i)}$. Thus, at most countably many values of $u^{(i)}$ will vanish at least one of $P_1, P_2, \ldots$. Since there are uncountably many complex numbers z with $|z| < 1$, there exists u_i satisfying both requirements.

We set $\hat{u}(t) := \sum\limits_{i=0}^{\infty} u_i \frac{t^i}{i!}$. Since $|u_i| < 1$ for every $i \geqslant 0$, $\hat{u}$ defines an element of $\mathbb{C}^{\infty}(0)$. For every $j \geqslant 1$, we have

$$P_j(\hat{u})|_{t=0} = P_j(u_0, u_1, \ldots, u_{h_j}) \neq 0.$$

Hence, $\hat{u} \in A_j$ for every $j \geqslant 1$. $\square$

Lemma 3.11. *For every extension $E \subset F$ of differential fields and $a \in F$ transcendental over E, there exists an extension of differential fields $F \subset K$ and infinitely many differential automorphisms $\alpha_1, \alpha_2, \ldots$ of K over E such that the elements $\alpha_1(a), \alpha_2(a), \ldots$ are all distinct.*

Proof. In this proof, we will use some methods and notions from model theory of differential fields [31]. Let K be a differential closure of F [31, Definition, p. 49]. For every $b \in K$, we denote the differential subfield generated by b and E in K by $E\langle b\rangle$. [31, Theorem 2.9(a)] implies that K is atomic over E, so there is a first-order formula $\varphi(x)$ in the language of differential fields with parameters from E such that, for all $b \in K$, $\varphi(b)$ is equivalent to

$$\exists \text{ differential field isomorphism } f : E\langle a\rangle \to E\langle b\rangle \text{ over } E \text{ such that } f(a) = b.$$

Since a is transcendental over E, [31, Lemma 5.1] implies that there are infinitely many elements $a_1, a_2, \ldots \in K$ such that, for every $i \geqslant 1$, $\varphi(a_i)$ is true. For every $i \geqslant 1$, we introduce an isomorphism $\alpha_i \colon E\langle a\rangle \to E\langle a_i\rangle$ sending a to a_i. Since K is a differential closure of both $E\langle a\rangle$ and $E\langle a_i\rangle$, [31, Corollary 2.10] implies that α_i can be extended to an isomorphism of differential fields $K \to K$. $\square$

3.4 Algebraic Criterion: Constructive Version

Although Proposition 3.4 provides us with an algebraic criterion, it involves the quotient field of an infinitely generated algebra, so is not constructive enough. In this section, we will show how to find the order $\boldsymbol{h}'$ of derivatives that is sufficient to consider to make conclusion about identifiability. This will reduce deciding identifiability to a question about the size of the generic fiber of a projection of finite-dimensional algebraic varieties.

Notation 3.12. Let z be an ℓ-tuple of differential indeterminates, and $\boldsymbol{h} \in \mathbb{Z}_{\geqslant 0}^{\ell}$. Then we define

$$z_{\boldsymbol{h}} := \{z_i^{(j)} \mid 0 \leqslant j < h_i,\ 1 \leqslant i \leqslant \ell\}.$$

Notation 3.13. Let $\boldsymbol{h} = (h_1, \ldots, h_m) \in \mathbb{Z}_{\geqslant 0}^{m}$. We construct a set of differential polynomials $\mathcal{S}_{\boldsymbol{h}}$ by the following procedure.

(a) We put $(Qy_k - G_k)^{(j)}$ (see the beginning of Section 3.2 for notation) into $\mathcal{S}_{\boldsymbol{h}}$ for every $1 \leqslant k \leqslant m$ and $0 \leqslant j < h_k$;

(b) While there exist $1 \leqslant i \leqslant n$ and $j \geqslant 1$ such that $x_i^{(j)}$ appears in some element of $\mathcal{S}_{\boldsymbol{h}}$ but $\left(Qx_i' - F_i\right)^{(j-1)} \notin \mathcal{S}_{\boldsymbol{h}}$, we put $\left(Qx_i' - F_i\right)^{(j-1)}$ into $\mathcal{S}_{\boldsymbol{h}}$.

Since the orders of all variables involved in the above procedure do not exceed $\max\{h_1, \ldots, h_m\}$, the second step will terminate after a finite number of iterations.

We introduce tuples $\operatorname{out}(\boldsymbol{h}) \in \mathbb{Z}_{\geqslant 0}^{m}$, $\operatorname{st}(\boldsymbol{h}) \in \mathbb{Z}_{\geqslant 0}^{n}$, and $\operatorname{in}(\boldsymbol{h}) \in \mathbb{Z}_{\geqslant 0}^{1}$ bounding the orders of derivatives of outputs, states, and input appearing in $\mathcal{S}_{\boldsymbol{h}}$:

$$\begin{aligned}
\operatorname{out}(\boldsymbol{h})_i &:= \max\{\operatorname{ord}_{y_i} P \mid P \in \mathcal{S}_{\boldsymbol{h}}\} + 1, && \text{for } i = 1, \ldots, m,\\
\operatorname{st}(\boldsymbol{h})_i &:= \max\{\operatorname{ord}_{x_i} P \mid P \in \mathcal{S}_{\boldsymbol{h}}\} + 1, && \text{for } i = 1, \ldots, n,\\
\operatorname{in}(\boldsymbol{h}) &:= \max\{\operatorname{ord}_{u} P \mid P \in \mathcal{S}_{\boldsymbol{h}}\} + 1.
\end{aligned}$$

Observe that, due to the construction of $\mathcal{S}_{\boldsymbol{h}}$, we will always have $\operatorname{out}(\boldsymbol{h}) = \boldsymbol{h}$, so we will use $\boldsymbol{h}$ instead of $\operatorname{out}(\boldsymbol{h})$ to keep the notation simple. Using Notation 3.12, we can express all the derivatives appearing in $\mathcal{S}_{\boldsymbol{h}}$ as $\boldsymbol{y}_{\boldsymbol{h}}, \boldsymbol{x}_{\operatorname{st}(\boldsymbol{h})}$, and $u_{\operatorname{in}(\boldsymbol{h})}$.

Finally, we introduce the smallest polynomial ring $\mathcal{R}_{\boldsymbol{h}}$ containing $\mathcal{S}_{\boldsymbol{h}}$ and the corresponding ideal $\mathcal{J}_{\boldsymbol{h}}$ as

$$\mathcal{R}_{\boldsymbol{h}} := \mathbb{C}[\boldsymbol{\mu}, \boldsymbol{x}_{\operatorname{st}(\boldsymbol{h})}, \boldsymbol{y}_{\boldsymbol{h}}, u_{\operatorname{in}(\boldsymbol{h})}] \quad \text{and} \quad \mathcal{J}_{\boldsymbol{h}} := (\mathcal{S}_{\boldsymbol{h}}) : Q^{\infty} \subset \mathcal{R}_{\boldsymbol{h}}\,.$$

Remark 3.14. For every $\boldsymbol{h} \in \mathbb{Z}_{\geqslant 0}^{m}$, $\mathcal{S}_{\boldsymbol{h}}$, being a subset of $\mathcal{S}$, is a triangular set for $\mathcal{J}_{\boldsymbol{h}}$ with respect to every ordering described in the proof of Lemma 3.2 such that the free variables are exactly $\boldsymbol{x}$, $\boldsymbol{\mu}$, and $u_{\operatorname{in}(\boldsymbol{h})}$.

Lemma 3.15. *For every tuple $\boldsymbol{h} \in \mathbb{Z}_{\geqslant 0}^{m}$, $\mathcal{J} \cap \mathcal{R}_{\boldsymbol{h}} = \mathcal{J}_{\boldsymbol{h}}$. In particular, $\mathcal{J}_{\boldsymbol{h}}$ is prime.*

Proof. By Lemma 3.2, $\mathcal{J}$ is prime. The statement now follows from [20, Proposition 4.5]. ☐

We denote the i-th standard basis vector in $\mathbb{Z}_{\geqslant 0}^{m}$ by $\mathbf{1}_i$ and set

$$\mathbf{1} := \mathbf{1}_1 + \ldots + \mathbf{1}_m.$$

For all $\boldsymbol{h} \in \mathbb{Z}_{\geqslant 0}^{m}$, we denote the zero set of $\mathcal{J}_{\boldsymbol{h}}$ by $Z_{\boldsymbol{h}}$. We will denote the d-dimensional affine $\mathbb{C}$-space by $\mathbb{A}^d$.

A morphism $f : X \to Y$ between two algebraic varieties is said to be dominant (or dominating) if its image $f(X)$ is dense in Y, i.e., $Y = \overline{f(X)}$ (see [35, Definition 1, p. 48]).

Theorem 3.16. *For all* $\boldsymbol{h} \in \mathbb{Z}^m_{\geqslant 0}$ *such that*

(a) *the projection of* $Z_{\boldsymbol{h}}$ *to* $(\boldsymbol{y}_{\boldsymbol{h}}, u_{\operatorname{in}(\boldsymbol{h})})$*-coordinates is dominant,*

(b) *the projection of* $Z_{\boldsymbol{h}+\mathbf{1}_i}$ *to the* $(\boldsymbol{y}_{\boldsymbol{h}+\mathbf{1}_i}, u_{\operatorname{in}(\boldsymbol{h}+\mathbf{1}_i)})$*-coordinates is not dominant for every* $1 \leqslant i \leqslant m$*,*

for every non-empty subset $\boldsymbol{\theta}^\# \subset \boldsymbol{\theta}$ *and every* $\boldsymbol{h}' \in \mathbb{Z}^m$ *such that* $\boldsymbol{h}' - \boldsymbol{h} \in \mathbb{Z}^m_{>0}$*, we have*

$$\begin{pmatrix} \textit{every parameter in } \boldsymbol{\theta}^\# \\ \textit{is } \text{globally identifiable} \end{pmatrix} \iff \begin{pmatrix} \textit{the generic fiber of the projection of } Z \textit{ to the} \\ (\boldsymbol{y}_{\boldsymbol{h}'}, u_{\operatorname{in}(\boldsymbol{h}')})\textit{-coordinates is of } \text{cardinality one} \end{pmatrix},$$

where Z *is the Zariski closure of the projection of* $Z_{\boldsymbol{h}'}$ *to the subspace with coordinates* $(\boldsymbol{\theta}^\#, \boldsymbol{y}_{\boldsymbol{h}'}, u_{\operatorname{in}(\boldsymbol{h}')})$*.*

Remark 3.17. One can check efficiently whether $\boldsymbol{h}$ satisfies requirements (a) and (b) from Theorem 3.16 using Proposition 3.20 (see next). Corollary 3.21 (see next) implies that such an $\boldsymbol{h}$ always exists. The intuition behind such an $\boldsymbol{h}$ is that we are looking for a prolongation that would determine all Taylor coefficients of $\boldsymbol{y}$ from the first $\boldsymbol{h} + \mathbf{1}$ and $\operatorname{in}(\boldsymbol{h} + \mathbf{1})$ Taylor coefficients of $\boldsymbol{y}$ and $\boldsymbol{u}$, respectively.

Lemma 3.18. *Let* $\boldsymbol{h}$ *be the tuple from the statement of Theorem 3.16. Then* $\mathcal{E}$ *is generated in* $\mathcal{F}$ *(see Notation 3.3) by the image of* $\mathbb{C}[\boldsymbol{y}_{\boldsymbol{h}+\mathbf{1}}]\{u\}$*.*

Proof. Consider i, $1 \leqslant i \leqslant m$. Since the projection of $Z_{\boldsymbol{h}+\mathbf{1}_i}$ to the $(\boldsymbol{y}_{\boldsymbol{h}+\mathbf{1}_i}, u_{\operatorname{in}(\boldsymbol{h}+\mathbf{1}_i)})$-coordinates is not dominant and the projection of $Z_{\boldsymbol{h}}$ to the $(\boldsymbol{y}_{\boldsymbol{h}}, u_{\operatorname{in}(\boldsymbol{h})})$-coordinates is dominant,

$$\mathcal{J}_{\boldsymbol{h}+\mathbf{1}_i} \cap \mathbb{C}[\boldsymbol{y}_{\boldsymbol{h}+\mathbf{1}_i}, u_{\operatorname{in}(\boldsymbol{h}+\mathbf{1}_i)}] \neq \{0\} \text{ and } \mathcal{J}_{\boldsymbol{h}} \cap \mathbb{C}[\boldsymbol{y}_{\boldsymbol{h}}, u_{\operatorname{in}(\boldsymbol{h})}] = \{0\}.$$

Consider a nonzero $P \in \mathcal{J}_{\boldsymbol{h}+\mathbf{1}_i} \cap \mathbb{C}[\boldsymbol{y}_{\boldsymbol{h}+\mathbf{1}_i}, u_{\operatorname{in}(\boldsymbol{h}+\mathbf{1}_i)}]$. Since $\mathcal{S}_{\boldsymbol{h}+\mathbf{1}_i}$ is a triangular set, $\mathcal{J}_{\boldsymbol{h}+\mathbf{1}_i} = (\mathcal{S}_{\boldsymbol{h}+\mathbf{1}_i}) : Q^\infty$, and the ideal $\mathcal{J}_{\boldsymbol{h}+\mathbf{1}_i}$ is prime, $\mathcal{S}_{\boldsymbol{h}+\mathbf{1}_i}$ is a characteristic set of $\mathcal{J}_{\boldsymbol{h}+\mathbf{1}_i}$ (see [20, Definitions 5.5 and 5.10, Theorem 5.13]). Hence, P can be reduced to zero with respect to $\mathcal{S}_{\boldsymbol{h}+\mathbf{1}_i}$. If P does not involve $y^{(h_i)}$ (which is an element of $\boldsymbol{y}_{\boldsymbol{h}+\mathbf{1}_i}$, see Notation 3.13), then the reduction uses only $\mathcal{S}_{\boldsymbol{h}}$, hence all coefficients of P as a polynomial in $u_{\operatorname{in}(\boldsymbol{h}+\mathbf{1}_i)} \setminus u_{\operatorname{in}(\boldsymbol{h})}$ are elements of $\mathcal{J}_{\boldsymbol{h}} \cap \mathbb{C}[\boldsymbol{y}_{\boldsymbol{h}}, u_{\operatorname{in}(\boldsymbol{h})}]$. Since the latter is zero, every nonzero element in $\mathcal{J}_{\boldsymbol{h}+\mathbf{1}_i} \cap \mathbb{C}[\boldsymbol{y}_{\boldsymbol{h}+\mathbf{1}_i}, u_{\operatorname{in}(\boldsymbol{h}+\mathbf{1}_i)}]$ depends on $y_i^{(h_i)}$. Consider such an element P_i of minimal possible degree in $y_i^{(h_i)}$.

