SECTIONS OF A DIFFERENTIAL SPECTRUM

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Abstract. We construct sections of a structure presheaf of rational functions on a differential spectrum using only localization and projective limits. For this purpose we introduce a special form of a multiplicative system generated by one differential polynomial and call it *D-localization*. This technique allows us to obtain sections of a differential spectrum of a differential ring $\mathcal{R}$ without computation of diffspec $\mathcal{R}$. We compare our construction with Kovacic's structure sheaf and with the results obtained by Keigher in [8] and discuss computational aspects. We prove that if $\mathcal{R}$ is a factorial differential ring then our construction gives the same answer as Kovacic's one. Moreover, if $\mathcal{R}$ is an RAAD ring then applying sheafification to our preasheaf one gets Kovacic's structure sheaf.

1. Introduction

In [13] Jerald Kovacic presented his approach to construction of differential schemes. He generalized Hartshorne's point of view on commutative schemes (see [3]). We propose another way (which was started in [15]) to define differential (affine) schemes using Shafarevich's approach to commutative schemes described in [17].

It is a natural problem to investigate when these approaches give the same answer. In [8] William Keigher solved this problem for *very nice* differential rings. We do this for a different class of differential rings. It is actually a different result that is shown in Example 1. Theorem 1 together with Theorem 2 from Section 4 tell us that Kovacic's approach is equivalent to ours for a certain class of differential rings. Namely, this is true for factorial differential rings also called unique factorization differential domains.

Our goal is to represent sections on principal open subsets, in particular global sections, as clear as it is possible. In commutative algebra these sections are localizations of a given ring over certain multiplicative systems. In Section 3 we generalize these multiplicative systems in order to represent sections in the simplest way. In Section 6 a method for computing these multiplicative systems is presented. Example 4 shows how easy one can compute global sections using our presentation of them.

Although the presheaf constructed in this paper not always a sheaf (it is demonstrated in Example 2 from Section 5), one can apply sheafification to obtain Kovacic's structure sheaf. This works fine for a big class of differential rings, namely, RAAD rings introduced in [12] that is a generalization of reduced rings. Section 7 is

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devoted to a study of this problem. The result about RAAD rings and sheafification is contained in Theorem 3.

2. Basic notions

We consider differential rings and fields. A differential ring is a commutative one with the unity and a basic set of differentiations $\Delta = \{\delta_1, \ldots, \delta_n\}$ on the ring. Differential ideals in differential rings are the ideals stable under the action of Δ.

We refer to [9, 10, 16, 18] for the basic definitions concerning differential algebra. Recent tutorials on constructive differential ideal theory that are very useful in our computations are presented in [5, 6, 18]. We denote elements of a differential ring by $f, g, h, \ldots$ and use the notation $\mathfrak{p}, \mathfrak{q}, \mathfrak{a}, \mathfrak{b}$ for ideals. Let $\mathcal{R}$ be a differential ring and $F \subset \mathcal{R}$. For the differential and radical differential ideal generated by F in $\mathcal{R}$, we use the notation $[F]$ and $\{F\}$, respectively.

Definition 1. Let $\mathcal{R}$ be a differential ring. Then $\operatorname{diffspec}\mathcal{R} = X$ is the set of prime differential ideals of $\mathcal{R}$ with the *Kolchin topology*: closed subsets are

$$\mathbb{V}(E) = \{\mathfrak{p} \in \operatorname{diffspec}\mathcal{R} : E \subset \mathfrak{p}\}$$

for $E \subset \mathcal{R}$. For any $f \in \mathcal{R}$ denote $\mathbb{D}(f) = \operatorname{diffspec}\mathcal{R} \setminus \mathbb{V}(f)$.

We use notions of projective (inverse) and injective (direct) limits (see, e.g., [3, Chapter II, Exercises 1.10, 1.11]). They are denoted by $\varprojlim$ and $\varinjlim$, respectively. We deal with presheaves, sheaves, and their stalks (see [3, pages 61, 62]).

3. Structure sheaf and presheaves on $\operatorname{diffspec}\mathcal{R}$

3.1. **D-localization.** Let $\mathcal{R}$ be a differential ring and $f \in \mathcal{R} \setminus \{0\}$. Let $\mathcal{D}$ be the set of linear differential operators with coefficients in $\mathcal{R}$. Construct a differential ring $\mathcal{R}_{S^\infty(f)}$ as follows. Let

$$\begin{aligned} S_f = \{ s \in \mathcal{R} \mid s \in D_1^{-1}((D_2^{-1}((\ldots(D_k^{-1}(f^{n_k}))^{n_{k-1}}\ldots)^{n_2}))^{n_1}),\\ D_i \in \mathcal{D}, n_i \in \mathbb{N}\cup\{0\}, 1 \leqslant i \leqslant k\}, \end{aligned}$$

where $D_i^{-1}(X)$ means the preimage of set X w.r.t the operator D_i. Let $S^\infty(f)$ be the multiplicative set generated by S_f.

This is a very natural way to define denominators. It comes from the problem of defining of differential ring homomorphisms between rings of sections on principal open subsets. Proposition 3 will show that our multiplicative systems $S^\infty(f)$ coincide with the multiplicative systems $S_{Keigher}(f)$ introduced by William F. Keigher (see [8, Section 2]).

Let $\mathbb{D}(f) \subset \mathbb{D}(g)$, that is $f \in \{g\}$. If we are in characteristic zero then $f^m = Dg$ for some differential operator $D \in \mathcal{D}$ and $m \in \mathbb{N}$. More generally, if $\mathcal{R}$ is a Keigher ring (in such a ring $\{F\} = \sqrt{[F]}$ for any set $F \subset \mathcal{R}$, see [13, Definition 2.3]) this is true. Hence, in this case the formula for S_f simplifies and needs just *one* differential operator for each $s \in \mathcal{R}$. This will become clear in the proof of Proposition 1.

Consider an arbitrary characteristic and $f, g \in \mathcal{R}$. Construct a differential ring homomorphism $\varphi_{g,f} : \mathcal{R}_{S^\infty(g)} \to \mathcal{R}_{S^\infty(f)}$. Let $\frac{a}{s} \in \mathcal{R}_{S^\infty(g)}$. Hence, $s = s_1 \cdot \ldots \cdot s_t$ and

$$s_j \in D_{1,j}^{-1}((D_{2,j}^{-1}((\ldots((D_{k_j,j}^{-1}(g^{n_{k,j}}))^{n_{k-1,j}})\ldots)^{n_{2,j}}))^{n_{1,j}})$$

for all $1 \leqslant j \leqslant t$.

