Projects for Differential Algebra
Due Wednesday, December 26, 2012, 11:59 pm EST

Choose one question from the list below, prepare your answer in LaTeX or write it very clearly by hand, produce a PDF of your answer, and send it by e-mail to `aovchinnikov@qc.cuny.edu` by the due date above. Do not forget to provide all details in the proofs. "One can show…", "a calculation shows…", "one may assume…" must be avoided.

1. Find the websites of Alexey Ovchinnikov, Michael Singer, and Lucia di Vizio, download the papers from there dated in the last three years, carefully read their introductions, and, based on this, present a list important open problems in the Differential Galois Theory and describe the partial cases in which these problems have been solved.

2. Develop an algorithm with input $a, b \in \mathbb{Q}(x)$ and output the differential Galois group of $y'' + ay' + by = 0$ over the field $\mathbb{C}(x)$ if the Galois group of $z'' - rz = 0$, where $r = a^2/4 + a'/2 - b$, is known. Hint: read §3.4 and the rest if needed in the paper by Carlos Arreche on the ArXiv:
`http://arxiv.org/abs/1208.2226`.

3. Calculate the differential Galois group of $y'' + 2xy = 0$ over $\mathbb{C}(x)$. Hint: read Kaplansky.

4. Read Kovacic's 1986 paper "An Algorithm for Solving Second Order Linear Homogeneous Differential Equations" (available via Blackboard under Content for our course, the proofs can be skipped) and write down an algorithm that calculates the differential Galois group over $\mathbb{C}(x)$ of $y'' - ry = 0$, where $r \in \mathbb{Q}(x)$, based on the statements in Kovacic's paper.

5. Present a proof of the cyclic vector theorem (if K is a non-constant differential field, then every differential module over K has a cyclic vector) based only on the facts proven in the course (see the lecture notes). Explain why this statement is important for computation with scalar and matrix differential equations.

6. Prove that the differential Galois group G of $Y' = AY$ over K is a subgroup of $SL_n(K^\partial)$ if and only if the differential equation $y' = trace(A)y$ has a solution in K.

7. Let $I \subset K[y_1, \ldots, y_n]$ be a prime ideal. Prove that $\{I\} \subset K\{y_1, \ldots, y_n\}$ is prime.

8. Prove that $[y'^2 - y] : y'^\infty \not\subset [y]$ but $[y'^2 - y^3] : y'^\infty \subset [y]$. Based on this, prove or disprove that
   (a) $\{y'^2 - y\}$ is prime,
   (b) $\{y'^2 - y^3\}$ is prime.

   Hint: among the papers by Evelyne Hubert, find the right one to use. All algorithms and statements from Hubert's paper that are used must be clearly quoted.

9. Prove that every autoreduced set is finite. Find the definition of a coherent autoreduced set in Kolchin's book and prove that, in characteristic zero, every characteristic set of a differential ideal is a coherent autoreduced set.

10. Produce a comprehensive list of all misprints (including the mathematical ones) in the course lecture notes.