We will prove by induction on k that the subfield of $\mathcal{F}$ generated by the image of $\mathbb{C}[\boldsymbol{y}_{\boldsymbol{h}+k\mathbf{1}}]\{u\}$ coincides with the subfield of $\mathcal{F}$ generated by the image of $\mathbb{C}[\boldsymbol{y}_{\boldsymbol{h}+\mathbf{1}}]\{u\}$ for every $k \geqslant 2$. Let $k = 2$. Consider i, $1 \leqslant i \leqslant m$, and $P_i' = S_i y_i^{(h_i+1)} + T_i$, where $S_i, T_i \in \mathbb{C}[\boldsymbol{y}_{\boldsymbol{h}+\mathbf{1}}]\{u\}$ and $S_i \notin \mathcal{J}_{\boldsymbol{h}+\mathbf{1}}$. Hence, the image of $y_i^{(h_i+1)}$ in $\mathcal{F}$ belongs to the subfield generated by $\mathbb{C}[\boldsymbol{y}_{\boldsymbol{h}+\mathbf{1}}]\{u\}$, so the base case is proved. Let $k > 2$. Consider i, $1 \leqslant i \leqslant m$. The inductive hypothesis implies that there are $A, B \in \mathbb{C}[\boldsymbol{y}_{\boldsymbol{h}+(k-1)\mathbf{1}}]\{u\}$ such that $Ay_i^{(h_i+k-1)} + B \in \mathcal{J}$, but $A \notin \mathcal{J}$. Taking the derivative of $Ay_i^{(h_i+k-1)} + B$, we obtain

$$Ay_i^{(h_i+k)} + A' y_i^{(h_i+k-1)} + B' \in \mathcal{J}.$$

This implies that the image of $y_i^{(h_i+k)}$ in $\mathcal{F}$ belongs to the subfield generated by the image of $\mathbb{C}[\boldsymbol{y}_{\boldsymbol{h}+k\boldsymbol{1}}]\{u\}$. By the inductive hypothesis, the latter coincides with the subfield generated by the image of $\mathbb{C}[\boldsymbol{y}_{\boldsymbol{h}+\boldsymbol{1}}]\{u\}$. $\square$

Proof of Theorem 3.16. If the generic fiber of the projection of Z to the $(\boldsymbol{y}_{\boldsymbol{h}'}, u_{\mathrm{in}(\boldsymbol{h}')})$-coordinates is of cardinality one, then every $\theta \in \boldsymbol{\theta}^{\#}$ is algebraic of degree one over $\mathbb{C}[\boldsymbol{y}_{\boldsymbol{h}'}, u_{\mathrm{in}(\boldsymbol{h}')}]$ modulo $\mathcal{J}_{\boldsymbol{h}'}$ (see [45, page 562]). Then the images of $\boldsymbol{\theta}^{\#}$ in $\mathcal{F}$ belong to the subfield generated by the image if $\mathbb{C}\{u\}[y_{\boldsymbol{h}'}]$ in $\mathcal{F}$, so they belong to $\mathcal{E}$. Proposition 3.4 implies that $\boldsymbol{\theta}^{\#}$ are globally identifiable.

Let $\boldsymbol{\theta}^{\#}$ be globally identifiable. Proposition 3.4 implies that the image of every $\theta \in \boldsymbol{\theta}^{\#}$ in $\mathcal{F}$ belongs to $\mathcal{E}$. Consider $\theta \in \boldsymbol{\theta}^{\#}$. Lemma 3.18 implies that $\mathcal{E}$ is generated by the image of $\mathbb{C}\{u\}[\boldsymbol{y}_{\boldsymbol{h}+\boldsymbol{1}}]$. Hence, θ is algebraic of degree one over $\mathbb{C}\{u\}[y_{\boldsymbol{h}'}]$ modulo $\mathcal{J}_{\boldsymbol{h}'} \cap \mathbb{C}\{u\}[\boldsymbol{y}_{\boldsymbol{h}'}]$. Let P_θ be such a relation. Since $\mathcal{S}_{\boldsymbol{h}'}$ is a triangular set, $\mathcal{J}_{\boldsymbol{h}'} = (\mathcal{S}_{\boldsymbol{h}'}) : Q^\infty$, and the ideal $\mathcal{J}_{\boldsymbol{h}'}$ is prime, $\mathcal{S}_{\boldsymbol{h}'}$ is a characteristic set of $\mathcal{J}_{\boldsymbol{h}'}$ (see [20, Definitions 5.5 and 5.10, Theorem 5.13]).

Therefore, P_θ can be reduced to zero with respect to $\mathcal{S}_{\boldsymbol{h}'}$. Consider P_θ as a polynomial in all derivatives of u that do not belong to $u_{\mathrm{in}(\boldsymbol{h}')}$, i.e., do not occur in $\mathcal{S}_{\boldsymbol{h}'}$. Then each of these coefficients can also be reduced to zero with respect to $\mathcal{S}_{\boldsymbol{h}'}$, and so each of these coefficients belongs to $\mathcal{J}_{\boldsymbol{h}'}$, therefore, to $\mathcal{J}_{\boldsymbol{h}'} \cap \mathbb{C}[\boldsymbol{y}_{\boldsymbol{h}'}, u_{\mathrm{in}(\boldsymbol{h}')}]$. Taking one of them involving θ, we obtain a polynomial equation of degree one for θ over $\mathbb{C}[y_{\boldsymbol{h}'}, u_{\mathrm{in}(\boldsymbol{h}')}]$ modulo $\mathcal{J}_{\boldsymbol{h}'} \cap \mathbb{C}[\boldsymbol{y}_{\boldsymbol{h}'}, u_{\mathrm{in}(\boldsymbol{h}')}]$. Since this holds for every $\theta \in \boldsymbol{\theta}^{\#}$, the cardinality of the generic fiber of the projection of Z onto $(\boldsymbol{y}_{\boldsymbol{h}'}, u_{\mathrm{in}(\boldsymbol{h}')})$-coordinates is one. $\square$

The following statement can be proved in the same way as Theorem 3.16 using the part of the statement of Proposition 3.4 about local identifiability.

Proposition 3.19. *We will use the notation from Theorem 3.16. Parameters $\boldsymbol{\theta}^{\#} \subset \boldsymbol{\theta}$ are locally identifiable if and only if the generic fiber of the projection of Z to the $\big(\boldsymbol{y}_{\boldsymbol{h}'}, u_{\mathrm{in}(\boldsymbol{h}')}\big)$-coordinates is finite.*

Proposition 3.20. *For all $\boldsymbol{h} \in \mathbb{Z}^m_{\geqslant 0}$, the projection of $Z_{\boldsymbol{h}}$ to the $(\boldsymbol{y}_{\boldsymbol{h}}, u_{\mathrm{in}(\boldsymbol{h})})$-coordinates is dominant if and only if the rank of the Jacobian of $\mathcal{S}_{\boldsymbol{h}}$ with respect to $(\boldsymbol{\mu}, \boldsymbol{x}_{\mathrm{st}(\boldsymbol{h})})$ is equal to $|\mathcal{S}_{\boldsymbol{h}}|$ on a dense open subset of $Z_{\boldsymbol{h}}$.*

Proof. We set

$$N := |\boldsymbol{y}_{\boldsymbol{h}}| + |\boldsymbol{x}_{\mathrm{st}(\boldsymbol{h})}| + |\boldsymbol{\mu}| + |u_{\mathrm{in}(\boldsymbol{h})}| \quad \text{and} \quad N_0 := |\boldsymbol{y}_{\boldsymbol{h}}| + |u_{\mathrm{in}(\boldsymbol{h})}|.$$

Then $Z_{\boldsymbol{h}} \subset \mathbb{A}^N$, and we denote the projection to the $(\boldsymbol{y}_{\boldsymbol{h}}, u_{\mathrm{in}(\boldsymbol{h})})$-coordinates by $\pi \colon \mathbb{A}^N \to \mathbb{A}^{N_0}$. Let M be the Jacobian matrix of $\mathcal{S}_{\boldsymbol{h}}$ with respect to $(\boldsymbol{x}_{\mathrm{st}(\boldsymbol{h})}, \boldsymbol{\mu})$. Since $\mathcal{S}_{\boldsymbol{h}}$ is a triangular set and $\mathcal{J}_{\boldsymbol{h}} = (\mathcal{S}_{\boldsymbol{h}}) : Q^\infty$, $\operatorname{codim} Z_{\boldsymbol{h}} = |\mathcal{S}_{\boldsymbol{h}}|$ by [20, Theorem 4.4].

Assume that $\pi(Z_{\boldsymbol{h}})$ is not dense in $\mathbb{A}^{N_0}$. Then, by [47, Theorem 1.25], for every $x \in Z_{\boldsymbol{h}}$,

$$\dim \pi^{-1}(\pi(x)) \cap Z_{\boldsymbol{h}} > \dim Z_{\boldsymbol{h}} - N_0. \tag{3.8}$$

Let $p := \left(\hat{\boldsymbol{y}}_{\boldsymbol{h}}, \hat{u}_{\mathrm{in}(\boldsymbol{h})}\right) \in \pi(Z_{\boldsymbol{h}})$. Then

$$\widehat{M} := M\left[\boldsymbol{y}_{\boldsymbol{h}} \leftarrow \hat{\boldsymbol{y}}_{\boldsymbol{h}}, u_{\mathrm{in}(\boldsymbol{h})} \leftarrow \hat{u}_{\mathrm{in}(\boldsymbol{h})}\right]$$

can be viewed as the Jacobian of the polynomials $\mathcal{S}_{\boldsymbol{h}}\left[\boldsymbol{y}_{\boldsymbol{h}} \leftarrow \hat{\boldsymbol{y}}_{\boldsymbol{h}}, u_{\mathrm{in}(\boldsymbol{h})} \leftarrow \hat{u}_{\mathrm{in}(\boldsymbol{h})}\right]$ in $(\boldsymbol{x}_{\mathrm{st}(\boldsymbol{h})}, \boldsymbol{\mu})$, which all vanish on $\pi^{-1}(p) \cap Z_{\boldsymbol{h}}$. Then the rank of $\widehat{M}$ at every point of $\pi^{-1}(p) \cap Z_{\boldsymbol{h}}$ does not exceed the codimension of $\pi^{-1}(p) \cap Z_{\boldsymbol{h}}$ in $\pi^{-1}(p)$ [15, Theorem 16.19], which, by (3.8), is less than

$$(N - N_0) - (\dim Z_{\boldsymbol{h}} - N_0) = \operatorname{codim} Z_{\boldsymbol{h}}.$$