One can define $\varphi_{g,f}(\frac{a}{s}) = \frac{a}{s'_1\cdot\ldots\cdot s'_t}$, where

$$s'_j \in D_{1,j}^{-1}((\ldots((D_{k_j,j}^{-1}((D^{-1}(f^m))^{n_k,j}))^{n_{k-1},j})\ldots)^{n_1,j})$$

for all $1 \leqslant j \leqslant t$. This definition is logical from the functional point of view, but it is not clear why $\varphi_{g,f}$ is correctly defined. In Defintion 2, we use a "more differential algebraic" approach *better* and *clearer* than the definition given by the previous formula.

Lemma 1. [7, Lemma 1.6] *Let $S, T \subset \mathcal{R}$. Then $\{S\}\{T\} \subset \{ST\}$.*

The proof of the following proposition is due to Jerald Kovacic in the case of an *arbitrary* characteristic.

Proposition 1. *Let $\mathcal{R}$ be a differential ring. For any elements $f, g \in \mathcal{R}$ we have $g \in S^\infty(f)$ iff $f \in \{g\}$.*

Proof. Rewrite the definition of S_f in the following way:

$$S_f^0 = f,$$
$$S_f^r = \{g \in \mathcal{R} \mid Dg = h^e \text{ for some } D \in \mathcal{D}, e \in \mathbb{N}\cup\{0\} \text{ and } h \in S_f^{r-1}\},$$
$$S_f = \bigcup_{r\geqslant 0} S_f^r.$$

Consider the construction of a smallest radical differential ideal containing a set Σ as in [9, page 122]:

$$\{\Sigma\}_0 = \Sigma,$$
$$\{\Sigma\}_1 = \sqrt{[\Sigma]},$$
$$\{\Sigma\}_r = \sqrt{[\{\Sigma\}_{r-1}]},$$
$$\{\Sigma\} = \bigcup_{r\geqslant 0}\{\Sigma\}_r.$$

Hence, by induction, we have $\{\sqrt{[g]}\}_{r-1} = \{g\}_r$ for all $r \geqslant 1$. Moreover,

$$g \in S_f^r \iff f \in \{g\}_r,$$
$$g \in S_f \iff f \in \{g\}.$$

Let us prove this by induction on r. For $r = 0$ we have nothing to prove.

Let $r > 0$ and $g \in S_f^r$. Then $Dg = h^e$ for some $D \in \mathcal{D}$, $e \in \mathbb{N}\cup\{0\}$, and $h \in S_f^{r-1}$. Hence, $h \in \sqrt{[g]}$. By induction, we have

$$f \in \{h\}_{r-1} \subset \{\sqrt{[g]}\}_{r-1} = \{g\}_r.$$

Suppose now that $f \in \{g\}_r$. Then, there exists $e \in \mathbb{N}$ such that $f^e \in [\{g\}_{r-1}]$, that is, there exist $D \in \mathcal{D}$ and $h \in \{g\}_{r-1}$ with $f^e = Dh$. Then, exactly by the definition of S_f^n and S_h^n, we have

$$S_f^n \supset S_h^{n-1}$$

for all $n \geqslant 1$. By the induction hypothesis, $g \in S_h^{r-1} \subset S_f^r$.

Now let us prove that $S_f = S^\infty(f)$. This is sufficient to finish the proof. We have $S_f \subset S^\infty(f)$. Let $g \in S^\infty(f)$, that is, there exist $g_1, \ldots, g_s \in S_f$ such that

$g = g_1 \cdot \ldots \cdot g_s$. We have just proved that $f \in \{g_i\}$ for all i, $1 \leqslant i \leqslant s$. Then, by Lemma 1,

$$f^s \in \{g_1\} \cdot \ldots \cdot \{g_s\} \subset \{g_1 \cdot \ldots \cdot g_s\} = \{g\}.$$

Thus, $f \in \{g\}$ and $g \in S_f$. □

Corollary 1. *One can use Proposition 1 to define the multiplicative system $S^\infty(f)$ associated with an element f of a differential ring $\mathcal{R}$.*

Corollary 2. *A differential unit $a \in \mathcal{R}$ is such an element that $1 \in \{a\}$. Hence, $a \in \mathcal{R}$ is a differential unit iff $a \in S^\infty(1)$. Moreover, for any $f \in \mathcal{R}$ we have $a \in S^\infty(f)$.*

Proof. The first assertion follows from Proposition 1 directly. For the second one since $1 \in \{a\}$, we have $f \in \{a\}$ and then $a \in S^\infty(f)$ again by Proposition 1. □

Proposition 2 shows that S "inverts" the inclusions of principal open subsets.

Proposition 2. *Let $\mathcal{R}$ be a differential ring, $f, g \in \mathcal{R} \setminus \{0\}$, and $\mathbb{D}(g) \supset \mathbb{D}(f)$. Then $S^\infty(g) \subset S^\infty(f)$.*

Proof. Let $h \in S^\infty(g)$. By Proposition 1 we obtain that $g \in \{h\}$. Since $\mathbb{D}(g) \supset \mathbb{D}(f)$ we have $f \in \{g\}$. Hence, $f \in \{h\}$ and Proposition 1 implies that $h \in S^\infty(f)$. □

Hence, we can construct well-defined morphisms.

Definition 2. Let $f, g \in \mathcal{R} \setminus \{0\}$ and $\mathbb{D}(g) \supset \mathbb{D}(f)$. Then $\varphi_{g,f}(\frac{a}{s}) = \frac{a}{s}$ for any $\frac{a}{s} \in \mathcal{R}_{S^\infty(g)}$.