Assume that $\pi(Z_{\boldsymbol{h}})$ is dense in $\mathbb{A}^{N_0}$. Let $A := \mathbb{C}[\boldsymbol{y}_{\boldsymbol{h}}, u_{\mathrm{in}(\boldsymbol{u})}]$ and $B := \mathrm{Quot}(A)$. Hence, $\mathcal{J}_{\boldsymbol{h}} \cap A = 0$. We introduce a new variable z and set $\widetilde{\mathcal{R}} := \mathcal{R}_{\boldsymbol{h}}[z]$ and $\widetilde{\mathcal{J}} := (\mathcal{S}_{\boldsymbol{h}}, Qz - 1)$. Then $\widetilde{\mathcal{J}} \cap \mathcal{R}_{\boldsymbol{h}} = \mathcal{J}_{\boldsymbol{h}}$ by [11, Theorem 14, page 205]. The ideal $\widetilde{\mathcal{J}}$ is prime because $\widetilde{\mathcal{R}}/\widetilde{\mathcal{J}}$ can be obtained from the domain $\mathcal{R}_{\boldsymbol{h}}/\mathcal{J}_{\boldsymbol{h}}$ by inverting the image of Q. Since $\widetilde{\mathcal{J}} \cap A = \{0\}$, $B \cdot \widetilde{\mathcal{J}}$ is a prime ideal in the B-algebra $B \otimes_A \widetilde{\mathcal{R}}$. Note that

$$\left(B \otimes_A \widetilde{\mathcal{R}}/B \cdot \widetilde{\mathcal{J}}\right)_{(0)}$$

is a regular local ring. Let c denote the codimension of $B \cdot \widetilde{\mathcal{J}}$ in $B \otimes_A \widetilde{\mathcal{R}}$. [15, Corollary 16.20] implies the ideal of $B \otimes_A \widetilde{\mathcal{R}}/B \cdot \widetilde{\mathcal{J}}$ generated by the $c \times c$ minors of $\widetilde{M}$, the Jacobian of $\{\mathcal{S}_{\boldsymbol{h}}, Qz - 1\}$ with respect to $(z, \boldsymbol{\mu}, \boldsymbol{x}_{\mathrm{st}(\boldsymbol{h})})$, strictly contains (0). Therefore, $\operatorname{rank} \widetilde{M} \geqslant c$. By [15, Theorem 16.19], $\operatorname{rank} \widetilde{M} \leqslant c$. Since $\widetilde{M}$ is of the form

$$\begin{pmatrix} Q & * \\ 0 & M \end{pmatrix},$$

its rank is equal to the rank of M modulo $\mathcal{J}_{\boldsymbol{h}}$ plus one. Since $\widetilde{\mathcal{J}} \cap A = \{0\}$, the codimension of $B\widetilde{\mathcal{J}}$ in $B \otimes_A \widetilde{\mathcal{R}}$ is equal to the codimension of $\widetilde{\mathcal{J}}$ in $\widetilde{\mathcal{R}}$, and the latter is equal to one plus the codimension of $\mathcal{J}_{\boldsymbol{h}}$ in $\mathcal{R}_{\boldsymbol{h}}$. Thus, the rank of M modulo $\mathcal{J}_{\boldsymbol{h}}$ is equal to $|\,\mathcal{S}_{\boldsymbol{h}}\,|$. ◻

Corollary 3.21. *There exists $\boldsymbol{h} \in \mathbb{Z}^m_{\geqslant 0}$ satisfying requirements (a) and (b) from Theorem 3.16. Moreover, for every $\boldsymbol{h} = (h_1, \ldots, h_m)$ satisfying requirement (a), $h_1 + \ldots + h_m \leqslant s = |\boldsymbol{\theta}|$.*

Proof. Let $\boldsymbol{h} = (h_1, \ldots, h_m)$ satisfy requirement (a) from Theorem 3.16. By Proposition 3.20, $|\,\mathcal{S}_{\boldsymbol{h}}\,| \leqslant |\boldsymbol{x}_{\mathrm{st}(\boldsymbol{h})}| + |\boldsymbol{\mu}|$. Since also $|\,\mathcal{S}_{\boldsymbol{h}}\,|$ is equal to the sum of $|\boldsymbol{y}_{\boldsymbol{h}}|$ and the number of the elements of $\boldsymbol{x}_{\mathrm{st}(\boldsymbol{h})}$ of nonzero order,

$$|\,\mathcal{S}_{\boldsymbol{h}}\,| \geqslant |\boldsymbol{y}_{\boldsymbol{h}}| + |\boldsymbol{x}_{\mathrm{st}(\boldsymbol{h})}| - n = h_1 + \ldots + h_m + |\boldsymbol{x}_{\mathrm{st}(\boldsymbol{h})}| - n.$$

Therefore, $h_1 + \ldots + h_m \leqslant n + |\boldsymbol{\mu}| = s$. Thus, $h_1 + \ldots + h_m \leqslant s$.

Let $\boldsymbol{H}$ be the set of all $\boldsymbol{h} \in \mathbb{Z}^m_{\geqslant 0}$ that satisfy requirement (a) from Theorem 3.16. Since $h_1 + \ldots + h_m \leqslant s$ for every $\boldsymbol{h} = (h_1, \ldots, h_m) \in \boldsymbol{H}$, $\boldsymbol{H}$ is finite. Then it contains a maximal element with respect to the coordinate-wise partial ordering. Such an element also satisfies requirement (b) from Theorem 3.16. ◻

4 Probabilistic Criterion

The goal of the present section is to provide theoretical grounds for our probabilistic algorithm for checking the fiber condition in the statement of Theorem 3.16 efficiently. For an example of using these results in Algorithm 1, we refer to Example 5.4.

Notation 4.1. Let $\boldsymbol{h} \in \mathbb{Z}^m_{\geqslant 0}$ be a tuple from the statement of Theorem 3.16, $\boldsymbol{\theta}^\#$ be any non-empty subset of $\boldsymbol{\theta}$, and $\boldsymbol{h'} \in \mathbb{Z}^m$ be a tuple such that $\boldsymbol{h'} - \boldsymbol{h} \in \mathbb{Z}^m_{>0}$. We introduce the following affine spaces:

(i) the ambient space V with coordinates $(\boldsymbol{\mu}, \boldsymbol{x}_{\mathrm{st}(\boldsymbol{h'})}, \boldsymbol{y}_{\boldsymbol{h'}}, u_{\mathrm{in}(\boldsymbol{h'})})$;
(ii) the input-output space $V_{io} \subset V$ with coordinates $(\boldsymbol{y}_{\boldsymbol{h'}}, u_{\mathrm{in}(\boldsymbol{h'})})$;
(iii) the identification space $V_\# \subset V$ with coordinates $\left(\boldsymbol{\theta}^\#, \boldsymbol{y}_{\boldsymbol{h'}}, u_{\mathrm{in}(\boldsymbol{h'})}\right)$, where $\boldsymbol{\theta}^\# \subset \boldsymbol{\theta}$ is the set of parameters whose global identifiability we would like to check.

Below are several natural projections between them:

- $\pi_{io}\colon V \to V_{io}$ is the projection of what we consider to what we observe;
- $\pi_\#\colon V \to V_\#$ is the projection of what we consider to what we care about;
- $\pi\colon V_\# \to V_{io}$ is the projection of what care about to what we observe.

Using this notation, we can define Z from the statement of Theorem 3.16 as

$$Z = \overline{\pi_\#(Z_{\boldsymbol{h'}})}.$$

Theorem 4.2. *If parameters $\boldsymbol{\theta}^\#$ are locally identifiable, then there exists a polynomial $P \in \mathbb{C}[V_\#]$ such that*

(i) P does not vanish everywhere on Z;
(ii) $\deg P \leqslant (2 + |\boldsymbol{\theta}^\#|) \deg Z_{\boldsymbol{h'}}$;
(iii) For all $a \in Z$ such that $P(a) \neq 0$, the following statements are equivalent:
  (a) every $\theta \in \boldsymbol{\theta}^\#$ is globally identifiable,
  (b) $\left(\mathcal{J}_{\boldsymbol{h'}} + I(\pi(a)) \cdot \mathbb{C}[V]\right) \cap \mathbb{C}[V_\#] = I(a)$,
  (c) the zero set of $\left(\mathcal{J}_{\boldsymbol{h'}} + I(\pi(a)) \cdot \mathbb{C}[V]\right) \cap \mathbb{C}[V_\#]$ is $\{a\}$.

Before proving Theorem 4.2, we formulate and prove two technical lemmas, in which we do *not* use any notation introduced beyond Section 3.1. We will denote the d-dimensional affine and projective $\mathbb{C}$-spaces by $\mathbb{A}^d$ and $\mathbb{P}^d$, respectively.

Lemma 4.3. *Let n, m, and r be non-negative integers such that $n = m + r$, $X \subset \mathbb{A}^n = \mathbb{A}^r \times \mathbb{A}^m$ an irreducible variety, and $\pi\colon \mathbb{A}^n \to \mathbb{A}^m$ the projection onto the second component. Assume that the generic fiber of $\pi|_X$ is finite.*

(i) Then there exists a proper subvariety $Y \subset \overline{\pi(X)}$ such that
  (a) $\deg Y \leqslant \deg X$ and
  (b) for every point $p \in \overline{\pi(X)} \setminus Y$, there exists a closed (in the standard topology) ball $B \subset \mathbb{A}^m$ centered at p such that $\pi^{-1}(B) \cap X$ is compact (in the standard topology) and $\pi^{-1}(p) \cap X \neq \varnothing$.

(ii) Then there exists a hypersurface $H \subset \mathbb{A}^n$ (possibly, empty) not containing X such that

  (a) $\deg H \leqslant r \cdot (\deg X - 1)$ *and*

  (b) for every $p \in X \setminus H$, there exists a closed ball (in the standard topology) $B \subset \mathbb{A}^m$ centered at $\pi(p)$ such that π defines a bijection between the connected component (in the standard topology) of $\pi^{-1}(B) \cap X$ containing p and its image.

(iii) If the generic fiber of $\pi|_X$ is of cardinality one, then there exists a hypersurface $H \subset \mathbb{A}^n$ (possibly empty) not containing X such that

  (a) $\deg H \leqslant r \cdot \deg X$ *and*

  (b) for every $p \in X \setminus H$,

$$I(X) + I(\pi(p)) \cdot \mathbb{C}[\mathbb{A}^n] = I(p).$$

Proof. We introduce coordinates $\boldsymbol{x} := (x_1, \ldots, x_r)$ in $\mathbb{A}^r$ and $\boldsymbol{y} := (y_1, \ldots, y_m)$ in $\mathbb{A}^m$. The condition that the generic fiber of $\pi|_X$ is finite implies that $\dim X = \dim \pi(X)$ [47, Theorem 1.25(ii)], so each element of $\boldsymbol{x}$ is algebraic over $\mathbb{C}[\boldsymbol{y}]$ modulo $I(X)$. We prove each claim.

(i) Embed $\mathbb{A}^r$ into $\mathbb{P}^r$, then π can be extended to $\pi_{\mathbb{P}} \colon \mathbb{P}^r \times \mathbb{A}^m \to \mathbb{A}^m$. Let

$$H_\infty := \mathbb{P}^r \times \mathbb{A}^m \setminus \mathbb{A}^r \times \mathbb{A}^m$$

and $\overline{X}^{\mathbb{P}}$ be the closure of X in $\mathbb{P}^r \times \mathbb{A}^m$. We set

$$Y := \pi_{\mathbb{P}}\big(\overline{X}^{\mathbb{P}} \cap H_\infty\big).$$

By [11, Corollary 9, page 431], $Y \subset \overline{\pi(X)}$. Since $\pi|_X$ has finite generic fiber, $\dim X = \dim \overline{\pi(X)}$ by [47, Theorem 1.25(ii)]. On the other hand,

$$\dim Y \leqslant \dim \overline{X}^{\mathbb{P}} \cap H_\infty \leqslant \dim X - 1,$$

hence Y is a proper subvariety in $\overline{\pi(X)}$. Also,

$$\deg Y \leqslant \deg \overline{X}^{\mathbb{P}} \cap H_\infty \leqslant \deg X.$$

Let $p \in \overline{\pi(X)} \setminus Y$ and $B \subset \mathbb{A}^m$ a closed ball (of a positive radius) centered at p such that $\pi^{-1}(B) \cap X$ is compact. Such a B exists by [23, Lemma 2]. Moreover, $\pi^{-1}(B) \cap X \neq \varnothing$. Indeed, [35, Theorem 1, page 58] implies that $\pi(X)$ is dense in $\overline{\pi(X)}$ with respect to the standard topology, so B contains at least one point of $\pi(X)$. Let us show that $\pi^{-1}(p) \cap X \neq \varnothing$. For this, let $p_1, p_2, \ldots \in \pi(X) \cap B$ be a sequence of points converging to p, which exists because $p \in \overline{\pi(X)}$. Let $q_1, q_2, \ldots \in \pi^{-1}(B) \cap X$ be such that $\pi(q_i) = p_i$, $i \geqslant 1$. Since $\pi^{-1}(B) \cap X$ is compact, there exists a converging subsequence of the sequence $q_1, q_2, \ldots$ with a limit $q \in X$. Since $p_1, p_2, \ldots$ converge to p and π is continuous, $\pi(q) = p$.

(ii) We choose a subset $S \subset \{\boldsymbol{y}\}$ that is a transcendence basis of $\mathbb{C}[\boldsymbol{y}]$ modulo $I(X)$ over $\mathbb{C}$. For every i, $1 \leqslant i \leqslant r$, the projection of X to the (x_i, S)-coordinates is an irreducible hypersurface of degree at most $\deg X$. We denote its defining irreducible polynomial by $P_i(x_i, \boldsymbol{y})$. For every i, $1 \leqslant i \leqslant r$, $\frac{\partial}{\partial x_i} P_i$ does not vanish everywhere on X. Hence,

$$P := \prod_{i=1}^{r} \frac{\partial}{\partial x_i} P_i$$

does not vanish everywhere on X. Let $H := Z(P)$. We prove that H satisfies the requirements.

Consider $p \in X \setminus H$, and let $p = (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$. Let $\widetilde{X} := Z(P_1, \ldots, P_r)$, a subvariety in $\mathbb{A}^r \times \mathbb{A}^m$. Then $X \subset \widetilde{X}$. Since $P(p) \neq 0$, the Jacobian of $P_1, \ldots, P_r$ with respect to $\boldsymbol{x}$ is invertible at p. The implicit function theorem [44, Theorem 3.1.4] (applied to $X = \mathbb{A}^m$, $Y = \mathbb{A}^r$, $f = (P_1, \ldots, P_r)$) implies that there exist neighborhoods $U_1 \subset \mathbb{A}^m$ and $U_2 \subset \mathbb{A}^r$ of $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{x}}$, respectively, such that $S(\widetilde{X} \cap U_2 \times U_1)$ is a graph of some function $\varphi\colon U_1 \to U_2$, where $S(\boldsymbol{a}, \boldsymbol{b}) := (\boldsymbol{b}, \boldsymbol{a})$, $\boldsymbol{a} \in \mathbb{A}^r$, $\boldsymbol{b} \in \mathbb{A}^m$. Hence, π defines a bijection between $\widetilde{X} \cap U_2 \times U_1$ and its image under π.

Let $B \subset U_1$ be a closed ball centered at $\pi(p)$ and C the connected component of $\widetilde{X} \cap \pi^{-1}(B)$ containing p. Let G_B denote the graph of $\varphi|_B$. We claim that $C \subset S(G_B)$. Consider the intersection

$$C \cap S(G_B) = C \cap \big(\widetilde{X} \cap (U_2 \times B)\big).$$

Since $C \subset \widetilde{X}$, the latter intersection is equal to $C \cap (U_2 \times B)$. Since $\pi(C) \subset B$, the latter intersection is the same as $C \cap (U_2 \times U_1)$. Hence,

$$C \cap S(G_B) = C \cap (U_2 \times U_1). \tag{4.1}$$

Since $S(G_B)$ is closed, $U_2 \times U_1$ is open, and C is connected,

$$C \cap (U_2 \times U_1) = \varnothing \quad \text{or} \quad C \cap (U_2 \times U_1) = C.$$

Since $p \in C \cap (U_2 \times U_1)$, by (4.1), we have $C \cap S(G_B) = C$, and so $C \subset S(G_B)$. Since $S(G_B)$ maps bijectively onto $\pi(S(G_B))$, C also maps bijectively onto $\pi(C)$. Since $X \subset \widetilde{X}$, the connected component C_0 of $\pi^{-1}(B) \cap X$ containing p is a subset of C, so π defines a bijection from C_0 to $\pi(C_0)$.

(iii) Since each element of $\boldsymbol{x}$ is algebraic over $\mathbb{C}[\boldsymbol{y}]$ modulo $I(X)$, there exists a representation of $I(X)$ as a triangular set $P_1, \ldots, P_r, \ldots$ with respect to an ordering of the form

$$x_1 > x_2 > \ldots > x_r > \boldsymbol{y} \text{ in some order}$$

so that x_i is the leading variable of P_i, $1 \leqslant i \leqslant r$. Since the cardinality of the generic fiber is one, $\deg_{x_i} P_i = 1$ for every i, $1 \leqslant i \leqslant r$ (see [45, page 562]). Since P_i is reduced with respect to $P_{i+1}, \ldots, P_r$, P_i does not depend on any variable in $\boldsymbol{x}$ except for x_i. [12, Theorem 2] implies that the triangular set can be chosen in such a way that the degrees of the coefficients of P_i as a polynomial with respect to x_i do not exceed $\deg X$. We set P to be the product of the leading coefficients of $P_1, \ldots, P_r$ and $H := Z(P)$ in $\mathbb{A}^n$. Then

$$\deg P \leqslant r \cdot \deg X.$$

Let $p \in X \setminus H$. Then $\pi(p)$ is not a zero of P. We observe that $P_1, \ldots, P_r \in I(X)$ since $I(X)$ is a prime ideal. Then the ideal

$$J := I(X) + I(\pi(p)) \cdot \mathbb{C}[\mathbb{A}^n]$$

contains a polynomial $P_i(\mathbf{x}, \pi(p)) = a_i x_i + b_i$, where $a_i \in \mathbb{C}^*$, $b_i \in \mathbb{C}$, for every $1 \leqslant i \leqslant r$. Since p is a common zero of all polynomials in J, $\frac{-b_i}{a_i}$ is the value of the x_i-th coordinate of p. Since $I(\pi(p))$ also contains a polynomial of the form $y_j - c_j$, where $c_j \in \mathbb{C}$ is the value of the y_j-th coordinate of p, for every j, $1 \leqslant j \leqslant m$, the ideal J is simply $I(p)$. □

Lemma 4.4. *Let n, m, and r be non-negative integers such that $n = m + r$, $X \subset \mathbb{A}^n = \mathbb{A}^r \times \mathbb{A}^m$ an irreducible variety, and $\pi\colon \mathbb{A}^n \to \mathbb{A}^m$ the projection onto the second component. Then there exists a proper subvariety $Y \subset \overline{\pi(X)}$ such that*

(i) $\deg Y \leqslant \deg X$ *and*
(ii) for every $p \in \overline{\pi(X)} \setminus Y$, $\pi^{-1}(p) \cap X \neq \varnothing$.

Proof. If $\dim \overline{\pi(X)} = 0$, then the irreducibility of X implies that $\overline{\pi(X)}$ is a single point. Hence, we can choose $Y = \varnothing$. Before finishing the proof, we will first prove the following.

Claim. *Assume that both the generic fiber of $\pi|_X$ and $\overline{\pi(X)}$ are not zero-dimensional. Then there exists a non-empty open subset $\mathcal{U}$ in the space of all hyperplanes in $\mathbb{A}^n$ such that, for all $H \in \mathcal{U}$,*

- *$X \not\subset H$,*
- *$H \cap X$ is irreducible, and*
- *$H \cap X$ projects dominantly onto $\pi(X)$.*

Proof. In this case, $\dim X \geqslant 2$ by [47, Theorem 1.25(ii)], so Bertini's theorem [3, III.7.(i)] implies that $H \cap X$ is irreducible for a generic H. Let U be a non-empty open subset of $\overline{\pi(X)}$ such that, for every $p \in U$,

$$\dim \pi^{-1}(p) \cap X > 0.$$

Such a U exists by [47, Theorem 1.25(ii)]. Since U is dense in $\overline{\pi(X)}$, there exists a set of points $p_1, \ldots, p_N \in U$ for some N such that, if a polynomial of degree at most $\deg X$ vanishes at $p_1, \ldots, p_N$, then it vanishes on $\overline{\pi(X)}$. Consider a hyperplane H such that

(a) $H \cap X$ is irreducible and

(b) for every i, $1 \leqslant i \leqslant N$,

$$H \cap \pi^{-1}(p_i) \cap X \neq \varnothing.$$

Since conditions (a) and (b) are generic, the conjunction is also generic. Let $Y := \overline{\pi(H \cap X)}$, then $p_1, \ldots, p_N \in Y$. Since $\deg Y \leqslant \deg X$, [16, Proposition 3] implies that Y can be defined by polynomials of degree at most $\deg X$. If $Y \subsetneq \overline{\pi(X)}$, then there exists a polynomial of degree at most $\deg X$ that vanishes on Y (and, in particular, at $p_1, \ldots, p_N$), but does not vanish on $\overline{\pi(X)}$. This is impossible. . ◻

We now return to the proof of Lemma 4.4. Assume that the generic fiber of $\pi|_X$ has dimension d. Applying the claim d times, we obtain an affine subspace L such that $X \cap L$ is irreducible, $X \cap L$ projects dominantly onto $\overline{\pi(X)}$, and the generic fiber of $\pi|_{X\cap L}$ is finite by [47, Theorem 1.25(ii)]. Applying statement (i) of Lemma 4.3 to $X \cap L$, we obtain a subvariety $Y \subset \overline{\pi(X)}$ of degree at most $\deg X$ such that every point in $\overline{\pi(X)} \setminus Y$ has a preimage in $X \cap L$. Then it has a preimage in X. ◻

Proof of Theorem 4.2. Since $\boldsymbol{\theta}^\#$ are locally identifiable $\pi|_Z$ has finite generic fiber due to Proposition 3.19. We will construct a polynomial $P \in \mathbb{C}[V_\#]$ as follows

- If $\boldsymbol{\theta}^\#$ is globally identifiable, then we set P to be the product of the polynomials P_1 and P_{lift} defined below.
  (a) Applying statement (iii) of Lemma 4.3 with $X = Z$ and $\pi = \pi$, we obtain a hypersurface $H \subset V_{io}$ of degree at most $|\boldsymbol{\theta}^\#| \deg Z$. We set P_1 to be the defining polynomial of H.
  (b) Applying Lemma 4.4 to $X = Z_{\boldsymbol{h}'}$ and $\pi = \pi_\#$, we obtain a proper subvariety $Y_2 \subset Z$ of degree at most $\deg Z_{\boldsymbol{h}'}$. Since the generic fiber of $\pi|_Z$ is finite, [47, Theorem 1.25(ii)] implies that $\dim Z = \dim \pi(Z)$, so $\overline{\pi(Y_2)}$ is a proper subvariety of $\overline{\pi(Z)}$. [16, Proposition 3] implies that $\overline{\pi(Y_2)}$ can be defined by polynomials of degree at most $\deg Z_{\boldsymbol{h}'}$. We set P_{lift} to be one of these polynomials that does not vanish everywhere on Z.
- If $\boldsymbol{\theta}^\#$ is not globally identifiable, then we set P to be the product of the polynomial P_{lift} defined above and the polynomials P_∞ and P_{mult} defined below.
  (a) Applying statement (i) of Lemma 4.3 with $X = Z$ and $\pi = \pi$, we obtain a proper subvariety $Y_1 \subset \overline{\pi(Z)}$ of degree at most $\deg Z$. [16, Proposition 3] implies that Y_1 can be defined by polynomials of degree at most $\deg Z$. We set P_∞ to be one of these polynomials that does not vanish everywhere on $\overline{\pi(Z)}$.
  (b) Applying statement (ii) of Lemma 4.3 with $X = Z$ and $\pi = \pi$, we obtain a hypersurface $H \subset V_\#$ of degree at most $|\boldsymbol{\theta}^\#|(\deg Z - 1)$. We set P_{mult} to be the irreducible defining polynomial of H.

Summing up the degree bounds, we obtain

$$\deg P \leqslant \max(\deg(P_1 \cdot P_{\text{lift}}), \deg(P_\infty \cdot P_{\text{mult}} \cdot P_{\text{lift}})) \leqslant \left(2 + |\boldsymbol{\theta}^{\#}|\right) \cdot \deg Z_{\boldsymbol{h}'}.$$

In order to prove that P satisfies requirement (iii), we consider $a \in Z$ such that $P(a) \neq 0$.

To prove (a) $\Longrightarrow$ (b), assume that the parameters $\boldsymbol{\theta}^{\#}$ are globally identifiable. Since $P_1(a) \neq 0$, the choice of P_1 (see statement (iii) of Lemma 4.3) implies that

$$I(Z) + I(\pi(a)) \cdot \mathbb{C}[V_{\#}] = I(a). \tag{4.2}$$

Since $I(Z) = I(\mathcal{J}_{\boldsymbol{h}'}) \cap \mathbb{C}[V_{\#}]$,

$$I(Z) + I(\pi(a)) \cdot \mathbb{C}[V_{\#}] \subset (\mathcal{J}_{\boldsymbol{h}'} + I(\pi(a)) \cdot \mathbb{C}[V]) \cap \mathbb{C}[V_{\#}]. \tag{4.3}$$

Since $P_{\text{lift}}(a) \neq 0$,

$$\pi_{\#}^{-1}(a) \cap Z_{\boldsymbol{h}'} \neq \varnothing.$$

Hence, the ideal $\mathcal{J}_{\boldsymbol{h}'} + I(\pi(a)) \cdot \mathbb{C}[V]$ is proper, and so the right-hand side of (4.3) is a proper ideal of $\mathbb{C}[V_{\#}]$. Since, by (4.2), it contains the maximal ideal $I(a)$, it coincides with $I(a)$.

The implication (b) $\Longrightarrow$ (c) follows from $Z(I(a)) = \{a\}$.

To prove (c) $\Longrightarrow$ (a), we assume that the parameters $\boldsymbol{\theta}^{\#}$ are not globally identifiable. Denote the cardinality of the generic fiber of $\pi|_Z$ by $d > 1$. We define

$$C(a) := \pi_{\#}\left(\pi_{io}^{-1}(\pi(a)) \cap Z_{\boldsymbol{h}'}\right).$$

A direct computation shows that

$$J := (\mathcal{J}_{\boldsymbol{h}'} + I(\pi(a)) \cdot \mathbb{C}[V]) \cap \mathbb{C}[V_{\#}]$$

vanishes at all the points of $C(a)$. Thus, if we prove that $|C(a)| > 1$, this would imply that the zero set of J in not $\{a\}$.