We are going to construct a *structure presheaf* on $\operatorname{diffspec} \mathcal{R}$. For this purpose, we define sections $\mathcal{O}(\mathbb{D}(f))$ on $\mathbb{D}(f)$ for all $f \in \mathcal{R}$ as follows: $\mathcal{O}(\mathbb{D}(f)) = \mathcal{R}_{S^\infty(f)}$. Then, we use the following formula for all open subsets U of $\operatorname{diffspec} \mathcal{R}$:

$$\mathcal{O}(U) = \varprojlim_{\mathbb{D}(f) \subset U} \mathcal{O}(\mathbb{D}(f)),$$

where the projective system $\{\mathcal{O}(\mathbb{D}(f))\}$ is supplied with the set of morphisms

$$\{\varphi_{g,f} : \mathcal{O}(\mathbb{D}(g)) \to \mathcal{O}(\mathbb{D}(f)) \text{ if } \mathbb{D}(g) \supset \mathbb{D}(f)\}.$$

Remark 1. Note that this construction is correctly defined on principal open subsets:

$$\mathcal{R}_{S^\infty(f)} = \varprojlim_{\mathbb{D}(g) \subset \mathbb{D}(f)} \mathcal{O}(\mathbb{D}(g)).$$

Indeed, as we have already emphasized, there exist homomorphisms $\varphi_{f,g} : \mathcal{R}_{S^\infty(f)} \to \mathcal{R}_{S^\infty(g)}$ if $\mathbb{D}(g) \subset \mathbb{D}(f)$. The universal property of $\mathcal{R}_{S^\infty(f)}$ is a consequence of the fact that $\mathcal{O}(\mathbb{D}(f))$ is an element of the projective system of rings of this projective limit.

The following proposition is due to Jerald Kovacic. In [8, page 164] the following multiplicative systems were introduced:

$$S_{Keigher}(f) = \bigcap \{\mathcal{R} \setminus \mathfrak{p} \mid \mathfrak{p} \in \mathbb{D}(f)\}.$$

Proposition 3. *If $f \in \mathcal{R}$ then $S_{Keigher}(f) = S^\infty(f)$.*

Proof. Let $g \in S^\infty(f)$. Then $f \in \{g\}$ by Proposition 1. Suppose that $\mathfrak{p} \in \mathbb{D}(f)$, that is, $f \notin \mathfrak{p}$. If $g \in \mathfrak{p}$ we would get $f \in \{g\} \subset \mathfrak{p}$ that cannot happen. Hence, $g \in \mathcal{R} \setminus \mathfrak{p}$ and $g \in S_{Keigher}(f)$.

Suppose now that $g \in S_{Keigher}(f)$. If $f \notin \{g\}$ then by [13, Proposition 2.6] there exists a prime differential ideal $\mathfrak{p}$ with $\{g\} \subset \mathfrak{p}$ and $f \notin \mathfrak{p}$, that is, $\mathfrak{p} \in \mathbb{D}(f)$. Since $g \in S_{Keigher}(f)$, we have $g \in \mathcal{R} \setminus \mathfrak{p}$, which is a contradiction. □

3.2. **Properties of the structure presheaf.** Consider the restriction of our construction to the case of commutative rings.

Let S be a multiplicative set. Then the *saturation* of S is the smallest multiplicative set $\tilde{S} \supset S$ such that whenever $xy \in \tilde{S}$ we have both x and y belong to the set $\tilde{S}$.

Proposition 4. *If $\mathcal{R}$ is a differential ring with $\delta g = 0$ for all $g \in \mathcal{R}$ and $\delta \in \Delta$ (this is also called a ring of constants) then $S^\infty(f)$ is the saturation of the multiplicative system generated by the powers of f for any $0 \neq f \in \mathcal{R}$.*

Proof. The condition $\delta f = 0$ for all $f \in \mathcal{R}$ and $g \in S^\infty(f)$ implies that $f \in \{g\} = \sqrt{(g)}$ and $f^n = bg$ for some $n \in \mathbb{N}$ and $b \in \mathcal{R}$. In this case g is the preimage of f w.r.t. the differential operator of multiplication by b.

If there are no derivations then inverse differential operators factorize $f \in \mathcal{R}$. Thus, $S^\infty(f)$ is of the form $ag_1^{r_1} \cdot \ldots \cdot g_k^{r_k}$ for some $r_i \in \mathbb{N}$, $a \in \mathcal{R}$ invertible, and g_i are factors of f with $1 \leqslant i \leqslant k$. □

Proposition 5. *If $\mathcal{R}$ is a differential ring of constants then $\mathcal{R}_{S^\infty(f)} \cong \mathcal{R}_f$ for any $0 \neq f \in \mathcal{R}$.*

Proof. If $f = ab$ then $\mathcal{R}_f = \mathcal{R}_{ab} \cong \mathcal{R}_{ab,a,b}$. The last localization is the localization of the ring $\mathcal{R}$ w.r.t. the multiplicative system generated by ab, a, and b. This is Exercise 1 in Chapter II, Section 2 of Bourbaki, Commutative Algebra. So, the result follows from Proposition 4. □

Consider Kovacic's approach to differential schemes.

Definition 3. [12, Definition 3.2] Kovacic's structure sheaf is constructed as follows. For each open set U of X, let $\mathcal{O}_X(U)$ be the set of functions

$$s : U \to \coprod_{\mathfrak{p} \in U} \mathcal{R}_\mathfrak{p}$$

satisfying the following conditions:

(1) $s(\mathfrak{p}) \in \mathcal{R}_\mathfrak{p}$;
(2) there is an open cover U_i of U, and a_i, $b_i \in \mathcal{R}$ such that for each $\mathfrak{q} \in U_i$, $b_i \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a_i/b_i \in \mathcal{R}_\mathfrak{q}$.

So, it is natural to *compare* the definition of sections introduced in this paper with Definition 3.

Corollary 3. *If $\mathcal{R}$ is a differential ring with $\delta f = 0$ for all $f \in \mathcal{R}$ then the structure presheaf $\mathcal{O}$ defined above is a sheaf of commutative rings and, as a result, coincides with Kovacic's structure sheaf.*

Let us investigate the properties of $\operatorname{diffspec} \mathcal{R}$ under localization that will be useful for studying the properties of the structure presheaf $\mathcal{O}$.