Since $P_{\text{lift}}(a) \neq 0$ and $P \in C[V_{io}]$, for all $b \in \pi^{-1}(\pi(a))$, $P_{\text{lift}}(b) \neq 0$. Hence, the choice of P_{lift} implies (see Lemma 4.4) that, for all $p \in \pi^{-1}(\pi(a)) \cap Z$,

$$\pi_{\#}^{-1}(p) \cap Z_{\boldsymbol{h}'} \neq \varnothing. \tag{4.4}$$

Since $\pi_{\#}^{-1}(p) \subset \pi_{io}^{-1}(\pi(p))$ and $\pi(a) = \pi(p)$, we have $\pi_{\#}^{-1}(p) \subset \pi_{io}^{-1}(\pi(a))$. Hence,

$$\pi_{\#}^{-1}(p) \cap Z_{\boldsymbol{h}'} \subset \pi_{io}^{-1}(\pi(a)) \cap Z_{\boldsymbol{h}'}.$$

Therefore, using (4.4),

$$\pi^{-1}(\pi(a)) \cap Z \subset \pi_{\#}\left(\pi_{io}^{-1}(\pi(a)) \cap Z_{\boldsymbol{h}'}\right).$$

Thus,

$$|C(a)| \geqslant |\pi^{-1}(\pi(a)) \cap Z|.$$

Let B_1 and B_2 be closed balls in V_{io} centered at $\pi(a)$ such that

- $\pi^{-1}(B_1) \cap Z$ is compact and
- π defines a bijection between the connected component of $\pi^{-1}(B_2) \cap Z$ containing a and its image.

The existence of such B_1 and B_2 is implied by statements (i) and (ii) of Lemma 4.3 because $P_{\text{mult}}(a) \neq 0$ and $P_\infty(a) \neq 0$. We set $B = B_1 \cap B_2$. Let $C(B)$ be the set of connected components of $\pi^{-1}(B) \cap Z$. Since $\pi^{-1}(B) \cap Z$ is compact, the set $C(B)$ is finite. Let D be the union of all $C \in C(B)$ such that

$$C \cap \pi^{-1}(\pi(a)) \cap Z = \varnothing.$$

Suppose that $D \neq \varnothing$. Since D is compact, $\pi(D)$ is compact, therefore, closed. Moreover, $\pi(a) \notin \pi(D)$. Let B' be a closed ball centered at $\pi(a)$ such that $\pi(D) \cap B' = \varnothing$. We have $B' \subset B$ and, for every $C \in C(B')$,

$$C \cap \pi^{-1}(\pi(a)) \cap Z \neq \varnothing.$$

If $D = \varnothing$, we set $B' = B$. Assume that

$$|\pi^{-1}(\pi(a)) \cap Z| = 1.$$

Hence, $\pi^{-1}(B') \cap Z$ has exactly one connected component. Therefore, for every $p' \in \pi(Z) \cap B'$, we have

$$|\pi^{-1}(p') \cap Z| = 1.$$

Since $\pi(Z) \cap B'$ is Zariski dense in $\pi(Z)$, we arrive at a contradiction with the assumption that the generic fiber is of cardinality $d > 1$. Hence, $|C(a)| > 1$. $\square$

Notation 4.5. Recall that Q is the common denominator of $\boldsymbol{f}$ and $\boldsymbol{g}$. Let $d_0 = \max(\deg Q\boldsymbol{g}, \deg Q\boldsymbol{f})$.

The following statement (with Theorem 4.2 and Proposition 3.20) is used in our design of Algorithm 1.

Proposition 4.6. *For all $\boldsymbol{h}' \in \mathbb{Z}^m_{\geqslant 0}$ and $P \in \mathcal{R}_{\boldsymbol{h}'}$ such that P does not vanish everywhere on $Z_{\boldsymbol{h}'}$, there exists a polynomial $\widetilde{P} \in \mathbb{C}[\boldsymbol{\theta}, u_{\text{in}(\boldsymbol{h}')}]$ such that*

- $\deg \widetilde{P} \leqslant (1 + d_0 \cdot (2 \max \boldsymbol{h}' - 1)) \cdot \deg P$ *and*
- *for all $\tilde{\boldsymbol{\theta}}$ and $\tilde{u}_{\text{in}(\boldsymbol{h}')}$,*

$$\widetilde{P}(\tilde{\boldsymbol{\theta}}, \tilde{u}_{\text{in}(\boldsymbol{h}')}) \neq 0 \implies P \text{ does not vanish at } \pi_{ip}^{-1}(\tilde{\boldsymbol{\theta}}, \tilde{u}_{\text{in}(\boldsymbol{h}')}) \cap Z_{\boldsymbol{h}'},$$

where π_{ip} is the projection from the space with coordinates $(\boldsymbol{\mu}, \boldsymbol{x}_{\text{st}(\boldsymbol{h}')}, \boldsymbol{y}_{\boldsymbol{h}'}, u_{\text{in}(\boldsymbol{h}')})$ to the space with coordinates $(\boldsymbol{\theta}, u_{\text{in}(\boldsymbol{h}')}) = (\boldsymbol{\mu}, x_1, \ldots, x_n, u_{\text{in}(\boldsymbol{h}')})$.

Proof. Consider the ranking on $\mathcal{R}_{\boldsymbol{h}'}$ defined in the proof of Lemma 3.2. We will define a linear operator $r\colon \mathcal{R}_{\boldsymbol{h}'} \to \mathcal{R}_{\boldsymbol{h}'}$. Consider a monomial $m \in \mathcal{R}_{\boldsymbol{h}'}$. Let v be the leading variable of m, so $m = v \cdot \tilde{m}$. Then

$$r(m) := \begin{cases} \left(Qx_i^{(j+1)} - (Qx_i' - F_i)^{(j)}\right) \tilde{m}, & v = x_i^{(j+1)}, \\ \left(Qy_i^{(j)} - (Qy_i - G_i)^{(j)}\right) \tilde{m}, & v = y_i^{(j)}, \\ Q \cdot m, & \text{otherwise.} \end{cases}$$

By the definition of $r(P)$ and $\mathcal{J}_{\boldsymbol{h}'}$,

$$QP - r(P) \in \mathcal{J}_{\boldsymbol{h}'} \text{ for every } P \in \mathcal{R}_{\boldsymbol{h}'}. \tag{4.5}$$

We introduce the following weight function w by

$$\begin{cases} w\big(u^{(i)}\big) = 0, & i \geqslant 0,\\ w\big(x_j^{(i)}\big) = \max(0, 2i-1), & 1 \leqslant j \leqslant n,\ i \geqslant 0,\\ w\big(y_j^{(i)}\big) = 2i+1, & 1 \leqslant j \leqslant m,\ i \geqslant 0,\\ w(\mu_i) = 0, & \mu_i \text{ in } \boldsymbol{\mu}, \end{cases}$$

(extending w multiplicatively to monomials and as the max to sums of monomials). A direct computation shows that

$$r(P) \neq QP \implies w(P) > w(r(P)).$$

Thus, there exists a finite sequence $P_0, \ldots, P_q$ such that

$$P_0 = P, \quad P_{i+1} = r(P_i) \neq QP_i \ \text{ for all } \ 0 \leqslant i < q, \ \text{ and } \ r(P_q) = QP_q.$$

We set $\widetilde{P} := P_q$. Since $w(P) \leqslant \deg P \cdot (2 \cdot \max \boldsymbol{h}' - 1)$, we have

$$q \leqslant \deg P \cdot (2 \cdot \max \boldsymbol{h}' - 1).$$

Therefore,

$$\deg \widetilde{P} \leqslant \deg P + \deg P \cdot d_0 \cdot (2 \cdot \max \boldsymbol{h}' - 1) = (1 + d_0 \cdot (2 \cdot \max \boldsymbol{h}' - 1)) \cdot \deg P.$$

Since $r(\widetilde{P}) = Q\widetilde{P}$, we have $\widetilde{P} \in \mathbb{C}[\boldsymbol{\theta}, u_{\mathrm{in}(\boldsymbol{h}')}]$. Due to (4.5), we have

$$Q^q \cdot P - \widetilde{P} \in \mathcal{J}_{\boldsymbol{h}'}.$$

Therefore, for all $\tilde{\boldsymbol{\theta}}$, $\tilde{u}_{\mathrm{in}(\boldsymbol{h}')}$, and $p \in \pi_{ip}^{-1}\big(\tilde{\boldsymbol{\theta}}, \tilde{u}_{\mathrm{in}(\boldsymbol{h}')}\big) \cap Z_{\boldsymbol{h}'}$, we have

$$\widetilde{P}\big(\tilde{\boldsymbol{\theta}}, \tilde{u}_{\mathrm{in}(\boldsymbol{h}')}\big) = Q^q\big(\tilde{\boldsymbol{\theta}}, \tilde{u}_{\mathrm{in}(\boldsymbol{h}')}\big) \cdot P(p).$$

Hence, if $\widetilde{P}\big(\tilde{\boldsymbol{\theta}}, \tilde{u}_{\mathrm{in}(\boldsymbol{h}')}\big) \neq 0$, then $P(p) \neq 0$. □

5 Algorithm

In this section, by integrating Theorems 3.16 and 4.2 and Propositions 3.20 and 4.6, we provide a probabilistic algorithm for checking global identifiability (Algorithm 1) and prove its correctness (Theorem 5.5).

Remark 5.1. Note that the input-output specification of Algorithm 1 states that $\boldsymbol{\theta}^g$ is equal, with probability at least p, to the set of all globally identifiable parameters in $\boldsymbol{\theta}^\ell$. Let us state it more formally. Let $\boldsymbol{\theta}^{ga}$ stand for the set of all globally identifiable parameters in $\boldsymbol{\theta}^\ell$. Imagine running the algorithm n times with the same input. Let c_n stand for the number of runs with the correct output, that is, $\boldsymbol{\theta}^g = \boldsymbol{\theta}^{ga}$. Then the following is guaranteed

$$\lim_{n \to \infty} \frac{c_n}{n} \geqslant p.$$

Notation 5.2. In the steps of Algorithm 1, we use Notation 3.12 and

$$\boldsymbol{u} = \left(u, u^{(1)}, \ldots, u^{(s)}\right), \qquad \text{where } s = |\boldsymbol{\theta}|.$$

Remark 5.3. In Step 4 of Algorithm 1, we check the consistency of a system of equations and inequations. This can be done in many different ways. We list a few.

Method (i) For each $\theta \in \boldsymbol{\theta}^{\ell}$, check the inconsistency of the following system of equations

$$\widehat{E^t} \cup \left\{z \cdot \hat{Q} - 1, w(\theta - \hat{\theta}) - 1\right\},$$

where z and w are new variables introduced for the Rabinowitsch trick. This can be done by using, for instance, Gröbner bases.

Method (ii) For each $\theta \in \boldsymbol{\theta}^{\ell}$, check

$$\theta - \hat{\theta} \in \mathrm{Ideal}\left(\widehat{E^t} \cup \{z \cdot \widehat{Q} - 1\}\right). \tag{5.1}$$

This can be done by using, for instance, Gröbner bases. Due to Theorem 4.2, condition (5.1) is equivalent to the consistency condition from Step 4 of Algorithm 1 under assumption (5.7) on the sampled point made in the proof of Theorem 5.5 for the index i of this particular θ.

Method (iii) Check it directly using regular chains, which also allows inequations (see, e.g., [56, Section 2.2]). In practice, we observed that this method is generally slower than Method (i) and Method (ii) for the problems that we considered.

Method (iv) Use homotopy continuation methods (for example, in Bertini [6]) as follows

(a) Take a square subsystem $\widehat{E^{ts}} = 0$ from the over-determined system $\widehat{E^t} = 0$ and find all of its roots using homotopy continuation;
(b) Let S be the set of all the roots of $\widehat{E^{ts}} = 0$ that are also roots of $\widehat{E^t} = 0$ and are not roots of $\widehat{Q} = 0$;
(c) For each parameter $\theta \in \boldsymbol{\theta}^{\ell}$, we put θ into $\boldsymbol{\theta}^{g}$ if and only if the θ-coordinate of every point in S is equal to $\hat{\theta}$.

However the number of paths in problems of moderate size (like Examples 6.1 and 6.2) is too large for practical computation.

Example 5.4. We will illustrate the steps of the algorithm on a system of small size.

IN: $\qquad \Sigma := \begin{cases} x' = \mu_2 x + \mu_1, \\ y = x^2, \\ x(0) = x^*. \end{cases}$

$\boldsymbol{\theta}^{\ell} := \{\mu_1, \mu_2, x^*\}$ *(one can show that these parameters are locally identifiable)*

$p := 0.8$

Algorithm 1: *Global_Identifiability*

IN: Σ: an algebraic differential model
given by rational functions $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}, u)$ and $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\mu}, u)$
$\boldsymbol{\theta}^{\ell}$: a subset of $\boldsymbol{\theta} = \boldsymbol{\mu} \cup \boldsymbol{x}^*$ with every parameter in $\boldsymbol{\theta}^{\ell}$ locally identifiable
p: an element of $(0, 1)$
OUT: $\boldsymbol{\theta}^{g}$: a subset of $\boldsymbol{\theta}^{\ell}$ that is equal, with probability at least p, to the set of all globally identifiable parameters in $\boldsymbol{\theta}^{\ell}$ (see Remark 5.1)

1 [Construct the maximal system E of algebraic equations]
 (a) $s \leftarrow |\boldsymbol{\theta}|$, $Q \leftarrow$ the common denominator of $\boldsymbol{f}$ and $\boldsymbol{g}$
 (b) $X_{i0} \leftarrow x_i^{(0)} - x_i^*$ for $1 \leqslant i \leqslant n$
 (c) $X_{ij} \leftarrow \left(Qx_i' - Qf_i\right)^{(j-1)}$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant s$
 (d) $Y_{ij} \leftarrow (Qy_i - Qg_i)^{(j)}$ for $1 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant s$
 (e) $E \leftarrow$ the set of all X_{ij} and Y_{ij} computed in Steps 1b, 1c, and 1d.

2 [Truncate the system E, obtaining E^t]
 (a) $d_0 \leftarrow \max(\deg Q\boldsymbol{f}, \deg Q\boldsymbol{g}, \deg Q)$
 (b) $D_1 \leftarrow 2d_0 s(n + 1)(1 + 2d_0 s)/(1 - p)$
 (c) $\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{u}} \leftarrow$ random vectors of integers from $[1, D_1]$ and $Q(\hat{\boldsymbol{\theta}}, \hat{u}_0) \neq 0$
 (d) Find the unique solution of the triangular system $E\left[\boldsymbol{\theta} \leftarrow \hat{\boldsymbol{\theta}}, \boldsymbol{u} \leftarrow \hat{\boldsymbol{u}}\right]$ (in which each equation is linear in its leader) for all the variables. Denote the $x_i^{(j)}$- and $y_i^{(j)}$-components of the solutions by $\hat{x}_{ij}$ and $\hat{y}_{ij}$.
 (e) Let $\boldsymbol{\alpha} \leftarrow (1, \ldots, 1) \in \mathbb{Z}^n_{\geqslant 0}$, $\boldsymbol{\beta} \leftarrow (0, \ldots, 0) \in \mathbb{Z}^m_{\geqslant 0}$, $E^t \leftarrow \{X_{10}, \ldots, X_{n0}\}$
 (f) While there exists k such that the rank of $\mathrm{Jac}_{\boldsymbol{\theta}, \boldsymbol{x}_{\boldsymbol{\alpha}}}\left(E^t \cup \{Y_{k\beta_k}\}\right)$ at $\left(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{x}}_{\boldsymbol{\alpha}}, \hat{\boldsymbol{y}}_{\boldsymbol{\beta}}, \hat{\boldsymbol{u}}\right)$ is equal to $|E^t| + 1$
  (i) Add Y_{k,β_k} to E^t and then increment β_k
  (ii) While $x_i^{(j)}$ appears in $E^t \cup \left\{Y_{1\beta_1}, \ldots, Y_{m\beta_m}\right\}$ but $X_{ij} \notin E^t$, add X_{ij} to E^t
  (iii) Set $\alpha_i \leftarrow \max_{P \in E^t} \mathrm{ord}_{x_i} P + 1$ for every $1 \leqslant i \leqslant n$
 (g) While for some $1 \leqslant i \leqslant m$ all $\boldsymbol{x}$-variables in $Y_{i(\beta_i+1)}$ belong to $\boldsymbol{x}_{\boldsymbol{\alpha}}$, add $Y_{i(\beta_i+1)}$ to E^t and increment β_i

3 [Randomize some variables in E^t, obtaining $\widehat{E^t}$]
 (a) $D_2 \leftarrow 6|\boldsymbol{\theta}^{\ell}|\left(\prod_{P \in E^t} \deg P\right)(1 + 2d_0 \max \boldsymbol{\beta})/(1 - p)$
 (b) $\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{u}} \leftarrow$ random vectors of integers from $[1, D_2]$ and $Q(\hat{\boldsymbol{\theta}}, \hat{u}_0) \neq 0$
 (c) Find the unique solution of the triangular system $E^t\left[\boldsymbol{\theta} \leftarrow \hat{\boldsymbol{\theta}}, \boldsymbol{u} \leftarrow \hat{\boldsymbol{u}}\right]$ (in which each equation is linear in its leader) for all the variables. Denote the $x_i^{(j)}$- and $y_i^{(j)}$-components of the solutions by $\hat{x}_{ij}$ and $\hat{y}_{ij}$.
 (d) $\widehat{E^t} \leftarrow E^t\left[\boldsymbol{y}_{\boldsymbol{\beta}} \leftarrow \hat{\boldsymbol{y}}_{\boldsymbol{\beta}},\ \boldsymbol{u} \leftarrow \hat{\boldsymbol{u}}\right]$, $\quad \widehat{Q} \leftarrow Q[\boldsymbol{u} \leftarrow \hat{\boldsymbol{u}}]$

4 [Determine $\boldsymbol{\theta}^{g}$ from $\widehat{E^t}$]
 (a) $\boldsymbol{\theta}^{g} \leftarrow \left\{\theta \in \boldsymbol{\theta}^{\ell} \,\middle|\, \text{the system } \widehat{E^t} = 0 \ \&\ \widehat{Q} \neq 0 \ \&\ \theta \neq \hat{\theta} \text{ is inconsistent}\right\}$

1. [Construct the maximal system E of algebraic equations]
   (a) $s \leftarrow 3, Q \leftarrow 1$
   Since there is only one state variable and only one output variable, from now on, we will drop the first index from X and Y for brevity, for instance X_0 and Y_0 instead $X_{1,0}$ and $Y_{1,0}$, etc.
   (b) $X_0 \leftarrow x - x^*$
   (c) $X_1 \leftarrow x^{(1)} - \mu_2 x - \mu_1$, $X_2 \leftarrow x^{(2)} - \mu_2 x^{(1)}$, $X_3 \leftarrow x^{(3)} - \mu_2 x^{(2)}$
   (d) $Y_0 \leftarrow y - x^2, \dots, Y_3 \leftarrow y^{(3)} - 2xx^{(3)} - 6x^{(1)}x^{(2)}$
   (e) $E \leftarrow \{X_0, \dots, X_3, Y_0, \dots, Y_3\}$.
2. [Truncate the system E, obtaining E^t]
   (a) $d_0 \leftarrow \deg(\mu_2 x + \mu_1) = 2$.
   (b) $D_1 \leftarrow 1560$
   (c) $(\hat{\mu}_1, \hat{\mu}_2, \hat{x}^*) \leftarrow (1210, 896, 453)$
   (d) Solve the triangular system

$$E\left[\boldsymbol{\theta} \leftarrow \hat{\boldsymbol{\theta}}, \boldsymbol{u} \leftarrow \hat{\boldsymbol{u}}\right] = E\left[\mu_1 \leftarrow \hat{\mu}_1, \mu_2 \leftarrow \hat{\mu}_2, x^* \leftarrow \hat{x}^*\right]$$

$$\begin{aligned} &y^{(3)} - 2xx^{(3)} - 6x^{(1)}x^{(2)} = 0, && x^{(3)} - 896x^{(2)} = 0,\\ &y^{(2)} - 2xx^{(2)} - 2(x^{(1)})^2 = 0, && x^{(2)} - 896x^{(1)} = 0,\\ &y^{(1)} - 2xx^{(1)} = 0, && x^{(1)} - 896x - 1210 = 0,\\ &y - x^2 = 0, && x - 453 = 0.\end{aligned}$$

   iteratively: first the right column from the bottom to the top, then the left column. We obtain

$$\begin{aligned} &\hat{x}_0 \leftarrow 453 && \hat{y}_0 \leftarrow 205209\\ &\hat{x}_1 \leftarrow 407098 && \hat{y}_1 \leftarrow 368830788\\ &\hat{x}_2 \leftarrow 364759808 && \hat{y}_2 \leftarrow 661929949256\\ &\hat{x}_3 \leftarrow 326824787968 && \hat{y}_3 \leftarrow 1187061187802112\end{aligned}$$

   (e) $\boldsymbol{\alpha} \leftarrow (1), \boldsymbol{\beta} \leftarrow (0), E^t \leftarrow \{X_0\}$.
   (f) Iteration 1: Since $\boldsymbol{\alpha} = (0)$ and

$$\operatorname{Jac}_{(\mu_1,\mu_2,x^*,x)}(X_0, Y_0) = \begin{pmatrix} 0 & 0 & -1 & 1\\ 0 & 0 & 0 & -2x \end{pmatrix}$$

   at $(\hat{\mu}_1, \hat{\mu}_2, \hat{x}^*, \hat{x}_0)$ has rank equal to the number or rows, we add Y_0 to E^t and set $\boldsymbol{\beta} \leftarrow (1)$. Since Y_1 involves $x^{(1)}$, we add X_1 to E^t and set $\boldsymbol{\alpha} \leftarrow (2)$.
   Iteration 2: Since

$$\operatorname{Jac}_{(\mu_1,\mu_2,x^*,x,x^{(1)})}(X_0, X_1, Y_0, Y_1) = \begin{pmatrix} 0 & 0 & -1 & 1 & 0\\ -1 & -x & 0 & -\mu_2 & 1\\ 0 & 0 & 0 & -2x & 0\\ 0 & 0 & 0 & -2x^{(1)} & -2x \end{pmatrix}$$

at $(\hat{\mu}_1, \hat{\mu}_2, \hat{x}^*, \hat{x}_0, \hat{x}_1)$ has rank equal to the number of rows, we add Y_1 to E^t and set $\boldsymbol{\beta} \leftarrow (2)$. Since Y_2 involves $x^{(2)}$, we add X_2 to E^t and set $\boldsymbol{\alpha} \leftarrow (3)$. Carrying out similar computations, we obtain a Jacobian with rank equal to the number of rows in the next iteration, but obtain a Jacobian with rank less than the number or rows in the following iteration, so we stop the while loop. We obtain $\boldsymbol{\alpha} = (4)$, $\boldsymbol{\beta} = (3)$, and $E^t = \{X_0, X_1, X_2, X_3, Y_0, Y_1, Y_2\}$.

(g) We add Y_3 to E^t, set $\boldsymbol{\beta} \leftarrow (4)$.

3. [Randomize some variables in E^t, obtaining $\widehat{E^t}$]
   (a) $D_2 \leftarrow 195840$
   (b) $(\hat{\mu}_1, \hat{\mu}_2, \hat{x}^*) \leftarrow (2440, 171852, 68794)$
   (c) We perform the same computation as in Step 2.d, but with the new numbers, and thus find numerical values $\hat{y}_0, \hat{y}_1, \hat{y}_2, \hat{y}_3$.
   (d) $\widehat{E^t} \leftarrow E^t\left[y \leftarrow \hat{y}_0, y^{(1)} \leftarrow \hat{y}_1, y^{(2)} \leftarrow \hat{y}_2, y^{(3)} \leftarrow \hat{y}_3\right], \qquad \hat{Q} \leftarrow 1$
4. [Determine $\boldsymbol{\theta}^g$ from $\widehat{E^t}$]
   This can be done in several ways as described in Remark 5.3.
   - Following Method (i) from Remark 5.3, we compute a Gröbner basis for each of the following

$$\widehat{E^t} \cup \{z-1, (\mu_1 - 2440)w - 1\}, \quad \widehat{E^t} \cup \{z-1, (\mu_2 - 171852)w - 1\},$$
$$\widehat{E^t} \cup \{z-1, (x^* - 68794)w - 1\}.$$

     with respect to a monomial ordering (any choice). A computation shows that only the second Gröbner basis contains 1. Thus μ_2 is globally identifiable, and μ_1 and x^* are not.
   - Following Method (ii) from Remark 5.3, we compute the reduced Gröbner basis of $\widehat{E^t} \cup \{z-1\}$ with respect to (we make this particular choice here just to obtain a concrete answer) the degree reverse lexicographic ordering with the ordering $\mu_1 > \mu_2 > x^* > x > \ldots > x^{(3)} > z$ on the variables:

$$\begin{array}{ll} z-1, & 171852x^{(2)} - x^{(3)}, \\ 29533109904x^{(1)} - x^{(3)}, & 1745759557692283714 56x - 34397x^{(3)}, \\ \mu_2 - 171852, & 436439889423070928 64\mu_1 - 305x^{(3)}, \\ (x^{(3)})^2 - 349151911538456742912^2, & x^* - x, \end{array}$$

     Observe that the Gröbner basis contains $\mu_2 - 171852$ but does not contain $x^* - 68794$ or $\mu_1 - 2440$. Thus, μ_2 is globally identifiable and μ_1 and x^* are not.

OUT: $\boldsymbol{\theta}^g \leftarrow \{\mu_2\}$

Theorem 5.5. *Algorithm 1 produces correct output with probability at least p.*

Proof. In our probability analysis of random vectors $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{u}}$ sampled in steps 2c and 3b, we will use the following observation. For all polynomials $P, Q \in \mathbb{C}[\boldsymbol{\theta}, \boldsymbol{u}]$ such that $Q \neq 0$ and for every sample set of vectors $(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{u}})$, the probability of

$P(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{u}}) \neq 0$ given that $Q(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{u}}) \neq 0$ is greater than or equal to the probability of $P(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{u}})Q(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{u}}) \neq 0$.

We **claim** that the value of $\boldsymbol{\beta}$ right before Step 2g will satisfy the requirements for $\boldsymbol{h}$ of Theorem 3.16 with probability at least $\frac{1+p}{2}$. We run Algorithm 1 replacing checking of the rank of the Jacobian in Step 2f with checking the fact that $Z_{\boldsymbol{\beta}+\mathbf{1}_k}$ projects dominantly to the $(\boldsymbol{y}_{\boldsymbol{\beta}+\mathbf{1}_k}, u_{\mathrm{in}(\boldsymbol{\beta}+\mathbf{1}_k)})$-coordinates. Denote the number of iterations of the while loop in Step 2f by q. For all i, $1 \leqslant i \leqslant q$, we denote the values of $\boldsymbol{\beta}$ and E^t after the i-th such iteration by $\boldsymbol{\beta}_i$ and E^t_i, respectively, where $\boldsymbol{\beta}_i := (\beta_{i1}, \ldots, \beta_{im})$. Let $\boldsymbol{\beta}_0$ and E^t_0 denote the initial values. Due to the construction, $\boldsymbol{\beta}_q$ satisfies the requirements for $\boldsymbol{h}$ of Theorem 3.16.