Proposition 6. *Let $\mathcal{R}$ be a differential ring and $S \subset \mathcal{R} \setminus \{0\}$ be a multiplicative system. Then*

$$\operatorname{diffspec} \mathcal{R}_S = \{\mathfrak{p} \in \operatorname{diffspec} \mathcal{R} \mid \mathfrak{p} \cap S = \varnothing\}.$$

Proof. If $\mathfrak{p} \in \operatorname{diffspec} \mathcal{R}$ and $\mathfrak{p} \cap S = \varnothing$ then $\mathfrak{p}\mathcal{R}_S$ is a prime differential ideal of $\mathcal{R}_S$. Let $\mathfrak{p} \in \operatorname{diffspec} \mathcal{R}_S$ and $a/s \in \mathfrak{p}$. Consider a basic differential operator $\delta \in \Delta$. We have $\delta(a/s) = (s\delta(a) - a\delta(s))/s^2$. Since $a\delta(s))/s^2 = \delta(s)/s \cdot a/s \in \mathfrak{p}$, we have $\delta(a)/s = s\delta(a)/s^2 \in \mathfrak{p}$. Thus, the set of numerators of $\mathfrak{p}$ is a prime differential ideal of $\mathcal{R}$. □

3.3. **Stalks.** It turns out that $\mathcal{O}$ and $\mathcal{O}_X$ have the same stalks:

Proposition 7. *Let $\mathcal{R}$ be a differential ring. Then the stalk $\mathcal{O}_\mathfrak{p}$ of the structure presheaf $\mathcal{O}$ on* $\operatorname{diffspec} \mathcal{R} = X$ *at a point $\mathfrak{p} \in X$ is equal to $\mathcal{R}_\mathfrak{p}$.*

Proof. By definition, $\mathcal{O}_\mathfrak{p} = \varinjlim\limits_{U \ni \mathfrak{p}} \mathcal{O}(U)$. We have $\mathcal{O}(U) = \varprojlim\limits_{\mathbb{D}(f) \subset U} \mathcal{O}(\mathbb{D}(f))$. If there are morphisms mapping all $\mathcal{O}(\mathbb{D}(f)) \ni \mathfrak{p}$ to some ring $\mathcal{B}$ then one can extend them to morphisms from $\mathcal{O}(U) \ni \mathfrak{p}$ to $\mathcal{B}$. Hence, it is sufficient to take the injective limit using only principal open subsets:

$$\mathcal{O}_\mathfrak{p} = \varinjlim_{\mathbb{D}(f) \ni \mathfrak{p}} \mathcal{O}(\mathbb{D}(f)).$$

We have $\mathcal{O}(\mathbb{D}(f)) = \mathcal{R}_{S^\infty(f)}$ and there exist differential ring homomorphisms $\varphi_f : \mathcal{R}_{S^\infty(f)} \hookrightarrow \mathcal{R}_\mathfrak{p}$. Indeed, according to Proposition 1 if $g \in S^\infty(f)$ then $f \in \{g\}$. Since $\mathfrak{p} \in \mathbb{D}(f)$, we have $f \notin \mathfrak{p}$. Therefore, $g \notin \mathfrak{p}$ and $S^\infty(f) \subset \mathcal{R} \setminus \mathfrak{p}$. Thus, if $\mathfrak{p} \in \mathbb{D}(f) \subset \mathbb{D}(g)$ then the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{R}_{S^\infty(g)} & \xrightarrow{\varphi_{g,f}} & \mathcal{R}_{S^\infty(f)} \\ {\scriptstyle \varphi_g}\downarrow & & {\scriptstyle \varphi_f}\downarrow \\ \mathcal{R}_\mathfrak{p} & = & \mathcal{R}_\mathfrak{p} \end{array} \tag{1}$$

Let $\mathcal{B}$ be a differential ring. Suppose that, for any $\mathbb{D}(f) \ni \mathfrak{p}$, there exist differential ring homomorphisms $\varphi_{\mathcal{B},f} : \mathcal{O}(\mathbb{D}(f)) \to \mathcal{B}$. Moreover, suppose that the following diagram is commutative for any $f, g \in \mathcal{R}$ such that $\mathfrak{p} \in \mathbb{D}(f) \subset \mathbb{D}(g)$:

$$\begin{array}{ccc} \mathcal{R}_{S^\infty(g)} & \xrightarrow{\varphi_{g,f}} & \mathcal{R}_{S^\infty(f)} \\ {\scriptstyle \varphi_{\mathcal{B},g}}\downarrow & & {\scriptstyle \varphi_{\mathcal{B},f}}\downarrow \\ \mathcal{B} & = & \mathcal{B} \end{array} \tag{2}$$

If $h = a/b \in \mathcal{R}_\mathfrak{p}$ then $b \notin \mathfrak{p}$ and $\mathfrak{p} \in \mathbb{D}(b)$. By the construction, there exists a homomorphism $\varphi_{\mathcal{B},b} : \mathcal{R}_{S^\infty(b)} \to \mathcal{B}$. Let us use it in order to map the element h. It is possible, because there is an embedding $\mathcal{R}_{S^\infty(b)} \hookrightarrow \mathcal{R}_\mathfrak{p}$. Therefore, there exists a differential ring homomorphism $\varphi_\mathcal{B} : \mathcal{R}_\mathfrak{p} \to \mathcal{B}$ such that diagrams (1) and (2) are commutative. □

4. Essential results

4.1. **When the presheaf $\mathcal{O}$ is a sheaf.** In Theorem 1 we show that the structure presheaf $\mathcal{O}$ is a sheaf in the case of $\mathcal{R}$ is a *factorial* differential ring. Throughout this section we work in an arbitrary characteristic.

In Section 4 of [8] William Keigher introduced *nice* and *very nice* differential rings. A differential ring $\mathcal{R}$ is said to be nice if the two following things are true about $\mathcal{R}$:

(1) for any differential ideal $I \subset \mathcal{R}$ and multiplicative set $S \subset \mathcal{R}$ such that $I \cap S = \varnothing$ there exists $\mathfrak{p} \in \operatorname{diffspec} \mathcal{R}$ such that $I \subset \mathfrak{p}$ and $\mathfrak{p} \cap S = \varnothing$
(2) for any $a \in \mathcal{R}$ and any multiplicative set $S \subset \mathcal{R}$ such that $\operatorname{ann}(a) \cap S = \varnothing$ there exists a differential ideal $I \subset \mathcal{R}$ with $I \cap S = \varnothing$ and $\operatorname{ann}(a) \subset I$.