Comparing Step 2(f)ii of Algorithm 1 and Step (b) of Notation 3.13, we see that, for every $\boldsymbol{\beta}$,

$$E^t \sqcup \left\{Y_{1,\beta_1}, \ldots, Y_{m,\beta_m}\right\} = \mathcal{S}_{\boldsymbol{\beta}+\mathbf{1}} \sqcup \left\{X_{10}, \ldots, X_{n0}\right\} \tag{5.2}$$

This and Notation 3.13 imply that

$$\begin{aligned} E^t &= \mathcal{S}_{\boldsymbol{\beta}} \sqcup \left\{X_{10}, \ldots, X_{n0}\right\} \\ &\quad \sqcup \left\{X_{ij} \in \mathcal{S}_{\boldsymbol{\beta}+\mathbf{1}} \;\middle|\; x_i^{(j)} \text{ does not appear in } \mathcal{S}_{\boldsymbol{\beta}},\ 0 < j,\ 1 \leqslant i \leqslant m\right\}. \end{aligned} \tag{5.3}$$

Proposition 3.20 together with (5.3) imply that, for all k, $1 \leqslant k \leqslant m$, if $Z_{\boldsymbol{\beta}+\mathbf{1}_k}$ does not project dominantly to the $(\boldsymbol{y}_{\boldsymbol{\beta}+\mathbf{1}_k}, u_{\mathrm{in}(\boldsymbol{\beta}+\mathbf{1}_k)})$-coordinates, then the Jacobian condition of the while loop in Step 2f is also false.

Let P_1 denote a $(|E^t_q|+1)\times(|E^t_q|+1)$-minor of the Jacobian of $\mathcal{S}_{\boldsymbol{\beta}_q}$ with respect to $(\boldsymbol{\mu}, \boldsymbol{x}_{\boldsymbol{\alpha}_q})$ that is nonzero modulo $\mathcal{J}_{\boldsymbol{\beta}_q}$. Decomposition (5.3) implies that there exists a $(|E^t_q|+1)\times(|E^t_q|+1)$-minor in the Jacobian of E^t_q with respect to $(\boldsymbol{\theta}, \boldsymbol{x}_{\boldsymbol{\alpha}_q})$ with the determinant equal to $Q^a P_1$ for some a. If $Q \cdot P_1$ does not vanish after the substitution

$$\boldsymbol{x}_{\boldsymbol{\alpha}_q} \leftarrow \hat{\boldsymbol{x}}_{\boldsymbol{\alpha}_q},\ \boldsymbol{y}_{\boldsymbol{\beta}_q} \leftarrow \hat{\boldsymbol{y}}_{\boldsymbol{\beta}_q},\ \boldsymbol{u} \leftarrow \hat{\boldsymbol{u}},\ \boldsymbol{\theta} \leftarrow \hat{\boldsymbol{\theta}}, \tag{5.4}$$

then $\mathrm{Jac}_{\boldsymbol{\theta}, \boldsymbol{x}_{\boldsymbol{\alpha}_q}}(E^t_q)$ has rank $|E^t_q|+1$ at $(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{x}}_{\boldsymbol{\alpha}_q}, \hat{\boldsymbol{y}}_{\boldsymbol{\beta}_q})$. Since $E^t_i \subset E^t_q$ for every $1 \leqslant i \leqslant q$, the corresponding Jacobian of E^t_i has rank $|E^t_i|+1$ at the corresponding point for $1 \leqslant i \leqslant q$. Thus, with this choice of $(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{u}})$, the value of $\boldsymbol{\beta}$ right before Step 2g will be $\boldsymbol{\beta}_q$ and, therefore, satisfy the requirements for $\boldsymbol{h}$ of Theorem 3.16.

We will bound the probability of non-vanishing of QP_1 after substitution (5.4). Let $\boldsymbol{\beta}_q = (\beta_{q1}, \ldots, \beta_{qm})$. Corollary 3.21 implies that $\beta_{q1} + \ldots + \beta_{qm} \leqslant s$, so $|\mathcal{S}_{\boldsymbol{\beta}_q}| \leqslant s + ns$. Hence,

$$\deg P_1 \leqslant s(n+1)d_0.$$

Applying Proposition 4.6 to P_1 with this degree bound, we obtain $\widetilde{P}_1 \in \mathbb{C}[\boldsymbol{\theta}, \boldsymbol{u}]$ such that,

- $\deg \widetilde{P}_1 \leqslant d_0 s(n+1)(1 + d_0(2s-1))$ and
- for all $\tilde{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{u}}$, $\widetilde{P}_1(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{u}}) \neq 0$ implies that P_1 does not vanish at $\pi_{ip}^{-1}(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{u}}) \cap Z_{\boldsymbol{\beta}_q}$.

Since the coordinates of $(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{u}})$ are sampled from 1 to D_1, the Demillo-Lipton-Schwartz-Zippel lemma (see [57, Proposition 98]) implies that $Q(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{u}})\widetilde{P_1}(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{u}}) \neq 0$ with probability at least

$$1 - \frac{d_0 s(n+1)(1 + d_0(2s-1)) + d_0}{D_1} = 1 - \frac{D_1(1-p) - d_0^2 s(n+1) + d_0}{2D_1}$$
$$\geqslant 1 - \frac{1-p}{2} = \frac{1+p}{2}.$$

The **claim** is proved.

The values of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and E^t right after Step 2 of Algorithm 1 have the following properties:

(a) $\boldsymbol{\alpha} = \boldsymbol{\alpha}_q$;
(b) $\boldsymbol{\beta} - \boldsymbol{\beta}_q \in \mathbb{Z}^m_{>0}$;
(c) $E^t \supset E^t_q \cup \{Y_{1\beta_{q1}}, \ldots, Y_{m\beta_{qm}}\}$.

By (5.2) applied to $\boldsymbol{\beta}_q$ and by (c), after Step 2, we have

$$E^t \supset \mathcal{S}_{\boldsymbol{\beta}_q + \mathbf{1}} \sqcup \{X_{10}, \ldots, X_{n0}\}.$$

By (a), $\mathcal{S}_{\boldsymbol{\beta}}$ and $\mathcal{S}_{\boldsymbol{\beta}_q+\mathbf{1}}$ contain the same polynomials of the form $X_{i,j}$. Hence, by the construction of $\mathcal{S}_{\boldsymbol{\beta}}$ (see Step (b) of Notation 3.13) and Step 2g of Algorithm 1,

$$E^t = \mathcal{S}_{\boldsymbol{\beta}} \sqcup \{X_{10}, \ldots, X_{n0}\}. \tag{5.5}$$

Consider i, $1 \leqslant i \leqslant |\boldsymbol{\theta}^\ell|$. Let $P_{2,i}$ be a polynomial whose existence is proven in Theorem 4.2 applied to $\boldsymbol{\theta}^\# = (\theta_i^\ell)$, $\boldsymbol{h}' = \boldsymbol{\beta}$, and $\boldsymbol{h} = \boldsymbol{\beta}_q$ (see (b)). Consider $a := \left(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}_{\boldsymbol{\alpha}}, \hat{\boldsymbol{y}}_{\boldsymbol{\beta}}, \hat{\boldsymbol{u}}\right)$. Then (5.5) implies that the projection of the Zariski closure of the zero set of $\widehat{E^t} = 0$ & $\widehat{Q} \neq 0$ to the $(\boldsymbol{\mu}, \boldsymbol{x}_{\boldsymbol{\alpha}}, \boldsymbol{y}_{\boldsymbol{\beta}}, \boldsymbol{u})$-coordinates is the zero set of the ideal

$$\mathcal{J}_{\boldsymbol{\beta}} + I(\pi_{io}(a)) \cdot \mathbb{C}[\boldsymbol{\mu}, \boldsymbol{x}_{\boldsymbol{\alpha}}, \boldsymbol{y}_{\boldsymbol{\beta}}, \boldsymbol{u}].$$

Hence, θ_i^ℓ will be added to $\boldsymbol{\theta}^g$ if and only if

$$Z\left(\left(\mathcal{J}_{\boldsymbol{\beta}} + I(\pi_{io}(a)) \cdot \mathbb{C}[\boldsymbol{\mu}, \boldsymbol{x}_{\boldsymbol{\alpha}}, \boldsymbol{y}_{\boldsymbol{\beta}}, \boldsymbol{u}]\right) \cap \mathbb{C}[\theta_i^\ell]\right) = \{\hat{\theta}_i^\ell\}. \tag{5.6}$$

The choice of $P_{2,i}$ implies that (5.6) is equivalent to the fact that θ_i^ℓ is globally identifiable under the assumption that $P_{2,i}(a) \neq 0$. Thus, if

$$P_{2,i}(a) \neq 0, \tag{5.7}$$

then the decision of Algorithm 1 regarding global identifiability of θ_i^ℓ will be correct.

Let $P_2 := \prod_{i=1}^{|\boldsymbol{\theta}^\ell|} P_{2,i}$. Then $P_2(a) \neq 0$ implies that the output of Algorithm 1 is correct. Proposition 4.6 applied to $\prod_{i=1}^{|\boldsymbol{\theta}^\ell|} P_{2,i}$ provides a polynomial $\widetilde{P_2} \in \mathbb{C}[\boldsymbol{\theta}, \boldsymbol{u}]$ such that, for all $\tilde{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{u}}$, $\widetilde{P}_2(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{u}}) \neq 0$ implies that P_2 does not vanish at $\pi_{ip}^{-1}(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{u}}) \cap Z_{\boldsymbol{\beta}}$. Using the Bézout bound and the degree bound from Theorem 4.2, we obtain

$$\deg Q\widetilde{P}_2 \leqslant 3\left|\boldsymbol{\theta}^{\ell}\right| \deg Z_{\boldsymbol{\beta}} \cdot (1 + 2d_0 \max \boldsymbol{\beta}) \leqslant 3\left|\boldsymbol{\theta}^{\ell}\right| \left(\prod_{P \in E^t} \deg P\right) (1 + 2d_0 \max \boldsymbol{\beta}).$$

Since the coordinates of $(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{u}})$ are sampled from 1 to D_2, the Demillo-Lipton-Schwartz-Zippel lemma (see [57, Proposition 98]) implies that the point $(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{u}})$ is not a zero of $Q\widetilde{P}_2$ with probability at least

$$1 - \frac{\deg Q\widetilde{P}_2}{D_2} \geqslant 1 - \frac{D_2(1-p)}{2D_2} = \frac{1+p}{2}$$

Thus, with probability at least

$$1 - \left(\left(1 - \frac{1+p}{2}\right) + \left(1 - \frac{1+p}{2}\right)\right) = 2 \cdot \frac{1+p}{2} - 1 = p,$$

the output of Algorithm 1 is correct. □

6 Performance

In this section, we discuss the performance of an implementation of Algorithm 1 using both basic and challenging examples taken from the literature and also discuss how several other existing software packages perform at these examples. Briefly, Table 6.1 below shows that our running time compares favorably to existing software for several challenging problems taken from the literature. More details may be found in our companion paper [19].

We begin by giving a brief descriptions of the implementation of Algorithm 1 and three other software packages that are available to us.

ALGORITHM 1: The current implementation of Algorithm 1 is done on the computer algebra system MAPLE 2017. It takes advantage of parallel computing because (1) the built-in MAPLE function for computing Gröbner bases is parallelized at the thread level and (2) while using Method (i) for Step 4, we naturally compute each Gröbner basis in a separate process. The implementation and examples used in this paper are contained in SIAN v0.5 (available at `https://github.com/pogudingleb/SIAN/releases/tag/v0.5`). Note that the algorithm for Gröbner basis computation in MAPLE is also Monte Carlo with the probability of error at most 10^{-18}, so this probability should be subtracted from the probability of success of our implementation.

DAISY: This is software written in REDUCE [7]. For comparison, we used version 1.9. DAISY takes as an input a positive integer SEED, which is used for sampling random points. We used the default value 35. In our experiments, the software did not appear to compute in parallel. Since the core of the algorithm is a computation of a characteristic set decomposition of a radical differential ideal, it is not clear how the algorithm could be efficiently parallelized.

COMBOS: This is a web-based application [32]. For some of the examples below we received an error message saying "Model may have been entered incorrectly or cannot be solved with COMBOS algorithms". We will denote such cases by $**$ in Table 6.1.

GenSSI 2.0: This is a package written in Matlab [26]. The algorithm uses solve function from Matlab to solve systems of algebraic equations symbolically. For large examples (such as, for example, Examples 6.1 and 6.2), such a symbolic solution does not exists. In such cases, the solve function returns an empty set of solutions together with a warning "Warning: Unable to find explicit solution." This means that there might be solutions but they could not be found by Matlab. The algorithm is unable to conclude whether the parameters are globally identifiable due to this Matlab failure (see Examples 6.1 and 6.2). We will denote such cases by $*$ in Table 6.1.

We ran the program on a computer with 96 CPUs, 2.4 GHz each, under a Linux operating system (CentOS 6.9). The runtime is the elapsed time. The sequential time is the total CPU time spent by all threads of all processes during the computation. The former is measured as the real part of the output of the Linux time command. The latter is estimated as the sum of the user and sys parts of the output of the time command. We report the runtimes in Table 6.1 and the sequentual times in the text afterwards.

We determine locally identifiable parameters using software from [46]. In all our examples, it took less than 5 seconds; this is negligible compared to the other timings.

We ran Algorithm 1 and DAISY on *all* examples from [7, 43, 41, 39]. The runtimes of both programs were below 1 minute.[1] Thus, from now on, we will elaborate on several (four) more challenging problems taken from literature [5, 9, 10, 13, 25, 27] and compare the performance of Algorithm 1 and the three other software packages (DAISY, COMBOS, GenSSI 2.0) on them.

Table 6.1 summarizes the runtimes. In the table, our timing is the best of timings obtained by performing Step 4 of Algorithm 1 by Method (i) and Method (ii) from Remark 5.3.

Example 6.1. The following system of ODEs corresponds to a chemical reaction network [10, Eq. 3.4], which is a reduced fully processive, n-site phosphorylation

[1] The example built in [39, Section 6] looks challenging since it involves 42 state variables and parameters. However, it is actually not challenging since it can be straightforwardly divided into several non-challenging problems: one can first analyze the identifiability of the parameters that appear in the first equation (using only this first equation) and then add the other equations one-by-one to analyze the remaining parameters. The corresponding computation for the first equation takes less than 1 minute for both programs, and the computations for the other equations can be done after that even by hand.

TABLE 6.1. RUNTIMES (IN MINUTES) ON CHALLENGING PROBLEMS

Example	GenSSI 2.0	COMBOS	DAISY	Algorithm 1
Example 6.1	$*$	$**$	$>6{,}000$	<1
Example 6.2	$*$	85	30	3
Example 6.3	$>$ 12,000	$**$	$>6{,}600$	48
Example 6.4	$>$ 12,000	$**$	$>7{,}800$	993

$*$: GenSSI 2.0 returns "*Warning: Unable to find explicit solution.*"
$**$: COMBOS returns "*Model may have been entered incorrectly or cannot be solved with COMBOS algorithms.*"

network.

$$\begin{cases} \dot{x}_1 &= -\mu_1 x_1 x_2 + \mu_2 x_4 + \mu_4 x_6, \\ \dot{x}_2 &= -\mu_1 x_1 x_2 + \mu_2 x_4 + \mu_3 x_4, \\ \dot{x}_3 &= \mu_3 x_4 + \mu_5 x_6 - \mu_6 x_3 x_5, \\ \dot{x}_4 &= \mu_1 x_1 x_2 - \mu_2 x_4 - \mu_3 x_4, \\ \dot{x}_5 &= \mu_4 x_6 + \mu_5 x_6 - \mu_6 x_3 x_5, \\ \dot{x}_6 &= -\mu_4 x_6 - \mu_5 x_6 + \mu_6 x_3 x_5 \end{cases}$$

Setting the outputs $y_1 = x_2$ and $y_2 = x_3$, we obtain a system of the form (2.1). We run Algorithm 1 setting $\boldsymbol{\theta}^\ell := \{\mu_1, \ldots, \mu_6, x_1^*, \ldots, x_6^*\}$ (a calculation shows that these parameters are locally identifiable). The intermediate results are the following:

- the system E consists of 104 equations in 116 variables;
- $D_1 = 4{,}204{,}800$;
- $\boldsymbol{\beta} = (7, 7)$, $\boldsymbol{\alpha} = (6, 7, 7, 6, 6, 6)$;
- $D_2 = 4{,}936{,}445{,}783 \cdot 10^{11}$;
- the system $\widehat{E}^t$ consists of 52 equations in 50 variables.

The algorithm returns that all the parameters are globally identifiable with probability at least 99%. If we perform the last step of Algorithm 1 using Method (i) from Remark 5.3, the runtime is 0.4 minutes (the estimated sequential time is 0.9 minutes). If we use Method (ii), the runtime is 0.4 minutes (the estimated sequential time is 0.9 minutes). The timings for both methods are similar, and the improvement by computing in parallel is not that significant because the Gröbner bases computations are relatively easy in this case compared to the other steps of the algorithm.

- DAISY did not output any result in 100 hours.
- COMBOS returned "Model may have been entered incorrectly or cannot be solved with COMBOS algorithms".
- GenSSI 2.0 constructed a polynomial system that could not be solved symbolically by MATLAB. MATLAB returned "Warning: Unable to find explicit solution." and an empty set of solutions. As a result, the algorithm

reports that x_2^* and x_3^* are globally identifiable and the other parameters are locally identifiable. It was not able to determine whether these other parameters are globally identifiable.

Example 6.2. The following version of SIWR is an extension of the SIR model, see [25, Eq. 3]:

$$\begin{cases} \dot{s} & = \mu - \beta_I si - \beta_W sw - \mu s + \alpha r, \\ \dot{i} & = \beta_W sw + \beta_I si - \gamma i - \mu i, \\ \dot{w} & = \xi(i - w), \\ \dot{r} & = \gamma i - \mu r - \alpha r \end{cases}$$

where s, i, and r stand for the fractions of the population that are susceptible, infectious, and recovered, respectively. The variable w represents the concentration of the bacteria in the environment. The scalars $\alpha, \beta_I, \beta_W, \gamma, \mu, \xi$ are unknown parameters. Following [25], we assume that we can observe $y_1 = \kappa_1 i$, where κ_1 is one more unknown parameter. We will also assume that one can measure the total population $s + i + r$, so $y_2 = s + i + r$. We run Algorithm 1 setting $\boldsymbol{\theta}^\ell := \{\alpha, \beta_I, \beta_W, \gamma, \mu, \xi, \kappa_1, s^*, i^*, w^*, r^*\}$ (it was show in [25] that they are locally identifiable). The intermediate results are the following:

- the system E consists of 66 equations in 77 variables;
- $D_1 = 2{,}653{,}200$;
- $\boldsymbol{\beta} = (10, 8)$, $\boldsymbol{\alpha} = (9, 10, 9, 8)$;
- $D_2 = 1{,}744{,}556{,}312 \cdot 10^{12}$;
- the system $\widehat{E}^t$ consists of 54 equations in 47 variables.

The algorithm returns that all parameters are globally identifiable with probability at least 99%. If we perform the last step of Algorithm 1 using Method (i) from Remark 5.3, the runtime is 3 minutes (the estimated sequential time is 77 minutes). If we use Method (ii), the runtime is 15 minutes (the estimated sequential time is 114 minutes). Unlike in Example 6.1, in this case, the Gröbner bases computations are the most time-consuming part of the algorithm. Because of this, the parallelization of the Gröbner bases computations in MAPLE gives almost 8 times speed up while using Method (ii). Combined with performing different Gröbner bases computations in parallel while using Method (i), it gives an almost 25 times speed up.

- DAISY took 30 minutes to output the correct result.
- COMBOS took 85 minutes to output the correct result.
- GenSSI 2.0 constructed a polynomial system that could not be solved symbolically by MATLAB. MATLAB returned “Warning: Explicit solution could not be found” and an empty set of solutions. Because of this, the algorithm was not able to determine if the parameters are globally identifiable.

Example 6.3. Consider the model of NFκB regulatory module proposed in [27] (see also [5] and [9, Case 6]) defined by the following system [9, Equation 27]
(6.1)

$$\begin{cases}
\dot{x}_1 = k_{prod} - k_{deg}x_1 - k_1x_1u,\\
\dot{x}_2 = -k_3x_2 - k_{deg}x_2 - a_2x_2x_{10} + t_1x_4 - a_3x_2x_{13} + t_2x_5 + (k_1x_1 - k_2x_2x_8)u,\\
\dot{x}_3 = k_3x_2 - k_{deg}x_3 + k_2x_2x_8u,\\
\dot{x}_4 = a_2x_2x_{10} - t_1x_4,\\
\dot{x}_5 = a_3x_2x_{13} - t_2x_5,\\
\dot{x}_6 = c_{6a}x_{13} - a_1x_6x_{10} + t_2x_5 - i_1x_6,\\
\dot{x}_7 = i_1k_vx_6 - a_1x_{11}x_7,\\
\dot{x}_8 = c_4x_9 - c_5x_8,\\
\dot{x}_9 = c_2 + c_1x_7 - c_3x_9,\\
\dot{x}_{10} = -a_2x_2x_{10} - a_1x_{10}x_6 + c_{4a}x_{12} - c_{5a}x_{10} - i_{1a}x_{10} + e_{1a}x_{11},\\
\dot{x}_{11} = -a_1x_{11}x_7 + i_{1a}k_vx_{10} - e_{1a}k_vx_{11},\\
\dot{x}_{12} = c_{2a} + c_{1a}x_7 - c_{3a}x_{12},\\
\dot{x}_{13} = a_1x_{10}x_6 - c_{6a}x_{13} - a_3x_2x_{13} + e_{2a}x_{14},\\
\dot{x}_{14} = a_1x_{11}x_7 - e_{2a}k_vx_{14},\\
\dot{x}_{15} = c_{2c} + c_{1c}x_7 - c_{3c}x_{15}
\end{cases}$$

In the above system, u is the input function, $x_1, \ldots, x_{15}$ are the state variables, and the rest are scalar parameters. The outputs are $y_1 = x_2$, $y_2 = x_{10} + x_{13}$, $y_3 = x_9$, $y_4 = x_1 + x_2 + x_3$, $y_5 = x_7$, and $y_6 = x_{12}$. The values of some of the parameters are known from the existing literature (see [5, Table 1]), so we run Algorithm 1 with

$$\boldsymbol{\theta}^\ell = (t_1, t_2, c_{3a}, c_{4a}, c_5, k_1, k_2, k_3, k_{prod}, k_{deg}, i_1, e_{2a}, i_{1a}, x_1^*, \ldots, x_{14}^*).$$

The intermediate results are the following:

- the system E consists of 588 equations in 623 variables;
- $D_1 = 80{,}640{,}000$;
- $\boldsymbol{\beta} = (8, 7, 6, 7, 6, 6)$, $\boldsymbol{\alpha} = (7, 8, 7, 7, 7, 6, 6, 7, 6, 7, 6, 6, 7, 6, 1)$;
- $D_2 = 1{,}856{,}032{,}379 \cdot 10^{24}$;
- the system $\widehat{E}^t$ consists of 134 equations in 121 variables.

The algorithm returns that all parameters are globally identifiable with probability at least 99%. If we perform the last step of Algorithm 1 using Method (i) from Remark 5.3, the runtime is 48 minutes (the estimated sequential time is 926 minutes). If we use Method (ii), the runtime is 190 minutes (the estimated sequential time is 1,830 minutes).

- DAISY did not output any result in 110 hours.
- COMBOS returned “Model may have been entered incorrectly or cannot be solved with COMBOS algorithms”.
- GenSSI 2.0 did not output any result in 200 hours.

A version of this problem was solved by GenSSI-like method in [5] under an additional assumption on the initial conditions [5, p. 9], which we do not make.

Example 6.4. Consider the following model arising in pharmacokinetics [13]:

$$\begin{cases} \dot{x}_1 = a_1(x_2 - x_1) - \frac{k_a V_m x_1}{k_c k_a + k_c x_3 + k_a x_1}, \\ \dot{x}_2 = a_2(x_1 - x_2), \\ \dot{x}_3 = b_1(x_4 - x_3) - \frac{k_c V_m x_3}{k_c k_a + k_c x_3 + k_a x_1}, \\ \dot{x}_4 = b_2(x_3 - x_4). \end{cases} \tag{6.2}$$

The only output is $y = x_1$. Despite having moderate size, the global identifiability of (6.2) is out of reach for all software tools we are aware of, see also [9, Case 2]. We consider a simplified version of (6.2) with the additional assumption $a_1 = a_2$. We run Algorithm 1 with

$$\boldsymbol{\theta}^\ell = \{a_1, b_1, b_2, k_a, k_c, V_m, x_1^*, \dots, x_4^*\}.$$

The intermediate results are the following:

- the system E consists of 55 equations in 65 variables;
- $D_1 = 3{,}240{,}000$;
- $\boldsymbol{\beta} = (11)$, $\boldsymbol{\alpha} = (11, 10, 10, 9)$;
- $D_2 = 1{,}923{,}937{,}761 \cdot 10^{13}$;
- the system $\widehat{E}^t$ consists of 51 equations in 50 variables.

The algorithm returns that all parameters are globally identifiable with probability at least 99%. If we perform the last step of Algorithm 1 using Method (i) from Remark 5.3, the runtime is 993 minutes (the estimated sequential time is 11,944 minutes). If we use Method (ii), the runtime is 6,042 minutes (the estimated sequential time is 112,000 minutes).

- DAISY did not output any result in 130 hours.
- COMBOS returned “Model may have been entered incorrectly or cannot be solved with COMBOS algorithms”.
- GenSSI 2.0 did not output any result in 200 hours.

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