A differential ring $\mathcal{R}$ is called very nice if $\mathcal{R}$ is nice and for any finitely generated ideal $I \subset \mathcal{R}$ and any $0 \neq f \in \mathcal{R}$ such that $I \cap S^\infty(f) = \varnothing$ there exists a differential ideal J such that $I \subset J$ and $J \cap S^\infty(f) = \varnothing$. [8, Proposition 4.1] gives an important result that for very nice differential rings Kovacic's approach coincides with the one using localization.

We do this for factorial differential rings and this is a different thing. Let us restrict to Keigher rings (when $\{F\} = \sqrt{[F]}$ for any F) for a moment to understand what nice rings are. Consider a factorial Keigher differential ring $\mathcal{R}$. It is reduced and, hence, by [12, Proposition 6.2] must be nice.

Let us give an example of a *not* very nice factorial differential ring (which must be nice).

Example 1. Consider $\mathcal{R} = k\{y,z\}/[y'-y]$ with $(k)' = 0$ and $\operatorname{char} k = 0$. Take the multicative set $S^\infty(1)$ and the ideal $I = (y+z, z'-1) \subset \mathcal{R}$. We have $y'+1 = (y+z)' - (z'-1) \in \{I\} = J$. Then, $y+1 \in J$. Hence, $z-1 \in J$. Eventually, $1 \in J$. But $I \cap S^\infty(1) = \varnothing$ (see Example 4). Thus, by [12, Proposition 11.7] the ring $\mathcal{R}$ is not very nice. Nevertheless, it is factorial because the polynomial $y'-y$ is linear.

We can formulate and prove an essential result concerning the structure presheaf $\mathcal{O}$ now. We do *not* need a differential ring to be a Keigher one.

Theorem 1. *Let $\mathcal{R}$ be a factorial (unique factorization domain, UFD) differential ring. Then the structure presheaf $\mathcal{O}$ defined above is a sheaf of differential rings on* $\operatorname{diffspec} \mathcal{R}$.

Proof. We check the axioms of a sheaf for principal open subsets since $\mathcal{O}(U) = \varprojlim_{\mathbb{D}(f)\subset U} \mathcal{O}(\mathbb{D}(f))$. The prolongation of this technique to any open subset has been considered in [17, Sheaves, Theorem 1]. It is completely valid for our proof.

According to Proposition 6 and since $\operatorname{diffspec} \mathcal{R}$ is a quasi-compact topological space (see [13, Proposition, page 4]), we only need to check the properties of a sheaf for the case of $U = \operatorname{diffspec} \mathcal{R}$ and $U_i = \mathbb{D}(f_i)$ with $U = \bigcup_{i=1}^{n} U_i$. Indeed, if $U = \mathbb{D}(f)$ and $U_i = \mathbb{D}(f_i)$ then the axioms of a sheaf are satisfied if they are satisfied for $\operatorname{diffspec} \mathcal{R}_{S^\infty(f)}$ and $\overline{U_i} = \mathbb{D}(\overline{f_i})$ where $\overline{f_i}$ is the image of f_i w.r.t. the canonical homomorphism $\mathcal{R} \to \mathcal{R}_{S^\infty(f)}$ for all $1 \leqslant i \leqslant n$. Then we have $1 \in \{f_1, \ldots, f_n\}$.

First, let $u \in \mathcal{R}_{S^\infty(1)}$ and $u = 0$ in each $\mathcal{O}(\mathbb{D}(f_i)) = \mathcal{R}_{S^\infty(f_i)}$. Then there exist $s_1, \ldots, s_n$, where $s_i \in S^\infty(f_i)$, such that $s_i u = 0$ in $\mathcal{R}$. Note that $1 \in \{s_1, \ldots, s_n\}$. According to Lemma 1 we have the following:

$$\{u\} = \{s_1, \ldots, s_n\}\{u\} \subset \{s_1 u, \ldots, s_n u\} = \{0\}.$$

Since the ring $\mathcal{R}$ is *reduced*, we have $\{0\} = (0)$. Thus, $u = 0$ in $\mathcal{R}$.

Second, since $\mathcal{R}$ is an integral domain, we have the embedding to its field of fractions: $\mathcal{R} \subset \operatorname{Quot}\mathcal{R} = k$. Let $k \ni a/b = u_i/s_i$, $s_i \in S^\infty(f_i)$ and $1 \in \{s_1, \ldots, s_n\}$.

We can claim that a/b is an irreducible representation, that is, a and b are relatively prime. But $as_i = bu_i$ for each i, $1 \leqslant i \leqslant n$. This means that each as_i is divisible by b. Hence, we have $s_i = bc_i$ for some $c_i \in \mathcal{R}$ and every i, $1 \leqslant i \leqslant n$. Then all $s_i \in (b)$ and as a consequence $1 \in \{b\}$. Thus, $b \in S^\infty(1)$ by Proposition 1 and $a/b \in \mathcal{R}_{S^\infty(1)}$. □

An essential difference between differential and commutative algebra will be discussed in Section 5. More precisely, the structure presheaf constructed in this paper appears not to be a sheaf of differential rings in some cases. Additional restrictions should be stated in order to avoid this problem. Moreover, each axiom of a sheaf requires a particular additional condition.

Remark 2. To prove the first part of Theorem 1 we only need that $\mathcal{R}$ is a *reduced* ring. Moreover, one can show this first property of a sheaf in the case of $\mathcal{R}$ is an RAAD ring (when $\sqrt{\operatorname{ann}(a)}$ is a differential ideal for any $a \in \mathcal{R}$ and $\operatorname{ann}(a) = \{r \in \mathcal{R} \mid ra = 0\}$). Nevertheless, this condition is *not sufficient* for the second axiom of a sheaf (see Example 2).

4.2. **Comparison with Kovacic's structure sheaf.**

Theorem 2. *Let $\mathcal{R}$ be a factorial differential ring. Then structure sheaf $\mathcal{O}$ on* $\operatorname{diffspec}\mathcal{R}$ *is isomorphic to Kovacic's structure sheaf $\mathcal{O}_X$ on $X = \operatorname{diffspec}\mathcal{R}$.*

Proof. According to Theorem 1 the structure presheaf $\mathcal{O}$ is a sheaf of differential rings. By Proposition 7 the sheaves $\mathcal{O}$ and $\mathcal{O}_X$ have the same stalks. There exists a morphism $\phi : \mathcal{O} \to \mathcal{O}_X$ defined on the sections on principal open subsets.

Prove that this morphism induces an isomorphism of stalks. According to [13, Proposition 10.1] the homomorphism, mapping $\mathcal{O}(\mathbb{D}(f)) \to \mathcal{O}_X(\mathbb{D}(f))$, is injective for all $f \in \mathcal{R}$. Here, we use the fact that $\mathcal{R}$ is a domain. Hence, we have an embedding of stalks. We only need to show the surjectivity.

In order to do this, for each point $\mathfrak{p} \in X$ consider the stalks of the sheaves $\mathcal{O}$ and $\mathcal{O}_X$ at this point. Let $\mathfrak{p} \in U \subset X$ be an open subset and $f \in \mathcal{O}_X(U)$. Then there exists an open cover $\bigcup U_i = U$ such that f is regular on each U_i. Hence, there are $f = a_i/b_i$ and $b_i \notin \mathfrak{q}$ for all $\mathfrak{q} \in U_i$.

Since $\mathfrak{p} \in U$, there exists a number i such that $\mathfrak{p} \in U_i$. Hence, since $f = a_i/b_i$, we have $f \in \mathcal{O}(U_i)$. Taking the injective limit we see that our homomorphism is surjective. Thus, we have an isomorphism of stalks. Hence, the sheaves $\mathcal{O}$ and $\mathcal{O}_X$ are isomorphic. □

Remark 3. If $1 \in (f)$ in a commutative ring $\mathcal{R}$ then f is invertible in $\mathcal{R}$. If $\mathcal{R}$ is a differential ring and $1 \in [f]$ then f belongs to no prime differential ideal $\mathfrak{p} \in \operatorname{diffspec}\mathcal{R}$. Nevertheless, if $\mathfrak{p} \in \operatorname{Spec}\mathcal{R} \setminus \operatorname{diffspec}\mathcal{R}$ and $f \in \mathfrak{p}$ then f is not invertible. For instance, let $\mathcal{R} = \mathbb{C}[x]$, $x' = 1$. Then $\operatorname{diffspec}\mathcal{R} = \{(0)\}$. Though $1 \in [x]$ the polynomial x is not invertible in $\mathcal{R}$.

Let us study the properties of the presheaf $\mathcal{O}$ in the linear case.

Proposition 8. *Let $\mathcal{R} = k\{y_1, \ldots, y_l\}/\mathfrak{p}$ and $\mathfrak{p} = \{f_1, \ldots f_m\}$, where f_i is a linear differential polynomial for each $1 \leqslant i \leqslant m$. Then $\mathcal{R}$ is a factorial differential ring.*

Proof. We have $\mathcal{R}$ is a factor-ring of the ring of commutative polynomials in infinitely many variables over the prime ideal $\mathfrak{p}'$ generated by linear polynomials θf_i for all $\theta \in \Theta$ and $1 \leqslant i \leqslant m$. This ring is isomorphic to a ring $\mathcal{R}'$ of polynomials in infinitely many variables. In conclusion, the ring $\mathcal{R}'$ is factorial, because each its element depends only on finitely many variables. □

Corollary 4. *Let $\mathcal{R} = k\{y_1, \ldots, y_l\}/\mathfrak{p}$ and $\mathfrak{p} = \{f_1, \ldots f_m\}$, where f_i is a linear differential polynomial for each $1 \leqslant i \leqslant m$. Then $\mathcal{O}$ is a sheaf and is isomorphic to Kovacic's structure sheaf.*

Proof. This is a consequence of Theorem 2 and Proposition 8. □

The importance of this assertion comes from the Picard-Vessiot theory of differential field extensions, where a linear differential polynomial is one of the principal objects to investigate. The connection between the theory of differential schemes and the differential Galois theory is discussed in [11].

5. Counter-Examples

We show that Theorem 1 does not hold if we omit the condition that $\mathcal{R}$ is a factorial ring. First, we give reasons for such an example and propose the way how to obtain it logically.

Let $\mathcal{R}$ be a domain with $\mathcal{F} = \operatorname{Quot}\mathcal{R}$ with $\operatorname{char}\mathcal{F} = 0$. Suppose that a set v_i/s_i, $1 \leqslant i \leqslant n$, of functions on $\mathbb{D}(s_i)$ with $\bigcup \mathbb{D}(s_i) = \operatorname{diffspec}\mathcal{R}$ is given. Then, $1 \in \{s_1, \ldots, s_n\}$. To prove Theorem 1, we have to find $a/b \in k$ such that $v_i/s_i = a/b$, where a/b has a representation with a "good" denominator, i.e., the denominator must belong to $S^\infty(1)$.

Let $\mathfrak{p} \in \operatorname{diffspec}\mathcal{R}$. If $b \in \mathfrak{p}$ then $a \in \mathfrak{p}$. Indeed, for all $1 \leqslant i \leqslant n$ we have $v_i b = s_i a$ and if $a \notin \mathfrak{p}$ then $s_i \in \mathfrak{p}$ that is a contradiction. Thus, we have obtained that $a \in \{b\}$. Then $a^m = \sum a_j \theta_j b$. To simplify our consideration we assume that $m = 1$. If there exists such a representation without differential operators then $a/b \in \mathcal{R}$. So, we should consider the opposite case to obtain a counter-example. Thus, the simplest form of the numerator of the counter-example is b'.

Another reason for consideration of Example 2 is the following. The simplest case is the case of $n = 2$. We have $1 \in \{s_1, s_2\}$. Let us simplify the situation more. Suppose $\mathcal{R} = k\{y\}$ with $\operatorname{char} k = 0$. We do *not* restrict to the ordinary differential polynomials. The order of a differential polynomial f is the maximal order of differential variables in f.

If both s_1 and s_2 are of order zero (that is, $s_1, s_2 \in k[y]$) than we can divide them by their greatest common divisor in the ring $k[y]$ and assume that s_1 and s_2 are relatively prime. In this case $1 \in (s_1, s_2)$, i.e., $1 = as_1 + bs_2$ for some polynomials $a, b \in k[y]$. Since $s_1\dfrac{u_1}{s_1} = u_1, s_2\dfrac{u_1}{s_1} = u_2 \in \mathcal{R}$, we have $\dfrac{u_1}{s_1} = au_1 + bu_2 \in \mathcal{R} \subset \mathcal{R}_{S^\infty(1)}$.

Remark 4. Note that the same idea works for any $n \in \mathbb{N}$ when $1 \in (s_1, \ldots, s_n)$. Indeed, let $a_1 s_1 + \ldots + a_n s_n = 1$. Then

$$\frac{u_1}{s_1} = a_1 u_1 + \ldots + a_n u_n \in \mathcal{R} \subset \mathcal{R}_{S^\infty(1)}.$$

Thus, if one denominator is y then the other should be of order 1 and, for example, equals $y' + 1$, because 1 is not allowed to have a representation involving s_1 and s_2 without differentiations. The numerators of this example can be obtained by differentiating the denominators:

Example 2. Let $\mathcal{R} = k\{y\}$ and $\mathfrak{a} = \{yy'' - y'(y'+1)\}$. Denote $\mathcal{R}/\mathfrak{a} = \mathcal{R}'$ and $X = \operatorname{diffspec}\mathcal{R}'$. Then, $\mathcal{R}'$ is a domain. Indeed,

$$\{yy'' - y'(y'+1)\} = [yy'' - y'(y'+1)] : (y)^\infty \cap \{y\}.$$

Moreover, $[yy'' - y'(y'+1)] : (y)^\infty \subset \{y\}$. Thus,

$$\{yy'' - y'(y'+1)\} = [yy'' - y'(y'+1)] : (y)^\infty.$$

Since the polynomial $yy'' - y'(y'+1)$ is irreducible, the radical differential ideal $[yy'' - y'(y'+1)] : (y)^\infty$ is prime.

Let $f_1 = y$ and $f_2 = y' + 1$. Consider $U_1 = \mathbb{D}(f_1)$ and $U_2 = \mathbb{D}(f_2)$. Since $1 \in \{f_1, f_2\}$, we have $U_1 \cup U_2 = X$. Take

$$u_1/f_1 = y''/(y'+1) \quad \text{and} \quad u_2/f_2 = y'/y.$$

By the construction, $u_1/f_1 = u_2/f_2 = a/b$. Nevertheless, $a/b \notin \mathcal{R}'_{S^\infty(1)} = \mathcal{R}'$. This is completely shown in [12, Section 10].

Remark 5. Example 2 is also interesting because of the minimality of orders of differential polynomials in the case of a factor-ring of $k\{y\}$ when $\operatorname{char} k = 0$. This follows from the discussions before Example 2.

6. Computation of Sections

Consider constructive aspects of our theory. The following example illustrates the difference between commutative and differential schemes.

Example 3. Consider a prime differential ideal $I = \{y' - 1\}$ in the ring of differential polynomials $k\{y\}$. Let $\mathcal{R} = k\{y\}/I$. We have $\mathcal{R} \cong k[x]$ with $x' = 1$ and $S^\infty(1) = k[x] \setminus \{0\}$. Thus, $\mathcal{O}(\operatorname{diffspec}\mathcal{R}) = \mathcal{R}_{S^\infty(1)} = k(x) \ncong k[x]$.

Let $\mathcal{R} = k\{y_1, \ldots, y_l\}$. We show how to compute sections on $\operatorname{diffspec}\mathcal{R}/\mathfrak{a} = \mathcal{R}'$ for a differential ideal $\mathfrak{a}$. To compute sections means to compute $S^\infty(f)$ for every $f \in \mathcal{R}'$, i.e., to be able to test the membership to $S^\infty(f)$.

Proposition 9. *Let $g \in \mathcal{R}' = \mathcal{R}/\mathfrak{a}$. Then $g \in S^\infty(f)$ iff $f \in \{\mathfrak{a}, g\}$ in $\mathcal{R}$.*

Proof. Due to Proposition 1 this is true since $\{g\} = \{\mathfrak{a}, g\}$ in $\mathcal{R}'$. □

Let $\operatorname{char} k = 0$. In [1, 4], factorization free algorithms for testing the membership of a differential polynomial to a radical differential ideal are presented. These algorithms represent a radical differential ideal as an intersection of regular differential ideals and one can test membership to each component of this decomposition using partial pseudo-reduction and algebraic computations. These algorithms are also discussed in detail in [18].

Corollary 5. *One can test the membership to the multiplicative set $S^\infty(f)$ in the ring $\mathcal{R}'$ using only factorization free computations.*

Our computations become tractable if, in Corollary 5, we require that $\mathfrak{a}$ is a finitely generated or radical differential ideal. Thus, if one can describe the ring $\mathcal{R}'$ algorithmically then one can do that for $S^\infty(f)$ when $f \in \mathcal{R}'$.

Example 4. Let $\mathcal{R} = k\{y\}/[y' - y]$. Assume $\operatorname{char} k = 0$ and k consists of *constants* so that $f' = 0$ for any $f \in k$. This ring is nothing else as the ring of polynomials in one variable $k[y]$ with derivation $'$ defined on y by $(y)' = y$. It is obviously factorial and we can apply Theorem 2 constructing global sections $\mathcal{O}_X(\operatorname{diffspec}\mathcal{R})$.

So, we proved that $\mathcal{O}_X(\operatorname{diffspec}\mathcal{R}) = R_{S^\infty(1)}$. Hence, we need to know what the denominators are. Let us compute $S^\infty(1)$. According to Proposition 9 we need to describe all $g \in k\{y\}$ such that $1 \in \{y' - y, g\} = I$. Replacing derivatives $y^{(t)}$ by y, for all possible $t \in \mathbb{N}$, in g we may assume that $g \in k[y]$. Let $g = a_n y^n + a_{n-1}y^{n-1} + \ldots + a_0$ with $a_i \in k$. We have $g' = (na_n y^{n-1} + \ldots + a_1)y'$ because $a_i' = 0$ for all i, $0 \leqslant i \leqslant n$. Reducing g' w.r.t. $y' - y$ we get $p = (na_n y^{n-1} + \ldots + a_1)y \in I$.

Then, the only thing we need to do is to compute a gcd. Assume $h \in k\{y\} \setminus k$ divides both g and p. We may square-free the polynomial g. In this case such an h exists if and only if $a_0 = 0$, because if g is square-free it is relatively prime with its separant $S_g = na_n y^{n-1} + \ldots + a_1$. Moreover, h must be equal to y.

Thus, if $a_0 \neq 0$ then $1 \in I$ and $g \in S^\infty(1)$. If $a_0 = 0$ then 1 cannot be in I because $1 \notin [y] \supset [y' - y, g]$ in this case. Summarizing what we have done, $S^\infty(1)$ is the set of differential polynomials in $k\{y\}/[y' - y]$ with a non-zero constant term and

$$\mathcal{O}_X(\operatorname{diffspec}(k\{y\}/[y'-y])) = \{f/g \mid f, g \in k\{y\} \text{ with } g \notin y \cdot k\{y\} \mod [y'-y]\}.$$

7. Sheafification

Let us return to the case of an arbitrary characteristic. Example 2 shows that D-localization does not give us a sheaf of differential rings in general. We can just guarantee that $\mathcal{O}$ is a presheaf. So, we have a presheaf $\mathcal{O}$. Construct the "closest" to this presheaf sheaf $\mathcal{O}_+$ as follows.

A section of $\mathcal{O}_+$ on an open subset $U \subset X$ is the family of elements $\sigma_x \in \mathcal{O}_x$ satisfying the following property. For any point $x \in U$ there exists an open subset V, $x \in V \subset U$ and a section $s \in \mathcal{O}(V)$ such that σ_y coincides with the image of s w.r.t. the homomorphism $\mathcal{O}(V) \to \mathcal{O}_y$ for all $y \in V$.

The sheaf $\mathcal{O}_+$ is called the sheaf *associated* with the presheaf $\mathcal{O}$. If $\mathcal{O}$ is a sheaf then $\mathcal{O}_+$ coincides with $\mathcal{O}$. Remember the following fact. A sequence of sheaves on a space X and homomorphisms

$$\ldots \to \mathcal{O}_1 \xrightarrow{f} \mathcal{O}_2 \xrightarrow{g} \mathcal{O}_3 \to \ldots$$

is exact in $\mathcal{O}_2$ iff for all $x \in X$ the sequence of stalks in $(\mathcal{O}_2)_x$ is exact:

$$\ldots \to (\mathcal{O}_1)_x \xrightarrow{f} (\mathcal{O}_2)_x \xrightarrow{g} (\mathcal{O}_3)_x \to \ldots.$$

Recall the notion of an RAAD ring again.

Definition 4. [12, Definition 6.3] A differential ring $\mathcal{R}$ is called RAAD (radical annihilators are differential) if for every $m \in \mathcal{R}$ the ideal $\sqrt{\operatorname{ann}(m)}$ is differential.

We need to prove the following assertion about localization stability of RAAD rings that is very similar to [13, Proposition 9.4].

Proposition 10. *Let $\mathcal{R}$ be an RAAD differential ring and $0 \notin S \subset \mathcal{R}$ be a multiplicative system. Then the localization $\mathcal{R}_S$ is also RAAD.*

Proof. Let $a/s_1 \in \sqrt{\operatorname{ann}(m/s_2)}$ for $a, b \in \mathcal{R}$ and $s_1, s_2 \in S$. Hence, for some $n_1 \in \mathbb{N}$ and $s \in S$ we have $sa^{n_1}m = 0$ in $\mathcal{R}$. So, $a \in \sqrt{\operatorname{ann}(sm)}$. Therefore, for any $\delta \in \Delta$ we get $\delta a \in \sqrt{\operatorname{ann}(sm)}$, that is, $(\delta a)^{n_2} sm = 0$ in $\mathcal{R}$ for some $n_2 \in \mathbb{N}$. Thus, in $\mathcal{R}_S$ we have

$$(\delta(a/s_1))^{n_1+n_2}\frac{m}{s_2} = \frac{((\delta a)s_1 + a(\delta s_1))^{n_1+n_2} sm}{ss_1^{2(n_1+n_2)}s_2} = 0.$$

□

Proposition 11. *If $\mathcal{R}$ is an RAAD differential ring then there exists a natural differential ring homomorphism mapping injectively: $\mathcal{O}(\mathbb{D}(f)) \to \mathcal{O}_X(\mathbb{D}(f))$.*

Proof. This follows from Proposition 10, [12, Proposition 6.2], and [13, Proposition 7.2]. □

Therefore, as in the proof of Theorem 2, there exists a homomorphism between sections of $\mathcal{O}$ and Kovacic's structure sheaf $\mathcal{O}_X$ that is an isomorphism of stalks if X is a differential spectrum of an RAAD ring. Thus, we have an immediate corollary.

Theorem 3. *Let $X = \operatorname{diffspec} \mathcal{R}$ and $\mathcal{R}$ is an RAAD differential ring. Then the sheaf $\mathcal{O}_+$, associated to our structure presheaf $\mathcal{O}$ on X, is isomorphic to Kovacic's structure sheaf $\mathcal{O}_X$.*

Proof. If the following sequence of sheaves

$$0 \to \mathcal{O}_1 \to \mathcal{O}_2 \to 0$$

is exact then the correspondent sequence of rings

$$0 \to \mathcal{O}_1(U) \to \mathcal{O}_2(U) \to 0$$

is exact for all open $U \subset X$. So, the rings of sections are isomorphic. □

Proposition 12. [12, Proposition 6.4] *If $\mathcal{R}$ is reduced then it is RAAD.*

Corollary 6. *Let $X = \operatorname{diffspec} \mathcal{R}$ and $\mathcal{R}$ is a reduced differential ring. Then the sheaf $\mathcal{O}_+$, associated to the presheaf $\mathcal{O}$, is isomorphic to the sheaf $\mathcal{O}_X$.*

Proof. Follows immediately from Proposition 12 and Theorem 3. □

8. Conclusions

Theorem 1 shows that our approach is equivalent to Kovacic's one on a certain class of differential rings useful, for instance, in the Picard-Vessiot theory of differential fields. Nevertheless, Example 2 demonstrates that there are differential domains where these approaches are not equivalent. A very natural problem is to modify our D-localization in order to obtain a good form of presentation of sections in the case of, for instance, RAAD or reduced differential rings.

It would be also interesting to understand what conditions really make the presheaf $\mathcal{O}$ a sheaf. There are some classes of rings that are close to factorial ones, e.g., normal, half-factorial (see [2] for the definition). Most probably, Theorem 1 is not true for both normal and half-factorial differential rings and Example 2 should be a counter-example but this is just a conjecture.

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